maths with attitude

2nd Edition

Space & Logic Years 9 & 10

Charles Lovitt
Doug Williams

Mathematics Task Centre & Maths300
helping to create happy healthy cheerful productive inspiring classrooms

maths with attitude

2nd Edition

Space & Logic

Years 9 & 10

In this kit:

- *Hands-on problem solving tasks*
- *Detailed curriculum planning*

Access from Maths300:

- *Extensive lesson plans*
- *Software*

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Maths With Attitude includes reference to hands-on problem solving tasks and Maths300 material.

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- To support teachers, schools and systems wanting to create:

  happy, healthy, cheerful, productive, inspiring classrooms

Our Attitude

- to learning:

  learning is a personal journey stimulated by achievable challenge
- to learners:

  stimulated students are creative and love to learn
- to pedagogy:

  the art of choosing teaching strategies to involve and interest all students
- to mathematics:

  mathematics is concrete, visual and makes sense
- to learning mathematics:

  all students can learn to work like a mathematician
- to teachers:

  the teacher is the most important resource in education
- to professional development:

  teachers improve their teaching by re-enacting stories from the classrooms of their colleagues

Our Objective in Detail

What do we mean by creating:

happy, healthy, cheerful, productive, inspiring classrooms

Happy…

means the elimination of the unnecessary fear of failure that hangs over so many students in their mathematics studies. Learning experiences *can* be structured so that all students see there is something in it for them and hence make a commitment to the learning. In so many 'threatening' situations, students see the impending failure and withhold their participation.

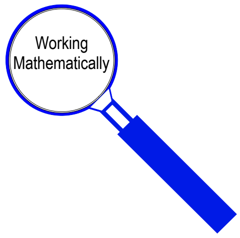
A phrase which describes the structure allowing all students to perceive something in it for them is *multiple entry points and multiple exit points*. That is, students can enter at a variety of levels, make progress and exit the problem having visibly achieved.

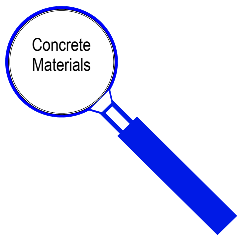
Healthy…

means *educationally healthy*. The learning environment should be a reflection of all that our community knows about how students learn. This translates into a rich array of teaching strategies that could and should be evident within the learning experience.

If we scrutinise the *exploration* through any lens, it should confirm to us that it is well structured or alert us to missed opportunities. For example, peering through a pedagogy lens we should see such features as:

- a story shell to embed the situation in a meaningful context
- significant active use of concrete materials
- a problem solving challenge which provides ownership for students
- small group work
- a strong visual component
- access to supportive software





Cheerful…

because we want 'happy' in the title twice!

Productive…

is the clear acknowledgment that students are working towards recognisable outcomes. They should know what these are and have guidelines to show they have either reached them or made progress. Teachers are accountable to these outcomes as well as to the quality of the learning environment.

Inspiring…

is about creating experiences that are uplifting or exalting; that actually *turn students on*. Experiences that make students feel great about themselves and empowered to act in meaningful ways.

Space & Logic Resources

To help you create

happy, healthy, cheerful, productive, inspiring classrooms

this kit contains

- 20 hands-on problem solving tasks from Mathematics Centre

and a Teachers' Manual which integrates the use of the tasks with

- 9 detailed lesson plans from Maths300

The kit offers **6 weeks** of Scope & Sequence planning in Space and Logic for *each* of Year 9 and Year 10. This is detailed in *Part 2: Planning Curriculum* which begins on Page 12. You are invited to map these weeks into your Year Planner. Together, the four kits available for these levels provide 25 weeks of core curriculum in Working Mathematically (working like a mathematician).

Note: Membership of Maths300 is assumed.
The kit will be useful without it, but it will be much more useful with it.

Tasks

- A Stacking Problem
- Cars In A Garage
- Chess Queens
- Coloured Squares
- Eight Queens
- Haberdasher's Problem
- Koala Carts
- Land Of ET
- Latin Squares
- McMahon's Triangles 1
- McMahon's Triangles 2
- Mirror Patterns 3
- Octaflex
- Pattern Cube
- Pentagon Triangles
- Pentominoes
- Reverse
- Sliding Tiles
- Soft Drink Crates
- Symmetric Shapes

Part 2 of this manual introduces each task. The latest information can be found at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm

Maths300 Lessons

- Cars In A Garage
- Chess Queens
- Eight Queens
- Farmyard Friends
- Four Cube Houses
- Haberdasher's Problem
- Land of ET
- Soft Drink Crates
- Where Do We Sit?

Lessons with Software

- Cars In A Garage
- Chess Queens
- Eight Queens

Part 2 of this manual introduces each lesson. Full details can be found at:

- http://www.maths300.com

Working Like A Mathematician

Our attitude is:

all students can learn to work like a mathematician

What does a mathematician's work actually involve? Mathematicians have provided their answer on Page 8. In particular we are indebted to Dr. Derek Holton for the clarity of his contribution to this description.

Perhaps the most important aspect of Working Mathematically is the recognition that *knowledge is created by a community and becomes part of the fabric of that community*. Recognising, and engaging in, the process by which that knowledge is generated can help students to see themselves as able to work like a mathematician. Hence Working Mathematically is the framework of **Maths With Attitude.**

Skills, Strategies & Working Mathematically

A Working Mathematically curriculum places learning mathematical skills and problem solving strategies in their true context. Skills and strategies are the tools mathematicians employ in their struggle to solve problems. Lessons on skills or lessons on strategies are not an end in themselves.

- **Our skill toolbox** can be added to in the same way as the mechanic or carpenter adds tools to their toolbox. Equally, the addition of the tools is not for the sake of collecting them, but rather for the purpose of getting on with a job. A mathematician's job is to attempt to solve problems, not to collect tools that might one day help solve a problem.
- **Our strategy toolbox** has been provided through the collective wisdom of mathematicians from the past. All mathematical problems (and indeed life problems) that have ever been solved have been solved by the application of this concise set of strategies.

About Tasks

Our attitude is:

mathematics is concrete, visual and makes sense

Tasks are from Mathematics Task Centre. They are an invitation to two students to work like a mathematician (see Page 8).

The Task Centre concept began in Australia in the late 1970s as a collection of rich tasks housed in a special room, which came to be called a Task Centre. Since that time hundreds of Australian teachers, and, more recently, teachers from other countries, have adapted and modified the concept to work in their schools. For example, the special purpose room is no longer seen as an essential component, although many schools continue to opt for this facility.

A brief history of Task Centre development, considerable support for using tasks, for example Task Cameos, and a catalogue of all currently available tasks can be found at:

- http://www.mathematicscentre.com/taskcentre

Key principles are:

- A good task is the tip of an iceberg
- Each task has three lives
- Tasks involve students in the Working Mathematically process

The Task Centre Room or the Classroom?

There are good reasons for using the tasks in a special room which the students visit regularly. There are also different good reasons for keeping the tasks in classrooms. Either system can work well if staff are committed to a core curriculum built around learning to work like a mathematician.

- A task centre room creates a focus and presence for mathematics in the school. Tasks are often housed in clear plastic 'cake storer' type boxes. Display space can be more easily managed. The visual impact can be vibrant and purposeful.
- However, tasks can be more readily integrated into the curriculum if teachers have them at their finger tips in the classrooms. In this case tasks are often housed in press-seal plastic bags which take up less space and are more readily moved from classroom to classroom.

Tip of an Iceberg

The initial problem on the card can usually be solved in 10 to 20 minutes. The investigation iceberg which lies beneath may take many lessons (even a lifetime!). Tasks are designed so that the original problem reveals just the 'tip of the iceberg'. Task Cameos and Maths300 lessons help to dig deeper into the iceberg.

We are constantly surprised by the creative steps teachers and students take that lead us further into a task. No task is ever 'finished'.

> *Most tasks have many levels of entry and exit and therefore offer an on-going invitation to revisit them, and, importantly, multiple levels of success for students.*

Three Lives of a Task

This phrase, coined by a teacher, captures the full potential and flexibility of the tasks. Teachers say they like using them in three distinct ways:

1. As on the card, which is designed for two students.
2. As a whole class lesson involving all students, as supported by outlines in the Task Cameos and in detail through the Maths300 site.
3. Extended by an Investigation Guide (project), examples of which are included in both Task Cameos and Maths300.

The first life involves just the 'tip of the iceberg' of each task, but nonetheless provides a worthwhile problem solving challenge - one which 'demands' concrete materials in its solution. This is the invitation to work like a mathematician. Most students will experience some level of success and accomplishment in a short time.

The second life involves adapting the materials to involve the whole class in the investigation, in the first instance to model the work of a mathematician, but also to develop key outcomes or specific content knowledge. This involves choosing teaching craft to interest the students in the problem and then absorb them in it.

The third life challenges students to explore the 'rest of the iceberg' independently. Investigation Guides are used to probe aspects and extensions of the task and can be introduced into either the first or second life. Typically this involves providing suggestions for the direction the investigation might take. Students submit the 'story' of their work for 'portfolio assessment'. Typically a major criteria for assessment is application of the Working Mathematically process.

About Maths300

Our attitude is:

> *teachers improve their teaching by re-enacting stories from the classrooms of their colleagues*

Maths300 is a subscription based web site. It is an attempt to collect and publish the 300 most 'interesting' maths lessons (K - 12).

- *Lessons have been successfully trialed in a range of classrooms.*
- *About one third of the lessons are supported by specially written software.*
- *Lessons are also supported by investigation sheets (with answers) and game boards where relevant.*
- *A 'living' Classroom Contributions section in each lesson includes the latest information from schools.*
- *The search engine allows teachers to find lessons by pedagogical feature, curriculum strand, content and year level.*
- *Lesson plans can be printed directly from the site.*
- *Each lesson supports teaches to model the Working Mathematically process.*

Modern internet facilities and computers allow teachers easy access to these lesson plans. Lesson plans need to be researched, reflected upon in the light of your own students and activated by collecting and organising materials as necessary.

