School Mathematics is Learning to Work Like a Mathematician

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This paper invites you to temporarily set aside your current interpretation of school mathematics to consider what your classroom might look like day to day if its primary purpose was learning to work like a mathematician, rather than learning maths.

Introduction

Dr. Sophie Carr is director of Bays Consulting, a company she founded in 2009. The company's current team of 12 provides data science and mathematical modelling for a wide range of clients. Sophie also:

- is Vice President for Education and Statistical Literacy at the Royal Statistical Society
- serves on the General Council for the Institute of Mathematics and its Applications and was named the World's Most Interesting Mathematician of 2019.

In an interview with Lynne McClure, Director, Cambridge Mathematics, published in Sip & Snack - Issue 40, July 2022 [Reference 1], Sophie was asked:

> *How would you change the school curriculum, if you had the chance? Why?*

The core of her answer is:

> *If there was a way to encourage maths to be learnt through exploration, where pupils realise that mathematicians are wrong far more than they are right (and that this is nothing to be ashamed of) and which helps develop their curiosity, then I think this would be fantastic.*

There is such a way and, using stories from the investigation Garden Beds collected from many classrooms over a long period, this article illustrates what that could look like. You are invited to suspend your current view of mathematics education for a while to consider (or reconsider) the question:

- What happens if learning to work like a mathematician IS the curriculum?

If you are ready to do that then let's start by imagining the Rationale statement of the National Mathematics Curriculum Document (NMCD) of this imagined time.

Rationale: Learning to Work Like a Mathematician

- *School mathematics is learning to work like a mathematician in classrooms engineered to fascinate, captivate and absorb learners.*
- *Teachers are the engineers and teaching craft (pedagogy) is their tool set.*

If you want to know what it means to work like a nurse, a carpenter, a disc jockey, or anything else, including a mathematician, you ask them. In the early '90s we began asking mathematicians and developed a one page synthesis of responses, Working Mathematically [2], presented in simple language using active verbs.

Working Mathematically

Learning to Work like a Mathematician

First give me an interesting problem.

When mathematicians become interested in a problem they:

- Play with the problem to collect & organise data about it.
- Discuss & record notes and diagrams.
- Seek & see patterns or connections in the organised data.
- Make & test hypotheses based on the patterns or connections.
- Look in their strategy toolbox for problem solving strategies which could help.
- Look in their skill toolbox for mathematical skills which could help.
- Check their answer and think about what else they can learn from it.
- Publish their results.

Questions which help mathematicians learn more are:

- Can I check this another way?
- What happens if …?
- How many solutions are there?
- How will I know when I have found them all?

When mathematicians have a problem they:

- Read & understand the problem.
- Plan a strategy to start the problem.
- Carry out their plan.
- Check the result.

A mathematician's strategy toolbox includes:

- Do I know a similar problem?
- Guess, check and improve
- Try a simpler problem
- Write an equation
- Make a list or table
- Work backwards
- Break the problem into smaller parts
- Act it out
- Draw a picture or graph
- Make a model
- Look for a pattern
- Try all possibilities
- Seek an exception
- ...

If one way doesn't work I just start again another way.

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Figure 1. Working Mathematically

Some have said that children should be learning to think like a mathematician. But thinking cannot be observed or assessed in the classroom. Only the outputs of presumed thinking can be observed and assessed. Hence the emphasis on active verbs.

In 1996, the BBC Horizon episode exploring Andrew Wiles' solution of Fermat's Last Theorem [3] convinced us that this refinement of our conversations with mathematicians was accurate. Teachers exploring Working Mathematically as the starting point for their teaching had already convinced us of its value as a learning framework with comments like "The kids have stopped asking 'Why do we have to do this?'. It just doesn't come up."

- *A mathematician's work begins with an interesting problem. The key words in this sentence are 'begins', 'problem' and 'interesting'.*

Lesson Features

- a checklist for encouraging learning -

- ☐ application focus
- ☐ assessment opportunities
- ☐ builds on personal student experience
- ☐ communicating mathematics
- ☐ concept focus
- ☐ concrete materials
- ☐ concurrent teaching of topics
- ☐ differentiation for ability range
- ☐ easy to state/easy to start
- ☐ estimation
- ☐ first hand data
- ☐ game context
- ☐ group work
- ☐ history of mathematics
- ☐ home/school links
- ☐ inclusive
- ☐ informal or incidental learning
- ☐ interdisciplinary connections
- ☐ investigative process
- ☐ ....
- ☐ ....
- ☐ kinaesthetic
- ☐ links to learning theory
- ☐ mathematical modelling
- ☐ mixed ability
- ☐ multiple entry & exit points
- ☐ non-threatening
- ☐ open-ended
- ☐ outdoor
- ☐ ownership
- ☐ recording & publishing
- ☐ skill development in context
- ☐ social issues
- ☐ story shell
- ☐ technology (calculators)
- ☐ technology (software)
- ☐ physical involvement
- ☐ visual (visualisation)
- ☐ whole class
- ☐ working mathematically process
- ☐ ....
- ☐ ....