Maths300 Software

Our attitude is:

> *stimulated students are creative and love to learn*

Pedagogically sound software is one feature likely to encourage enthusiastic learning and for that reason it has been included as an element in about one third of Maths300 lesson plans. The software is used to develop an investigation beyond its introduction and early exploration which is likely to include other pedagogical techniques such as concrete materials, physical involvement, estimation or mathematical conversation. The software is not the lesson plan. It is a feature of the lesson plan used at the teacher's discretion.

For school-wide use, the software needs to be downloaded from the site and installed in the school's network image. You will need to consult your IT Manager about these arrangements. It can also be downloaded to stand alone machines covered by the site licence, in particular a teacher's own laptop, from where it can be used with the whole class through a data projector.

Note:

- *Maths300 lessons and software may only be used by Maths300 members.*

Working Mathematically

First give me an interesting problem.

When mathematicians become interested in a problem they:

- Play with the problem to collect & organise data about it.
- Discuss & record notes and diagrams.
- Seek & see patterns or connections in the organised data.
- Make & test hypotheses based on the patterns or connections.
- Look in their strategy toolbox for problem solving strategies which could help.
- Look in their skill toolbox for mathematical skills which could help.
- Check their answer and think about what else they can learn from it.
- Publish their results.

Questions which help mathematicians learn more are:

- Can I check this another way?
- What happens if ...?
- How many solutions are there?
- How will I know when I have found them all?

When mathematicians have a problem they:

- Read & understand the problem.
- Plan a strategy to start the problem.
- Carry out their plan.
- Check the result.

A mathematician's strategy toolbox includes:

- Do I know a similar problem?
- Guess, check and improve
- Try a simpler problem
- Write an equation
- Make a list or table
- Work backwards
- Act it out
- Draw a picture or graph
- Make a model
- Look for a pattern
- Try all possibilities
- Seek an exception
- Break a problem into smaller parts
- ...

If one way doesn't work, I just start again another way.

Professional Development Purpose

Our attitude is:

the teacher is the most important resource in education

> *We had our first study group on Monday. The session will be repeated again on Thursday. I had 15 teachers attend. We looked at the task Farmyard Friends (Task 129 from the Mathematics Task Centre). We extended it out like the questions from the companion Maths300 lesson suggested, and talked for quite a while about the concept of a factorial. This is exactly the type of dialog that I feel is essential for our elementary teachers to support the development of their math background. So anytime we can use the tasks to extend the teacher's math knowledge we are ahead of the game.*
> District Math Coordinator, Denver, Colorado

Research suggests that professional development most likely to succeed:

- is requested by the teachers
- takes place as close to the teacher's own working environment as possible
- takes place over an extended period of time
- provides opportunities for reflection and feedback
- enables participants to feel a substantial degree of ownership
- involves conscious commitment by the teacher
- involves groups of teachers rather than individuals from a school
- increases the participant's mathematical knowledge in some way
- uses the services of a consultant and/or critical friend

Maths With Attitude has been designed with these principles in mind. All the materials have been tried, tested and modified by teachers from a wide range of classrooms. We hope the resources will enable teacher groups to lead themselves further along the professional development road, and support systems to improve the learning outcomes for students K - 12.

With the support of Maths300 ETuTE, professional development can be a regular component of in-house professional development. See:

- http://www.mathematicscentre.com/taskcentre/resource.htm#etute

For external assistance with professional development, contact:

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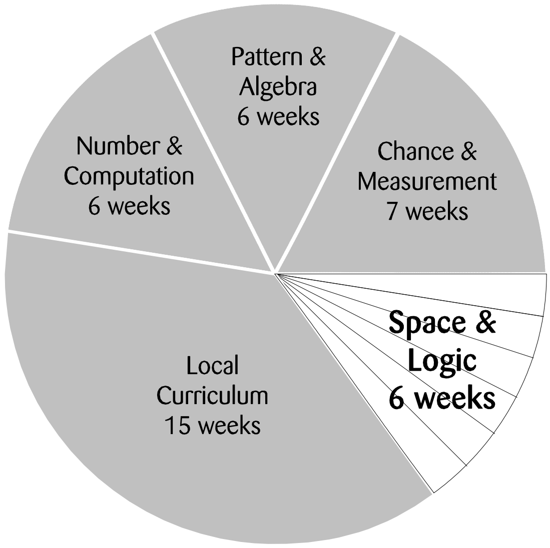
Part 2: Planning Curriculum

Curriculum Planners

Our attitude is:

learning is a personal journey stimulated by achievable challenge



Curriculum Planners:

- show one way these resources can be integrated into your weekly planning
- provide a starting point for those new to these materials
- offer a flexible structure for those more experienced

You are invited to map Planner weeks into your school year planner as the core of the curriculum.

Planners:

- detail each week lesson by lesson
- offer structures for using tasks and lessons
- are sequenced from lesson to lesson, week to week and year to year to 'grow' learning

Teachers and schools will map the material in their own way, but all will be making use of extensively trialed materials and pedagogy.

Using Resources

- Your kit contains 20 hands-on problem solving tasks and reference to relevant Maths300 lessons.
- Tasks are introduced in this manual and supported by the Task Cameos at:
  http://www.mathematicscentre.com/taskcentre/iceberg.htm
- Maths300 lessons are introduced in this manual and supported by detailed lesson plans at:
  http://www.maths300.com

In your preparation, please note:

- Planners assume 4 lessons per week of about 1 hour each.
- Planners are *not* prescribing a continuous block of work.
- Weeks can be interspersed with other learning; perhaps a **Maths With Attitude** week from a different strand.
- Weeks can sometimes be interchanged within the planner.
- Lessons can sometimes be interchanged within weeks.
- The four **Maths With Attitude** kits available at each year level offer 25 weeks of a Working Mathematically core curriculum.

A Way to Begin

- Glance over the Planner for your class. Skim through the comments for each task and lesson as it is named. This will provide an overview of the kit.
- Task Comments begin after the Planners. Lesson Comments begin after Task Comments. The index will also lead you to any task or lesson comments.
- Select your preferred starting week - usually Week 1.
- Now plan in detail by researching the comments and web support. Enjoy!

Research, Reflect, Activate

Curriculum Planner

Space & Logic: Year 9

<table>
<tr><th></th><th>Session 1</th><th>Session 2</th><th>Session 3</th><th>Session 4</th></tr>
<tr><td>Week 1</td><td colspan="2">Whole Class Investigation: Soft Drink Crates is perfect for reconnecting the students with the work of a mathematician. Pop the task under your arm and bring it to class to model what will be expected of students in the assessment week. The initial task is 'tweaked' into an extended investigation requiring number knowledge, spatial perception and logical thought.</td><td colspan="2" rowspan="2">Tasks & Text: The main reason for using the tasks in these two weeks is to give students a chance to 'look them over'. In the Assessment Week each pair will have to choose just one, investigate it in depth and publish a report.
At least one session per week needs to be used for this purpose. The other can focus on related work from the text.

It is likely that the class investigations in these weeks leave many challenges open. These may be picked up in this additional session if you wish, but are perhaps better included as choices in the Assessment Week.</td></tr>
<tr><td>Week 2</td><td colspan="2">Whole Class Investigation: Four Cube Houses is rich in 3d spatial challenges and their representations in 2d space. There are many spin-offs from the lesson; in particular cross-curriculum opportunities.</td></tr>
<tr><td>Week 3</td><td colspan="4">Assessment Week: See the section in Important Notes, Page 16</td></tr>
<tr><td>Week 4</td><td colspan="2">Whole Class Investigation: Chess tends to appeal to students of this age, and Chess Queens is an easy to establish investigation with an extensive iceberg. The lesson plan models Working Mathematically and supports students to publish reports.</td><td colspan="2" rowspan="2">Tasks & Text: As above.</td></tr>
<tr><td>Week 5</td><td colspan="2">Whole Class Investigation: Eight Queens is another classic problem offering many follow on studies.</td></tr>
<tr><td>Week 6</td><td colspan="4">Assessment Week: See notes on Page 16</td></tr>
</table>

- Weeks can be interchanged.
- An activity named in **bold** refers to a hands-on task.
- An activity named in *italic* refers to a lesson from Maths300.
- Text book style Toolbox Lessons can be interwoven or set for homework.

Curriculum Planner
Space & Logic: Year 10

<table>
<tr><th></th><th>Session 1</th><th>Session 2</th><th>Session 3</th><th>Session 4</th></tr>
<tr><td>Week 1</td><td colspan="2">Whole Class Investigation: This unit in 'Clever Counting' begins with Cars in a Garage. The lesson develops from the task and it is appropriate to show that connection. "In this form it is an invitation to work like a mathematician. Today we will see what we can learn from it together." The content begins to dig deeply into combination theory, but this is reached through the application of logical thought in context.</td><td colspan="2" rowspan="2">Tasks & Text: The main reason for using the tasks in these two weeks is to give students a chance to 'look them over'. In the Assessment Week each pair will have to choose just one, investigate it in depth and publish a report.
At least one session per week needs to be used for this purpose. The other can focus on related work from the text.

It is likely that the class investigations in these weeks leave many challenges open. These may be picked up in this additional session if you wish, but are perhaps better included as choices in the Assessment Week.</td></tr>
<tr><td>Week 2</td><td colspan="2">Whole Class Investigation: Farmyard Friends continues the unit. From a simple starting point, easy to act out, it further develops the ability, begun with the previous lesson, to investigate "What happens if...?" questions. Where Do We Sit? also strengthens this work.</td></tr>
<tr><td>Week 3</td><td colspan="4">Assessment Week: See the section in Important Notes, Page 16</td></tr>
<tr><td>Week 4</td><td colspan="2">Whole Class Investigation: Land of ET is a logic investigation leading to a structure which underpins many spatial and number situations.</td><td colspan="2" rowspan="2">Tasks & Text: As above.</td></tr>
<tr><td>Week 5</td><td colspan="2">Whole Class Investigation: Haberdasher's Problem is a classic which combines the interest of a spatial jigsaw with the rigour of Euclidean Geometry.</td></tr>
<tr><td>Week 6</td><td colspan="4">Assessment Week: See notes on Page 16</td></tr>
</table>

- Weeks can be interchanged.
- An activity named in **bold** refers to a hands-on task.
- An activity named in *italic* refers to a lesson from Maths300.
- Text book style Toolbox Lessons can be interwoven or set for homework.

Planning Notes

Enhancing Maths With Attitude

Resources to support learning to work like a mathematician are extensive and growing. There are more tasks and lessons available than have been included in this Space & Logic kit. You could use the following to enhance this kit.

Additional Tasks

- Task 21, Tactical
  A logic game with clear rules and a spatial dimension. The aim is to force your opponent to take the last counter.
- Task 185, Coloured Cubes
  A classic puzzle requiring a great deal of thought to solve without hint. Each of the four cubes have their faces coloured in a different way using four different colours. The challenge is to place the cubes in a line so that all 4 colours show along the surface of the 'square-section rod' created.

More information about these tasks may be available in the Task Cameo Library:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Additional Lessons

- Lesson 123, Mirror Bounce
  How were artists deep in the Egyptian pyramids able to find sufficient light to create the elaborate paintings on the walls of the burial chambers? The question generates interest in the properties of light, in particular in the intensity of light and its reflection. Students use mirrors to 'bounce' light around the classroom in a group activity with the challenge of 'hitting' a stated target.
- Lesson 134, Pentagon Triangles
  Take a regular pentagon and cut it into three triangles along its diagonals. Easy to state, easy to start and heaps of maths. At one level the three pieces can be used for creating spatial patterns and exploring shapes such as triangles, pentagons and decagons. At another there is work on angles, lengths, areas, fractions, decimals, Fibonacci Numbers and the Golden Ratio. Best of all, the joy of discovering this mathematical content develops in a problem posing and problem solving environment which reflects the work of a professional mathematician.

Keep in touch with new developments which enhance **Maths With Attitude** at:

- http://www.mathematicscentre.com/taskcentre/enhance.htm

Additional Materials

As stated, our attitude is that mathematics is concrete, visual and makes sense. We assume that all classrooms will have easy access to many materials beyond what we supply. For this unit you will need:

- Counters
- Cubes or other objects to use as queens in the chess lessons and *Soft Drink Crates*. If you have Poly Plug, the yellow/blue plugs work well in both cases. Yellow/blue Poly Plug are also useful for making 'words' in *Land of ET*.

- Multiple toy cars can be useful for *Cars In A Garage*, but are not essential at this level.
- Wooden cubes for *Four Cubes Houses*. Linking cubes can substitute, but do introduce issues related to cantilevered constructions which are not allowed in the puzzle.

A class set of Pentagon Triangles is not essential, but as you discover more about the task, you may see them as desirable. Pentagon Triangles and Poly Plug can be purchased through Mathematics Centre.

Find more information about Mathematics Centre resources at:

- http://www.mathematicscentre.com/taskcentre/resource.htm

Assessment Week

There is a clear focus on assessment in this kit because the tasks in it are more likely to support students in showing their ability to work like a mathematician, without having to simultaneously call on an extensive range of mathematical skills. For various reasons some students have a less extensive mathematical skill toolbox than we might wish. The challenges in this kit allow teachers to assess the strength of students' reasoning and communication skills as applied to mathematical challenges and thus develop a more balanced view of them as a mathematical learner than would be the case if assessment were based on skills only.

Mathematical skills are not being disregarded with this approach. Rather they are being put into perspective.

As explained in the planners, during the Assessment Week each pair will have to choose just one task, investigate it in depth and publish a report. Students prepare for this week in the two weeks prior. As they explore the various tasks, student pairs contribute to a class set of questions and challenges for each task which are beyond what is offered on the cards. It is the teachers role to collect, annotate and publish these challenges. Additional challenges will grow from the class lessons. It is from this set of challenges that students choose their assessment project.

Student Publishing

It is inappropriate to simply expect students to publish a report of their investigation. We have to devote lesson time to teaching how to keep an investigation journal and how to plan and present a report. The Recording & Publishing section of Mathematics Task Centre includes two different approaches to scaffolding this process with the class. Both include sample student work and suggest that a report can be presented in forms other than pencil and paper, for example PowerPoint. The links are titled 'Learning to Write a Maths Report' and 'Learning to Write a Maths Report 2' and can be found at:

- http://www.mathematicscentre.com/taskcentre/record.htm

Special Comments Year 9

- *Soft Drink Crates*, Planner Week 1. You will need many objects to use as soft drink cans, and copies of the grid available from the lesson plan. If you have Poly Plug, the red board can be masked to make the crate and the yellow/blue plugs can be the cans. See **Soft Drink Crates** Task Cameo at:
  http://www.mathematicscentre.com/taskcentre/iceberg.htm
  for an example and a recording sheet to suit Poly Plug.
- *Four Cube Houses*, Planner Week 2. You will need many wooden cubes to build four cube houses.
- *Chess Queens* & *Eight Queens*, Planner Weeks 4 & 5. You will need many objects to use as chess queens in both lessons, and copies of the grids available from the lesson plans.

Special Comments Year 10

- *Cars In A Garage*, Planner Week 1. There are materials to print from the lesson plan and printing in colour and laminating is a good way to make a permanent faculty set.
- *Farmyard Friends*, Planner Week 2. There is also material to print from this lesson plan which would be better prepared as a permanent class set.
- *Land of ET*, Planner Week 4. You may want to prepare sets of counters in two colours. If you have Poly Plug, they are perfect for the investigation.
- *Haberdasher's Problem*, Planner Week 5. You will need a class set of rulers and compasses. Scissors may also be useful.

Find Poly Plug information at:

- http://www.mathematicscentre.com/taskcentre/polyplug.htm

Task Comments

- Tasks, lessons and unit plans prepare students for the more traditional skill practice lessons, which we invite you to weave into your curriculum. Teachers who have used practical, hands-on investigations as the focus of their curriculum, rather than focussing on the drill and practice diet of traditional mathematics, report success in referring to skill practice lessons as Toolbox Lessons. This links to the idea of a mathematician dipping into a toolbox to find and use skills to solve problems.

A Stacking Problem

This is a tough problem for most students. It certainly doesn't need to be solved in one sitting and, equally, we need not offer hints too soon. It relates to Tower of Hanoi, but it is not identical. When the minimum number of moves is eventually counted (the answer being 60 moves), the challenge and the patterns in the problem may seem surprising for a problem so easy to begin.

The turning point of the problem is when Block 6 is the only one left on Square A, and Blocks 1 through 5 are in order from 1 down to 5 on Square B. Even when students have reached this stage and continued to the solution, it may be necessary to repeat the solution several times to realise the pattern in the movements.

Once understood, the description of how to solve it so that someone else might repeat their success leads to the use of algebraic notation.

The problem encourages the mathematician's strategy of breaking a problem into smaller parts. Further investigation develops by asking about whether similar problems (3, 9, 12, 15... blocks) on three squares, can be solved, and, if so, what is the minimum number of moves.

Cars In A Garage

This task is the starting point for a wealth of investigation which relates to applying the mathematician's questions:

- What happens if...?
- How many solutions are there?
- How do I know I have found them all?

The task makes a point of asking questions in which the number of cars matches the number of garages and in this way leads into combination theory. The answers to the questions on the card are 3!, 4!, 5! and 100!, although students may not yet know the operator ! as factorial. The opening to further investigation can be the phrase in each question *...one car in each garage*. It's not that we might consider more than one car in each garage, but,

- What happens if not all the cars turn up and there are some empty garages?
- How many different ways are there to park?
- How do we know when we have found them all?

Once the possibility of this variation has been realised, it opens the door to almost endless others.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Chess Queens

This problem is a classic recreational puzzle from Chess that opens into a considerable investigation which involves every aspect of the work of a mathematician. The problem is to place a given number of queens on a chess board so that every square can be reached by at least one queen. The task offers early success, by using 7 queens to do the job, then increases the difficulty with a challenge to repeat the task with 6 queens. The main challenge though is to tackle the problem using only 5 queens. So in essence the students are being led to the question, *What is the minimum number of queens needed, and how are they to be placed, so that every square can be reached?*. One could also ask, *What happens if we change the size of the board?*.

Coloured Squares

This easy to establish task has a depth that is waiting for further exploration. Much of what we know about it already is from one teacher, Markus Bucher from Tasmania, and we would welcome the contribution of further insights.

Most students tackle the problem by Guess & Check to find their first solution, which might be:

B	R	G	Y
G	Y	B	R
Y	G	R	B
R	B	Y	G

The second part of the card makes it clear that finding solutions is one thing, but checking whether solutions are unique is another. For example, a student might claim this as their second solution:

Y	R	B	G
G	B	R	Y
R	Y	G	B
B	G	Y	R

but inspection shows that a 90° clockwise rotation of this one produces the original. Sometimes it is easier to see these rotations by making and doing with two sets of grids and coloured tiles, or graph paper with the squares coloured.

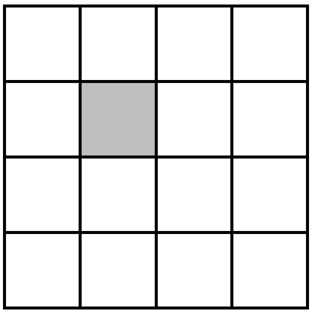
As more solutions develop so do strategies for finding them:

- Some might notice the placement of each tile of the same colour is a knight move from the previous position.
- Some might try a 'marching to the next possible position' strategy similar to that used in *Eight Queens*.
- Some might use the four cells in the top left corner as a 'master' and attempt reflection and rotation strategies.