- This list has been constructed through discussion with teachers in many workshop situations.
- It is our attempt to build a common language to debate the features of a classroom more likely to engage students in mathematics learning.
- Is there a place for some of these features in every lesson you plan?
- Please 'play' with the list in professional learning and team planning situations and let us know what support it gives you.
- Contact Doug. Williams: doug@blackdouglas.com.au

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Figure 2. Learning Features

NMCD includes many starting problems for each school level and at every level provides Example Lesson Sequences (ELS) from trial classrooms. The starting problem in itself was never enough. The success of the journey relied on teachers deliberately making teaching craft choices likely to fascinate, captivate and absorb learners. In these decisions they were guided by the Learning Features document [4], a checklist of features identified by teachers who observed trial lessons and asked
"What has helped to make this lesson work?".

NMCD doesn't exist, but if you are willing to extend the suspension of your current views a little longer we can explore a lesson sequence that does exist and could be used right now. The first section makes it clear how the lingua franca of learning to work like a mathematician is used as a planning framework.
As you read it would also be useful to:

- have Learning Features beside you to check off the choices made by the teachers involved in creating the example
- make a list with the heading 'What maths is involved in this lesson?'

Garden Beds: an example of learning to work like a mathematician

First give me an interesting problem.
You can find this problem in text books and 'give it' that way. Or you can make deliberate teaching craft (pedagogy) choices and introduce it to a class as follows.

In advance prepare a set of cards about 20cm square which are somehow different on each side. For example mark one side of each with a circle. You also need a collection of square tiles (about 2cm square) in various colours.

- Invite students to join you around a floor space somewhere in the room.
- As students gather, hand them a card each.

Alternatively ask students to fold the short side corner of an A4 page over to the opposite side to create the diagonal of a square, mark the point, fold the corner back flat and the extra rectangle flap in. Voila! a square.

Less than one minute has elapsed (a little longer for the DIY version) and students are experiencing a lesson that:

- doesn't start with something on the board at the front of the room;
- doesn't require them to be in their seats;
- doesn't need a text book.

They are also wondering about the purpose of the card.

Introduce the story shell.

> *"A gardener puts some plants in a garden side by side. Angela, Ricardo and Jo please put your cards on the floor side by side to show what she did. Then she built a pathway of tiles around the edge of the garden to make a border like this ... (put in three cards yourself making sure the 'other' side is used). Now any of you near our model can put down tiles to complete the border."*

Students are already involved in acting out a potential investigation and are beginning to take ownership of the problem in an environment where they are working together with the teacher.