The application of the strategy of breaking a problem into smaller parts might lead to other observations:

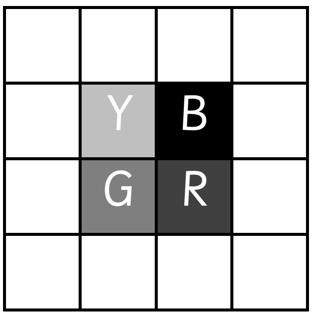
- Sets of four cells making a square in the corner must be four different colours.
- The four main corner cells must be four different colours.
- The central four cells must be four different colours.

Perhaps this last observation is the most important in terms of deciding how many solutions there are. If the rest of the square could be built by deduction from the centre, then finding all the different arrangements of the centre must lead to finding all the different solutions of the 4 x 4 square.



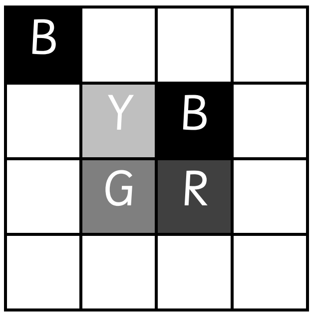
Whichever tile is placed here the other three (of the central four) must be different because they are linked by either the horizontal, or the vertical, or the main diagonal.

So a starting point for building a successful 4 x 4 grid might be:

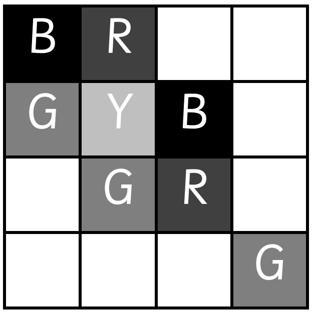


Consider the top left cell. It is on one of the main diagonals and two colours have already been used in that diagonal. So the top left cell can only be Blue or Green.

Choose one of these and explore the consequences:



- The bottom right must be green.
- The two missing tiles in the block of four at the top left must be green and red and they can only be arranged one way.



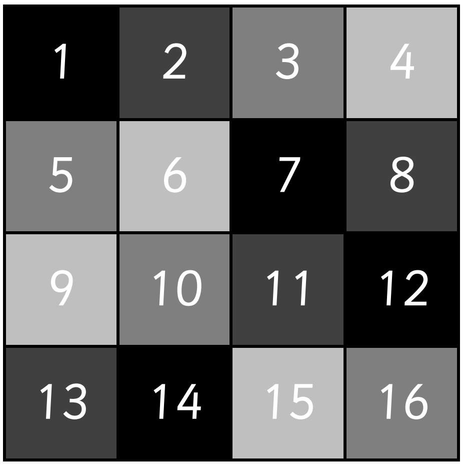
- Similar reasoning completes the four tiles in the bottom right.
- Once this corner is completed, the other main diagonal can be unambiguously defined.
- Finally there will be only one empty space on each outer edge and only one tile will be able to go in each position.

Therefore, obtaining a successful solution depends on only two decisions:
- Which tiles are chosen for the central four.
- Which tile is chosen for one of the four corner squares.

But there is more work to do yet before the questions:
- *How many solutions are there?*
- *How do you know you have found them all?*

can be answered.
- There are 24 ways to arrange 4 colours into 4 cells in a straight line. But how many ways in a square?
- Once the central four are set, do the two choices for the corner cell lead to unique solutions?
- Once the central four are set, does starting with different corner cells lead to different solutions?

The iceberg of this task can also be explored in other directions. For example, every visual pattern is likely to have a companion number pattern, and vice versa, so:
- *What happens if we number the squares 1 to 16?*



The sum of each colour is now the same!
- Does this always happen?
- Is the sum of the colours always the same number?
- Why does it happen anyway?

In the example the colour total is 34. This is also the total of the diagonals.
- 34 is the magic total of a 4 x 4 magic square using the numbers 1 to 16.
- Is there a connection?

A further consideration in the problem would be to ask about the validity of the restriction on reflected solutions being considered non-unique. After all, to physically turn one reflected solution into another requires flipping a board of counters up into the third dimension and over - just a bit difficult. On the other hand rotated solutions can be compared by a transformation in the plane.

Once we have explored 4 x 4 grids is there still more to investigate?

- What happens if we use three colours on a 3 x 3 grid?
- Or, five colours on a 5 x 5 grid?
- Or, six colours on a 6 x 6 grid?
- ...

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Eight Queens

Another recreational puzzle from Chess that opens into a considerable investigation involving every aspect of the work of a mathematician. The challenge is to place eight queens on the board so that none share the same row, column or diagonal with any other. Although there are many solutions to the task, finding the first can be quite a challenge. Students will have to give up 'guess and check' and try more refined reasoning to be able to solve the problem.

One approach is to break the problem into smaller parts by placing a queen on the top row first. Then a queen on the second row in the first 'safe' square; then the third row queen in a similar way; and so on. When a 'contradiction' arises, the problem has to be 'undone' to the last placement which had a choice and another alternative tried.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

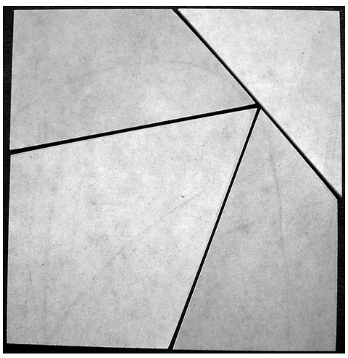
Haberdasher's Problem

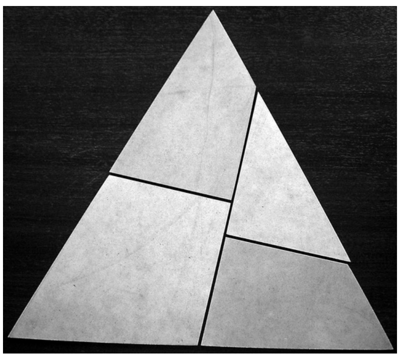
This classic 'jigsaw' from the Puzzlists' Era of the 1880s is the end point of some very impressive Euclidean Geometry. However, it is valuable for students to try the 'real thing' for themselves before exploring the geometry that led to it.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

This includes a short AVI file showing a group of Swedish trainee teachers struggling with the puzzle. It may only be a four piece jigsaw, but the two solutions - square and equilateral triangle - are elusive, even for young adults.

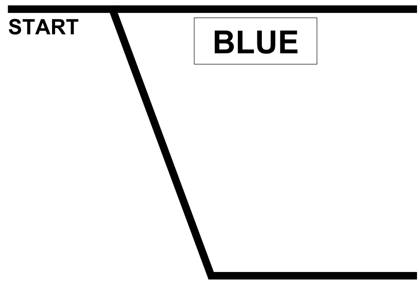




Koala Carts

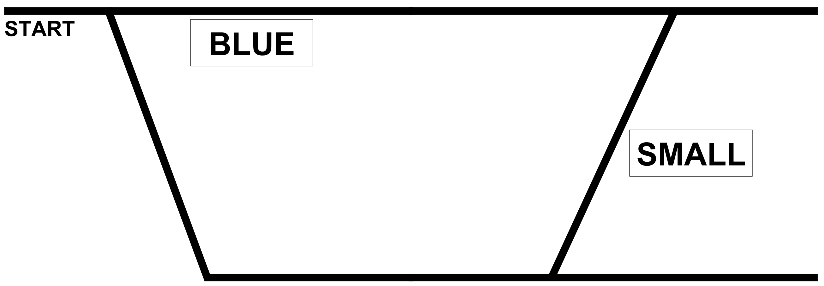
Mathematical Logic is a particular branch of mathematics and this task introduces students to some of the basic ideas from this study. Consistent with a multiple intelligences approach the ideas of the logical connectives 'and', 'or' and 'not' are introduced in a concrete, visual manner. As students create their own 'logic roads' the emphasis is on discussion and precise use of language.

The first question on the card is based on just one road section:

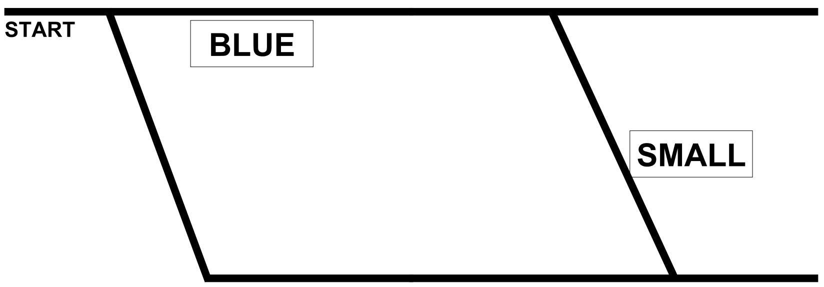


With the blue card in the position shown, only blue koalas can collect on the top road. If the blue card is on the other fork, one could say that red, yellow and green koalas of all sizes collect on the top road; or one could be more precise and say that 'not blue' koalas collect on the top road.

The next road on the card:



collects koalas which are blue or small on the top road. However, if the road network looked like this:



then the koalas collecting at the top road would be the large and medium blue ones, or more precisely, blue, not small koalas.

When students attempt the other challenges in the task, whether the specific ones in Question 4 or the ones they create themselves, encourage them to look for alternative ways to solve each one. This can lead to discovering the equivalence of roads which collect:

- large red koalas and small red koalas
- red and large or small koalas
- red and not medium koalas

Additionally, if you have other attribute materials in the school (there are various commercial sets in which the objects vary in colour, size, shape and thickness) the students can use the same roadway cards and these materials to explore situations where more attributes are involved.

Land Of ET

This task may seem like it comes from 'left field', however, it is included because the 'fantasy system' employed by the King and Queen to shorten words can also be used to illustrate important similarities to, and differences from, our number system. The task places in relief key laws of number such as closure, commutativity, inverse operations, associativity and the need for an identity element.

It turns out that the 'royal' system results in only 6 words and this opens a link to Modulo 5 (digits 0 - 5) arithmetic. There are also, perhaps unexpected, direct links to the symmetries of an equilateral triangle when operated on by flips and rotations. It turns out that the structures of the two systems and their operations are identical, while appearing at first glance to be totally unrelated. So a task which appears to be straightforward logic is actually illustrating fundamental number theory and linking to applications in other strands.

There is a further connection with the task Eight Queens, because in deciding the rotations and reflections in the solutions of that puzzle, attention has to be paid to the rotations and reflections of square. This aspect is explored in the companion Maths300 lesson plan for *Eight Queens*, and together these two tasks in conjunction with others call on many skills and concepts from transformation geometry.

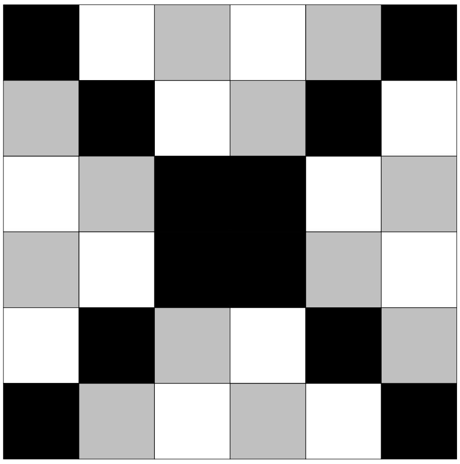
Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Latin Squares

In the first instance, students are likely to tackle this problem by guess and check. That will bring a solution for the 4 by 4 square, but it becomes a less successful strategy for 5 by 5 and beyond. Encourage them to look for a pattern by comparing the 3 by 3 solution on the card and their own 4 by 4 solution. The pattern is a movement based one. The next row is formed from the current row by shifting the right hand block to the left hand end and moving the others one space to the right. A visual check that results from this movement pattern is to look at the top left to bottom right diagonals. The colours along each should be the same.

An extension of the task is to use more cubes of your own and create repeats of the Latin Square which are joined to the original unit by reflection or rotation. For example using the 3 by 3 and rotation creates:

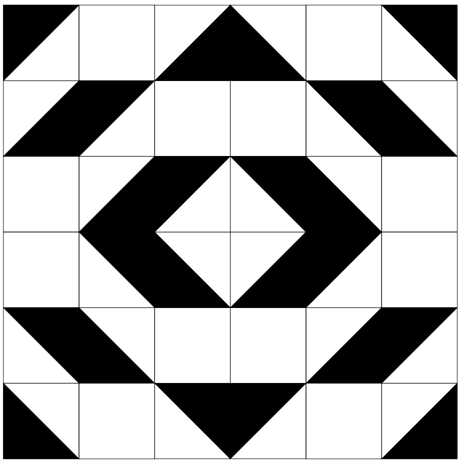


Think of the original unit as being in the bottom left quarter and the rotation as being 90° anti-clockwise.

If you don't have sufficient cubes, graph paper can be used either by colouring the 'cubes', or by assigning a piece of coloured design to each square in the order of a Latin Square. For example, assigning designs as follows:

Black above = *, White above =* *, Grey above =* 

and reflecting a 3 by 3 gives:



When extended in this way, the task fits with several others in this kit to make a firm basis for the transformation geometry section of the curriculum.

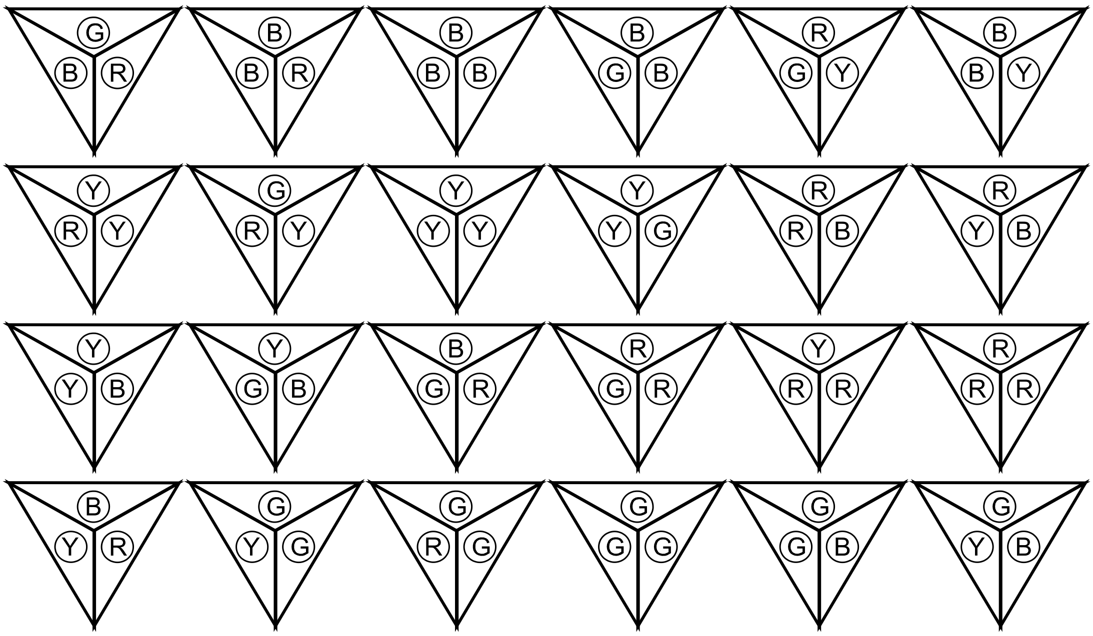
Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

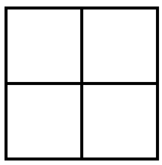
McMahon's Triangles 1

This task combines logic and spatial elements and, through its analysis of solutions that are reflected and rotated, links to tasks like **Eight Queens** and **Coloured Squares**. The task is also a partner to **McMahon's Triangles 2** which uses the coloured shapes produced in **McMahon's Triangles 1** and develops them into a deeper spatial problem.

The 24 triangles represent every possible combination of mapping 4 colours into 3 positions. To find them and be sure that you have them all is a serious mathematician's challenge. The solution set is:



Extensions to the task include finding all the ways of mapping three colours into the three positions. In turn, this relates to the mapping of four colours into four square positions:

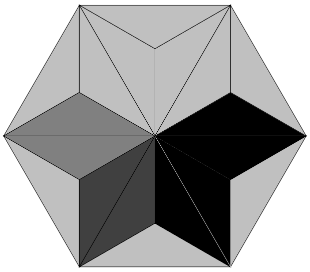


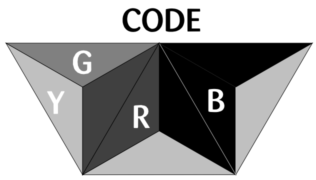
which is at the heart of the **Coloured Squares** task.

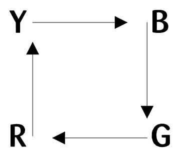
McMahon's Triangles 2

The 24 solutions of McMahon's Triangles 1 are put to use in this task to encourage spatial perception and systematic thinking. Students find the colourful pieces attractive and even if they don't solve the puzzles readily, they often enjoy the patterns created while trying. Many are struck by the 3D illusion of stepped cubes which sometimes develops in the solutions.

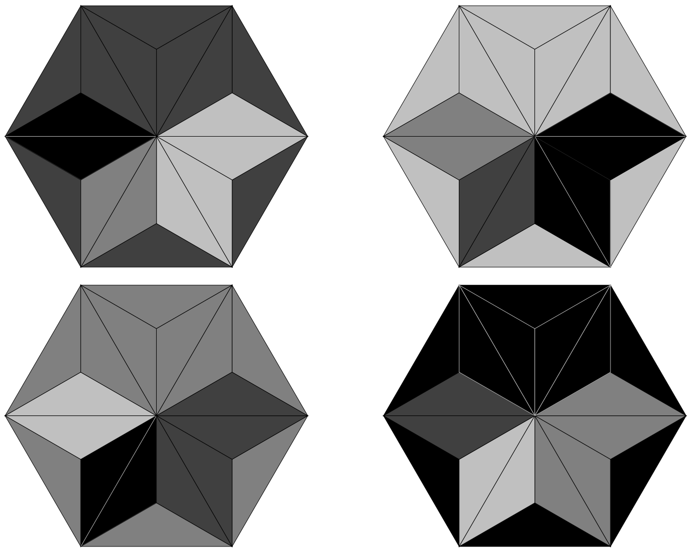
The first question on the task card has many answers because the colours aren't important. However, when the rules of Question 2 are applied it is not so easy to find a solution. One example is:



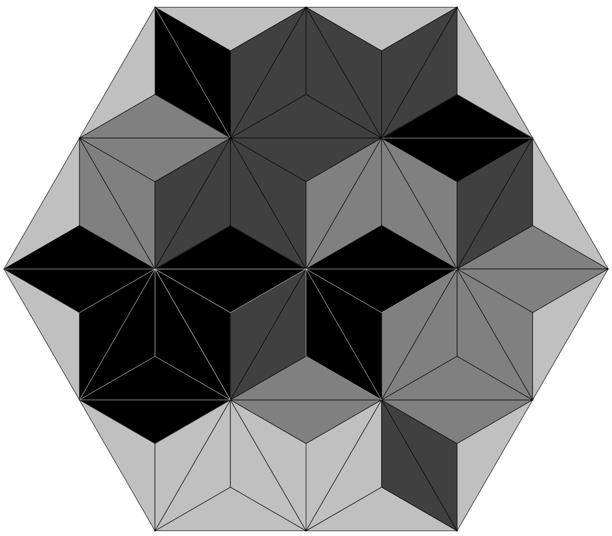




The other solutions can be found from this by systematically rotating the colours as shown. One set of four solutions is:



The final challenge on the card is to make one big hexagon with the same colour all around the edge *and* colours matching across all internal edges. One solution is shown, but there are others.



To find any of the solutions above requires considerable effort. Another way of using the pieces of the task is to challenge students to make and record as many shapes as possible which are symmetric either by rotation or reflection.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

where you will also be able to see (and print) solutions in colour.