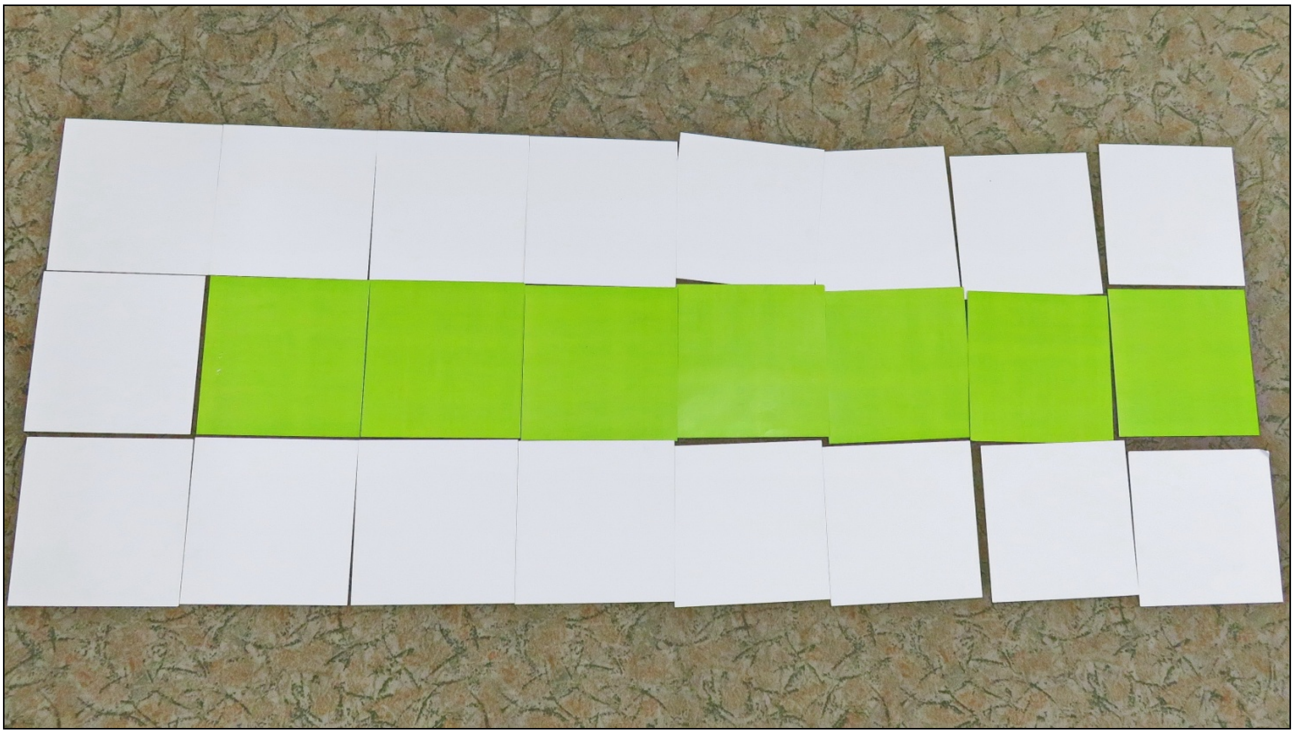

Figure 3. A garden bed of 7 plants with an incomplete border

Play with the problem to collect and organise data.

- Begin with the data pictured so far.
  "For three plants arranged like this how many tiles were needed for the border? ... Ina can you explain how you worked that out. ... Yep, that makes sense. ... Could someone suggest how we could check that calculation another way? ... Good, so now we can be pretty sure that in this problem the number 3 is a pair with.... We should record that piece of data in our journal. Tamer, use the board as our journal and record that for us please. Use the heading Garden Beds and today's date."
- Check that everyone understands the way Tamer has recorded.
- Suggest that the gardener likes the look of this garden bed so much that she decides to extend the line of plants.
- Encourage a student to suggest another number and encourage working together to model that example. But just as they start, ask for a pause.
  "Wait a second. Just before we do that, predict the number of tiles you think we will need and whisper your prediction to a person near you."
- Come to a consensus about the answer and include checking the answer another way.
  "Tamer, please add that data to our journal."

Discuss & record notes and diagrams.

- Students make a personal record of the information so far.
  "Please return to your place for 3 or 4 minutes and make your own journal record. Include a diagram and the data so far. Use the heading Garden Beds and today's date. Then I will have a challenge for you."

Seek & see patterns / Make & test hypotheses / Look in strategy & skill toolboxes

- What happens if our garden is 100 plants long?
  "Your first challenge is to calculate the number of border tiles for a garden bed like this with 100 plants in a line. ... When mathematicians tackle a problem they can't ask anyone else if their answer is correct. Because if it's a problem no one knows the answer.

Mathematicians have to check their own answer another way. So what I am really asking for is one answer, with two different ways to explain it.
You can go to our floorboard with your partner if you want, or you can choose square tiles from this collection I brought with me, or you can draw diagrams, or whatever. Keep notes, but all I want you to be able to do at the moment is talk about what you find out."