Mirror Patterns 3

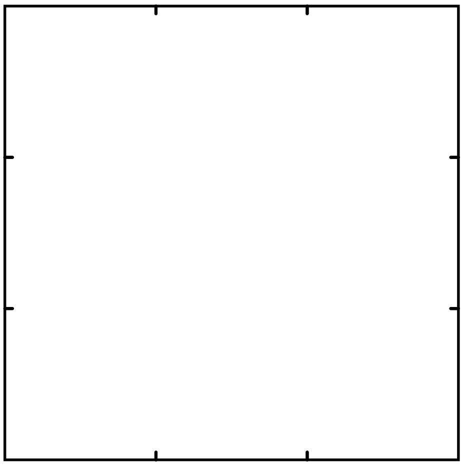
Many of the other tasks in this kit involve application of the concept of reflective symmetry to find the minimum number of solutions. However, we can't assume that all students have sufficient background experience with this concept to deal with the more sophisticated nature of those applications. This task is one of a number included to refresh and reinforce practical experience with symmetry. It was designed by members of a Year 8 class and, in the latter part of the card, it allows students to follow their own design directions. *The recording sheet students may need to store their explorations is at the end of this manual.*

If you check with the school's Design & Technology staff you are likely to find links between this task and material in their curriculum. Further links can be made to the Information Technology curriculum if the students are asked to report on their experiments supported by software such as PowerPoint.

Within this kit, **Mirror Patterns 3** involves the same mathematics as a suggested extension of **Latin Squares**.

An important *What happens if...?* question which will lead students into the iceberg of this task is:

- What happens if the starting shape is different? For example, what happens if the original design is based on each side of the square being divided into three like this...?



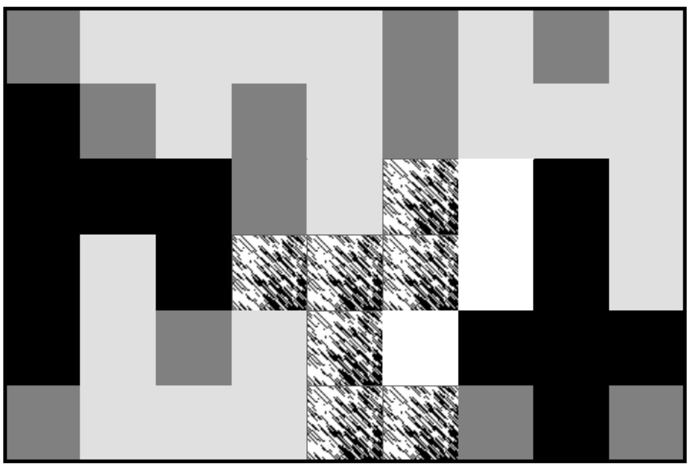
Following such threads may even lead students into the mathematics of kaleidoscopes.

Octaflex

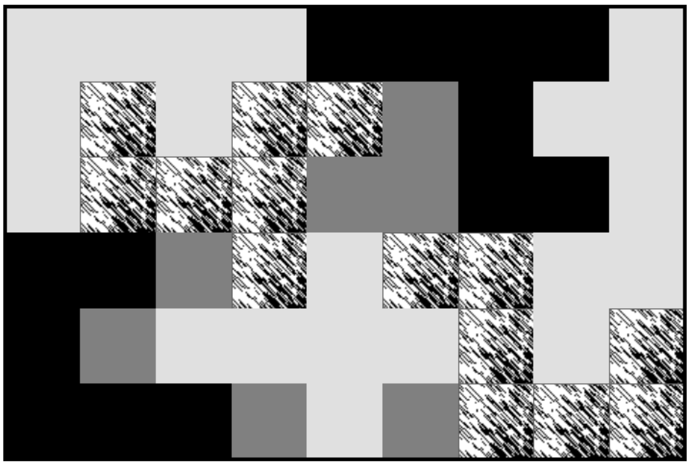
A mathematician frequently visualises a solution to a problem before being able to solve it. **Octaflex** is a puzzle which contributes to the development of student's spatial perception and therefore to their developing ability to work like a mathematician.

The first challenge on the card is reasonably straightforward. The second is harder and may take several attempts over time to solve.

Solution: Diagram 1



Solution: Diagram 2



Can the students make up similar puzzles which become a class set?

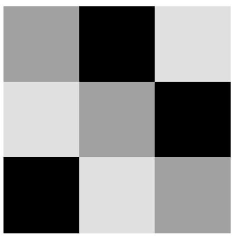
Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Pattern Cube

Pattern Cube grows out of the task **Latin Squares**. A Latin Square is a square made from the same number of colours as the number of units in the base *and* has no colour the same in any row or column. For example:

3x3 Latin Square



To create the Pattern Cube, begin with a Latin Square in the base. Notice that each row is formed from the previous one by a cyclic movement of colours. The top row is:

Red, Black, Yellow

The next row is formed by taking the Yellow from the right end, putting it on left end and 'pushing' the other two blocks to the right to make:

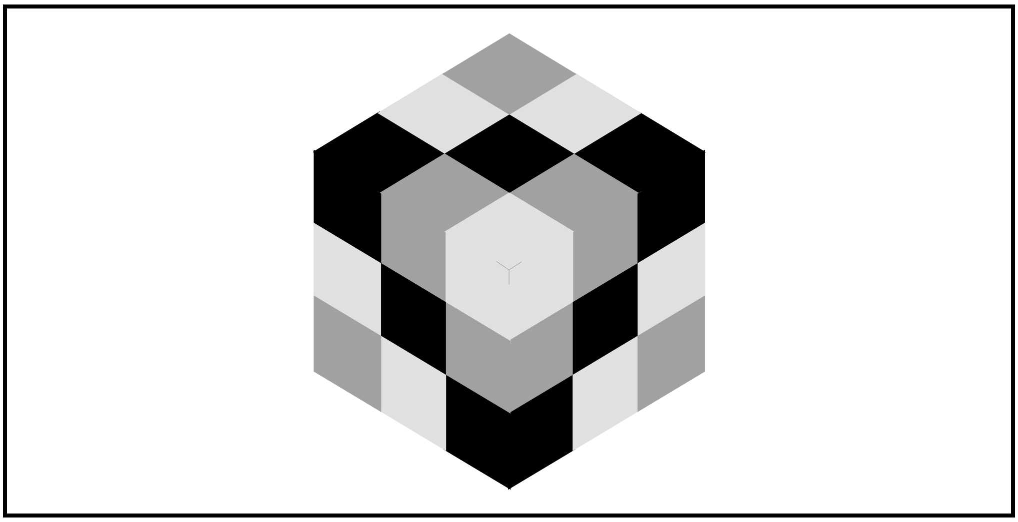
Yellow, Red, Black

Repeating this process with the Black from the right end makes the bottom row:

Black, Yellow Red

To make the cube, build a Latin Square upward from each row. The three Latin Square 'walls' you make will become a pattern cube.

One solution is:



There are variations on this solution depending on how the cube is rotated in 2 or 3 dimensions, however, these are essentially the same solution. To highlight this, students could be asked to sit opposite each other with a completed cube between them. They then draw and shade the cube on the isometric paper supplied at the end of the manual. They are looking at the same solution, but their diagrams will appear to be different.

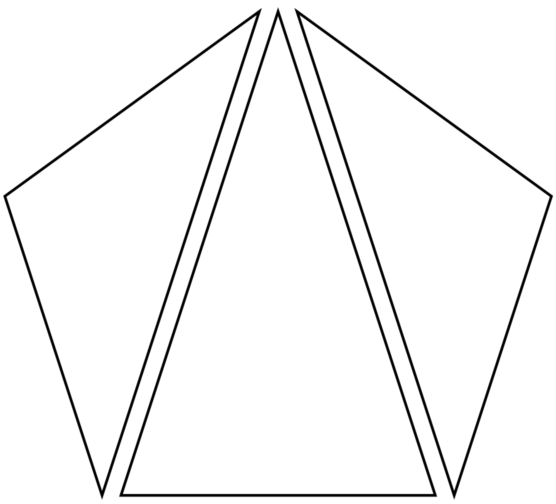
So, are there other unique solutions for **Pattern Cube**? Perhaps the students could begin to answer this, as a mathematician might, by breaking the problem into smaller parts and asking if there are other unique Latin Squares that could be used in the base.

An additional investigation is to start with a 4x4 Latin Square and try to make a 4x4 pattern cube from it.

Pentagon Triangles

The material in this task - the foam triangles - was designed by well known Scottish educator Geoff Giles. The information below draws on his extensive knowledge of this task.

The isosceles triangle pieces are made from a regular pentagon by making two cuts:



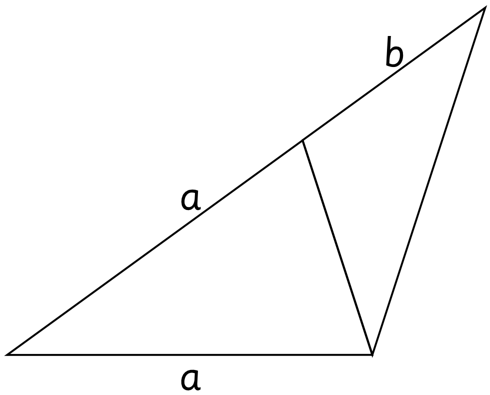
The task card encourages playing with these shapes as a starting point, which is just what a mathematician does once they find an interesting problem. Due to their construction, the tiles don't allow much room for regular tiling, but there are many interesting ways of fitting them together, some of which are symmetric, or rotationally symmetric. The fact that the two shapes are made from only two different lengths not only limits the possible ways to combine them, but makes it possible to explore all the ways they might be combined.

The mathematics waiting to be discovered within the shapes is far from trivial, as indicated by the question on the card referring to the Fibonacci Numbers, 2, 3, 5, 8, 13..., and the card is structured to encourage students to ask some of these questions for themselves. Even Question 8, which adds a restriction that same colours cannot touch edge to edge gently opens the door to a larger investigation.

Further investigations could be launched by asking questions like:

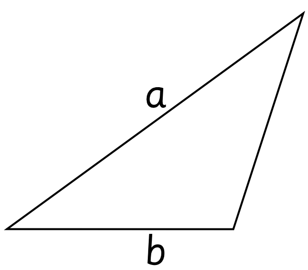
- Question 7 on the card tells us we can make new triangles with 2, 3, 5, 8, 13, ... pieces. Can we make triangles with the missing numbers, ie: 4, 6, ...?
- Can we work out the angles in the triangles?
- What can we find out about the lengths of the sides?
- What about areas?

Constructing a new triangle from two pieces like this:



and applying a little mathematics related to isosceles triangles and angles in a triangle leads to finding the angle values.

Realising the similarity between this construction and the smaller tile:



leads to the equation:

$$\frac{a}{b} = \frac{(a+b)}{b}$$

The resulting quadratic equation:

$$\left(\frac{a}{b}\right)^2 - \frac{a}{b} - 1 = 0$$

has the solution:

$$^a/_b = {}^{(1 + \sqrt{5})}/_2$$
$$= 1.61803398...$$

which is the Golden Ratio. In its turn this ratio relates to the Fibonacci Numbers which have already appeared in the task. Try dividing:

3 by 2, 5 by 3, 8 by 5, 13 by 8

and so on. What do you notice about the list of answers?

This may be a task you would like to turn into a whole class lesson. If so, there is a sheet provided at the end of this manual from which you can make a class set of cardboard cut-outs. Alternatively, class sets of the foam pieces can be obtained from Mathematics Centre Resources.

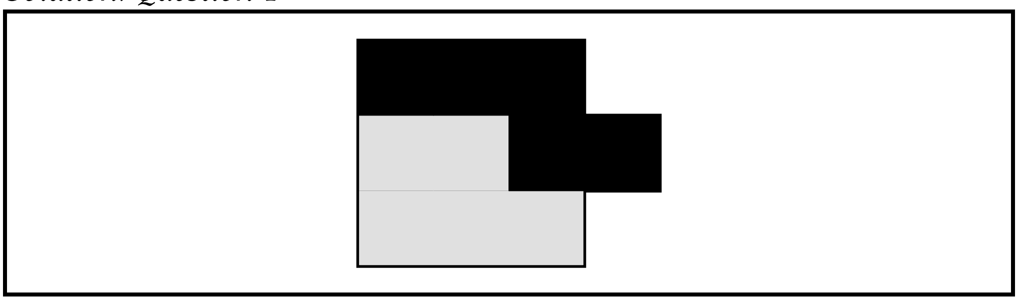
Find more information about Mathematics Centre resources at:
- http://www.mathematicscentre.com/taskcentre/resource.htm

Find more information about this task in the Task Cameo Library at:
- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

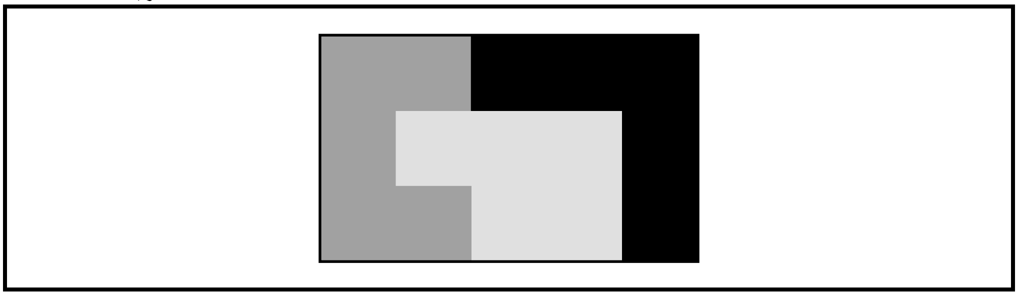
Pentominoes

This is a task with a long history as a mathematical recreation. The three challenges on the card are only a few of the seemingly endless number that have been created over centuries. Even the additional puzzles suggested as extensions below still only represent the tip of the iceberg.

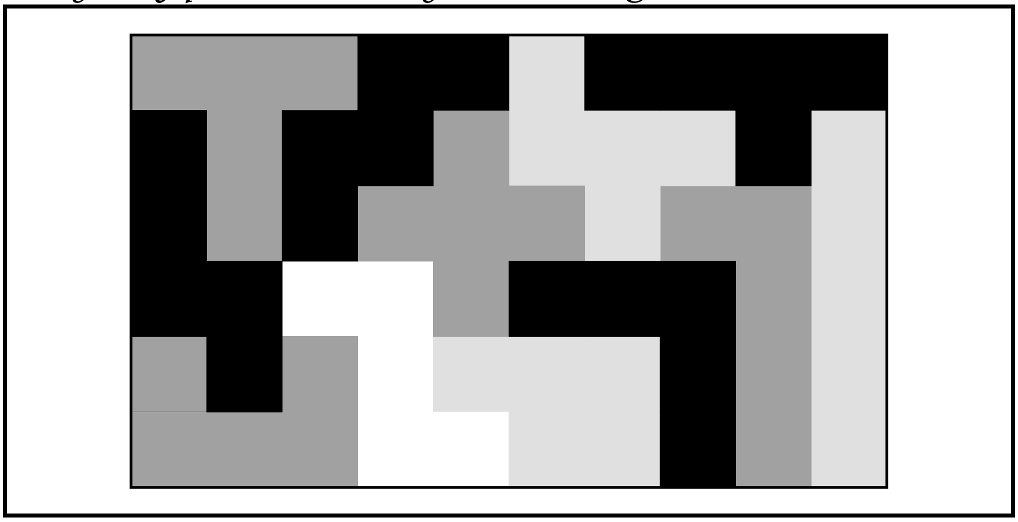
Solution: Question 1



Solution: Question 2



One of many possible solutions for the Challenge:



There is no need for the Challenge to be solved at the first attempt. A mathematician takes as much time as necessary to solve a problem. Confirm for the students that it is okay to put the problem away and try it again another time.

Additional challenges:

- Question 3 can become a game. Players take turns to select from the twelve pieces until they have six each. Then they take turns to place the pieces on the loungeroom floor one at a time. The first player unable to place a piece loses.
- Each of the twelve pentominoes is 5 square units. Their total area is 60 square units. The loungeroom floor is 60 square units as a 10 x 6 board. Other factors of 60 are 4 x 15 and 5 x 12. Students could make these boards and try to place all the pieces on them.
- Can the pieces be placed on an 8 x 8 board so that there is either a 2 x 2 gap in the centre or an uncovered square in each corner?
- One piece is a **+** shape. Nine of the other pieces can be put together to make a **+** which has each length three times longer than the single piece **+**.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Reverse

The task is easy to understand. Place five numbered counters on the six places as shown:

	1	2	3	4	5

Now reverse them to this position:

	5	4	3	2	1

given that you can:

- only move one piece at a time
- only move one position at a time in either direction
- only jump over one piece at a time in either direction.

From this beginning an investigation grows which illustrates every aspect of the work of a mathematician. A conversation between two teachers that records this richness is stored in the Task Cameo Library.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Sliding Tiles

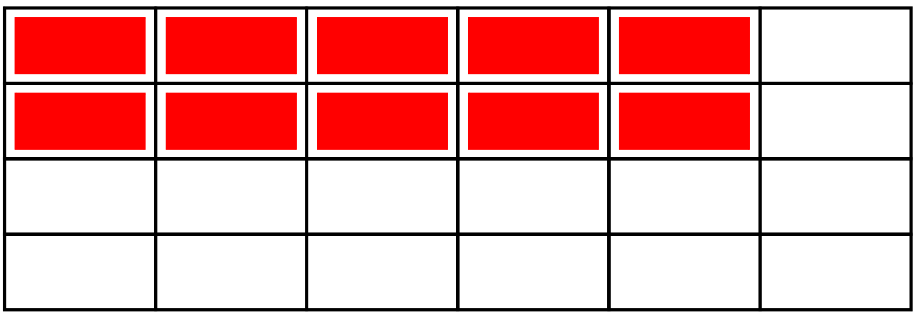
This task is a simplified version of a puzzle that has been very popular over the years, namely 15 tiles that can slide within a 4x4 frame. The user is challenged to slide the tiles into certain configurations and is also given 'impossible' challenges. Using a 3 x 2 frame with 5 tiles, this simplified version explores the mathematics behind the challenge and shows why some arrangements are 'impossible'.

First, reduce the 6 piece puzzle (5 tiles and a blank) to 4 (by covering up two positions) and only using the tiles numbered 1 to 3. Students can discover there are only 24 configurations (or arrangements). Once the tiles are randomly placed in any positions to start, only 12 of the combinations can be made - the other 12 are 'impossible'. Indeed the combinations turn into two sets of 12. The 12 of the 'opposite' set are like reflections of each combination of the first 12.

Similarly with the 6 pieces of the task (5 & blank). It is a worthwhile challenge to discover that there are 720 (6!) arrangements of the tiles. These form into two sets of 360. Each arrangement within the 360 is like a 'rotation' of the others, and the 360 'impossible' arrangements are reflections of the first set of 360. So, if the tiles are placed down at random then there is a 50:50 chance that they can be moved into any given arrangement.

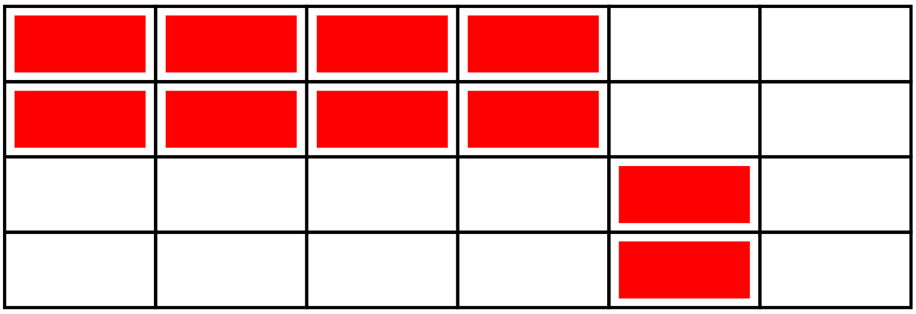
Soft Drink Crates

This task is simply stated, but the solution of the first problem on the card may challenge teachers and students alike. Thinking in terms of breaking the problem into smaller parts can help.