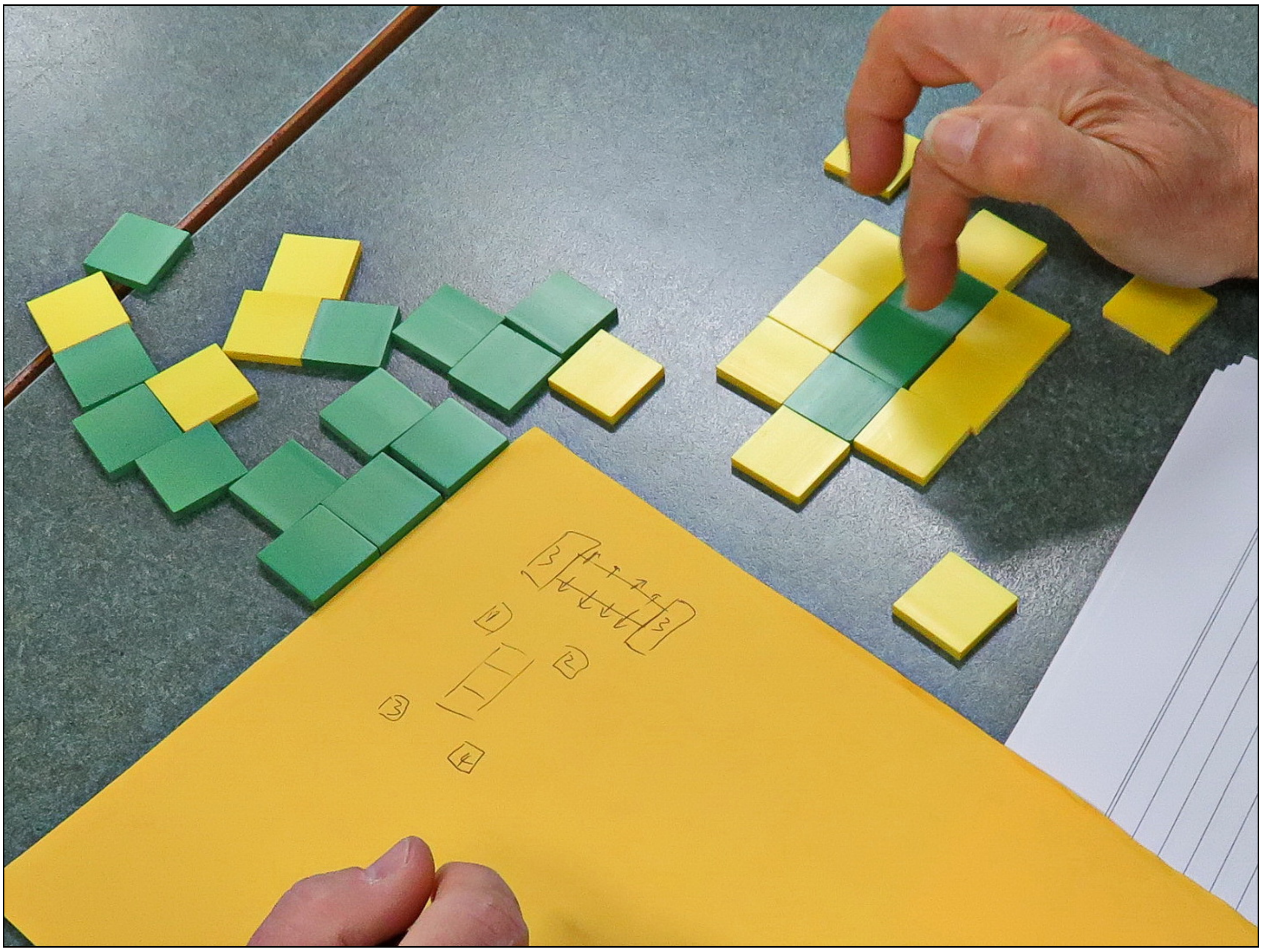

Figure 4. Teacher workshop with improvised journal

Check answers / What can we learn?

- Allow time to pack up. Round off this first lesson by gathering at the floorboard. Invite various students to explain how they worked out the 100 challenge, with the aim of identifying the range of methods in the class.

The Next Lessons

The second lesson begins with the challenge:

"If I tell you any number of plants can you tell me how to find the number of tiles?"

Before reading further, picture how you would structure this second lesson from here. Better still work with your team to sketch out its lesson plan.

One Way Is...

The students' oral explanations from the previous lesson are refreshed and transcribed by the teacher to *written* statements on the board such as these:

- To find the number of tiles, double the number of plants and add six.

- To find the number of tiles, start with 8, then add twice the number of plants less 1.
- To find the number of tiles, add 3 to the number of plants and double that.
- To find the number of tiles, add 2 to the number of plants, multiply that by 3, then subtract the number of plants.

Notes:
1. All these explanations have been suggested by learners from Year 2 to Year 10 in the past.
2. At the end of each explanation add in brackets the names of the students who suggested it-ownership of the problem.

These explanations are algebraic generalisations. Not symbolic, but algebraic none-the-less. The explanations you have will depend on your class. It doesn't matter how many you have, but you are guaranteed to have more than one. If you want to introduce an explanation that hasn't been thought of by your group, you can use this device:

> *"In my class last year someone thought of a way we haven't mentioned yet. Would you like to see it?"*

As soon as you ask students to record these, they will begin to see benefit in shortening them while keeping the meaning.

> *"Do we have to write 'To find the number of tiles' each time? Can't we write T or N or something?"*

Then, for example, the four generalisations above can, with student help, be meaningfully shortened to:
- T = 2xP + 6
- T = 8 + 2x(P - 1)
- T = 2x(P + 3)
- T = 3x(P + 2) - P

and the lesson can round off with various oral and written substitution questions which confirm that it doesn't matter which one you use, the answer is always the same.

The next lesson might open by refreshing a few substitutions and then asking the backwards question.

> *"If I tell you the number of tiles I found in the shed, can you tell me the number of plants I should buy to make a garden bed like ours?"*

List at least three ways students might tackle this challenge. Imagine how you would structure the lesson from here to highlight the process of learning to work like a mathematician. Perhaps a role play where students visit a pretend garden shop and ask assistants to tell them the number of plants if they have found ... tiles?

This lesson, and perhaps the next, could now easily turn to text book questions involving extracting an equation from a word problem. Also substitution and solving equations exercises might be practised. Obviously a skill toolbox has a place in the work of a mathematician, and skills do need to be honed. Teachers trying the Working Mathematically context have often commented on how quickly and successfully the students work through the text exercises after an introductory sequence such as that above.

Garden Beds Lesson Suite

The sequence above is only the start of the Garden Beds suite of lessons. Others in the suite are:
- From the equations above develop experience with equivalent algebraic expressions. "Are you trying to tell me that all these equations give the same answer. How can that be? They all look different." A taunt such as this has often led to students showing the teacher how to expand, for

example, 2x(P + 3). Can you visualise the two L-shaped P + 3 elements in the tile garden on the floorboard?

- Back in the first lesson Tamer recorded the numbers 3 and 12 as a pair. There have been many such pairs calculated since. Paired numbers can be represented in a table - ordering the elements of the table might seem a reasonable way to make sense of such data. It could also lead to asking how many tiles would be needed for zero plants.
- Paired numbers can also be represented like this: (3, 12). Suppose each of 10 students is randomly given an incomplete Garden Beds number pair card, such as (__, 12) or (7, __). First they each calculate their missing number, then work with the rest of the class to show all ten as points on a class grid. What do students predict? The door is opening to the chapter on graphing linear equations and making sense of domain, range, gradient and intercepts in a context already meaningful to students.
- What happens if we plot the change in the number of tiles as we move along the graph from plant to plant?

(Note: The outcome is derived from the linear graph. A constant function of course and perhaps not very surprising. The important thing is the validity, accessibility and transferability of the question. The same question applied in the Jumping Kangaroos investigation [6], for example, leads to a very different sort of derived function.)

- What happens if the gardener plants a rectangular garden of multiple equal rows? Or a square garden? Perhaps an L-shaped garden?

Conclusions

In accepting the invitation to suspend what you knew about school mathematics and ponder the Garden Beds story for a while, perhaps your knowledge of what school mathematics could be has grown. Perhaps you will consider (or reconsider) that there is *no* reason for mathematics to be taught the way it 'always has been'.

In the example the assumed learning situation is a teacher working with a whole class. Multiple pedagogical decisions have been shown to be important in developing insightful, meaningful and successful learning in this context. But does Garden Beds have to be used this way? Are there other pedagogic alternatives?

You can delve into different learning contexts for this problem by following the Maths At Home (MAH) reference [5]. The basis of the MAH approach to Garden Beds is 2 students working in isolation in a self-directed environment. It includes:

- making their own materials by printing and cutting a template provided;
- a task card from the Mathematics Centre Task Library;
- a Picture Puzzle using one screen, two learners, concrete materials and a challenge;
- an additional Investigation Guide into ordered pairs and graphing only available at this link;
- integrated self-assessment;
- photos and lesson notes from a home schooling mum who used the investigation for a whole day in that context with her 7, 10 and 12 year old children;
- a link to the Garden Beds Task Cameo (teaching notes) which includes a photo story from a Year 2 class tackling Garden Beds in their afternoon session because their teacher was part of a Year 9 observation lesson that morning;
- teaching notes for the Garden Beds Picture Puzzle.

All these resources are complementary. Plenty there to help you and your team plan a trial Working Like A Mathematician (WLAM) unit run over time at the best possible venue with the best possible people - your students in your classrooms. In fact, as one teacher suggested with some amazement, at Year 7 or 8 you could plan a whole term around Garden Beds. Using time in this different way might be just the resetting needed to achieve Dr. Carr's learning through exploration curriculum.

References

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