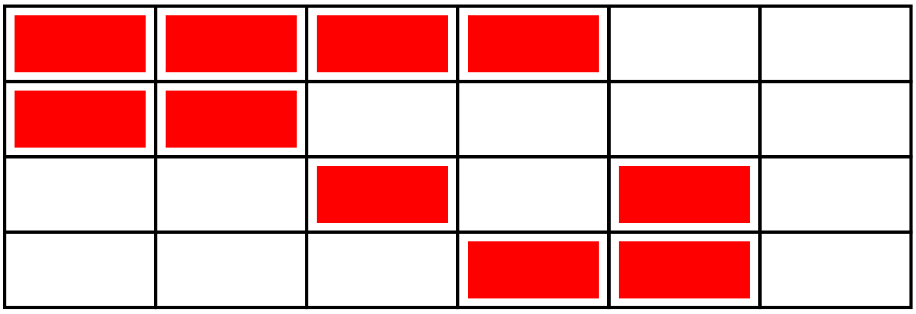


1. Make the columns correct as shown here.

2. Keep cans in the same column but alter their row...



3. Until the rows are also correct:



There are many variations on this solution, and the strategy is now applicable to other problems on the card. The task encourages high level thinking without a correspondingly high demand for mathematical skills. The emphasis in on the concept of proof and the application of strategies to achieve proof.

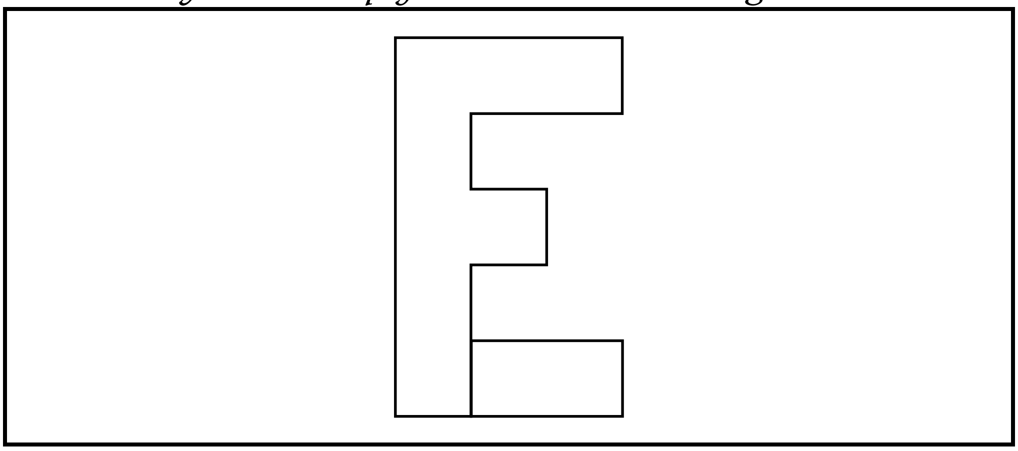
Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

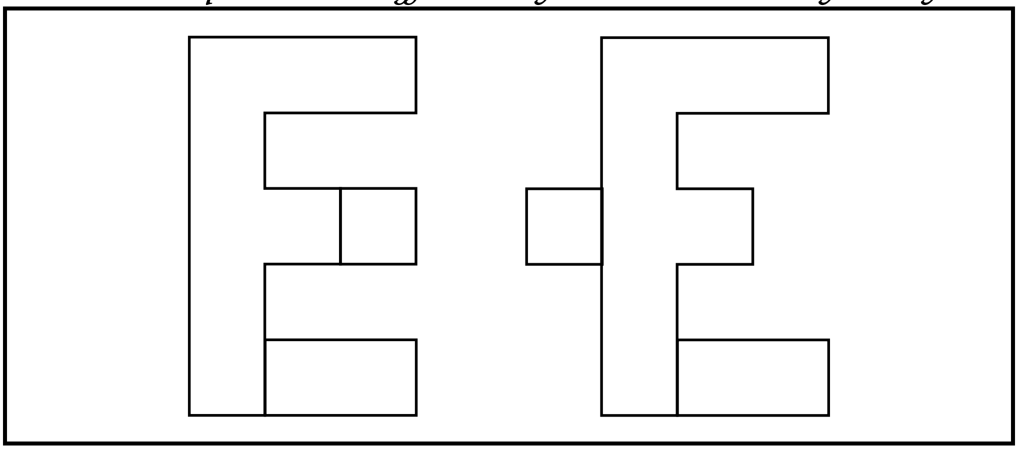
Symmetric Shapes

This task also has its origins in the work of Geoff Giles and the DIME (Developments in Mathematics Education) Project. As with **Mirror Patterns 3** it is a task we have included to refresh and reinforce practical experience with symmetry. The task starts with challenges which are quite achievable.

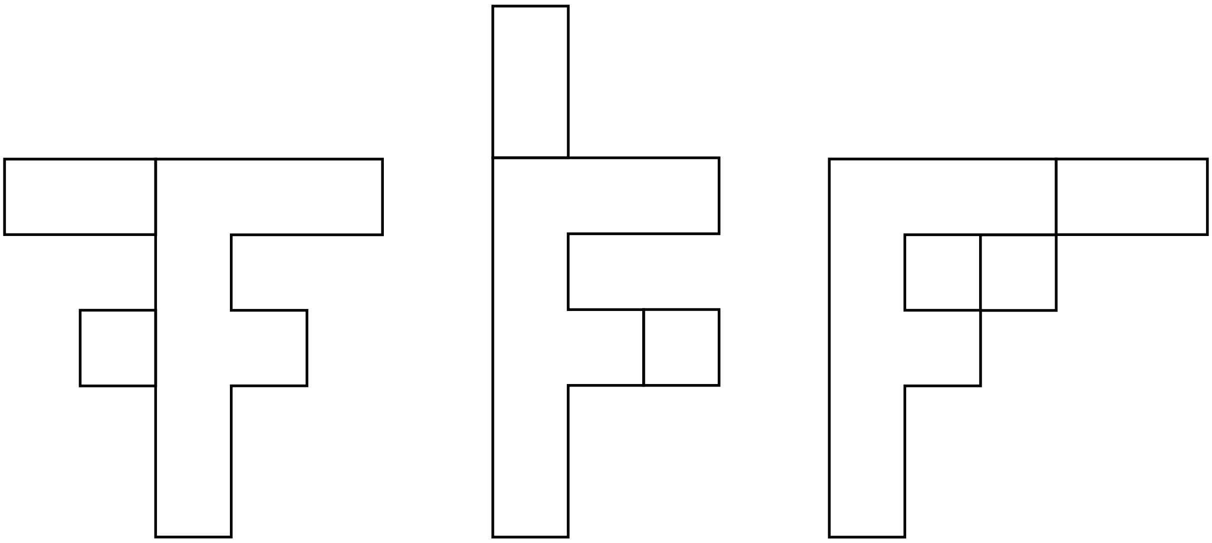
Make a line symmetric shape from the F and the rectangle:



Now add the square in two different ways to retain the line symmetry:



So far so good, but there are three more ways to combine the F, the rectangle and the square to make shapes with line symmetry. These are much more of a challenge, especially since we may have been led into expecting the line of symmetry to be horizontal or vertical. It takes a deal of thinking 'outside the square' to discover:



If you would like to take the task a little further you might ask:

- How did the writer of the card know that these were the only solutions?
- How many rotationally symmetric shapes could you make with these three pieces?

The sheet students may need to record their solutions is at the end of this manual.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Lesson Comments

- These comments introduce you to each Maths300 lesson. The complete plan is easily accessed through the lesson library available to members at:
  http://www.maths300.com
  where they are listed alphabetically by lesson name.

Cars In A Garage

The context - parking cars - is familiar to everyone. Kindergarten children know what you mean and can act out the problem while they 'play cars'. In secondary school, students are challenged to find the number of ways of parking in a line if:

- the number of cars is changed, or
- there are conditions limiting which car is able to park next to which.

The mathematician's questions:

- How many solutions are there?
- How do we know when we have found them all?

focus the investigation. The lesson is non-threatening, can be physically involving and can easily be made concrete by raiding a child's toy box, or printing the cars provided. The lesson helps the often dry, symbolic study of selections and arrangements (permutations and combinations) to become an engaging challenge in clever counting. It is complemented by software that offers more exercises than a text book in a much more supportive way.

Chess Queens

Place five queens on a chessboard so that between them they cover every square on the board. Then change the size of the board and change the number of queens and the problem offers opportunities for an extended investigation involving spatial visual thinking, the development of specific problem solving strategies and notions of generalisation and algebraic patterns. The computer software allows the puzzle to be explored at significantly greater depth than if only pencil and paper were available.

Eight Queens

This intriguing chess puzzle offers opportunities for an extended investigation involving spatial visual thinking, the development of specific problem solving strategies, notions of symmetry and the search for all possible solutions. The companion software allows the puzzle to be explored at significantly greater depth than if pencil and paper alone were available. The challenge this time is to place eight queens on a chessboard so that none can be captured by any other. There are 92 different solutions, but only 12 unique solutions that are not symmetry transformations (rotations or reflections) of each other.

Farmyard Friends

Five animals have to be arranged into five pens according to certain rules such as *the pig comes after the cow*. The task itself is not included in this kit because it is designed to be accessible to, and suitable for, quite young children. However, the search for ALL the solutions quickly shifts the focus onto the mathematical investigative process. The many challenging extensions are appropriate for senior students who are being introduced to the topic of combinations and arrangements. Like *Cars In A Garage* the lesson is driven by the mathematician's questions:

- What happens if...?
- How many solutions are there?
- How do I know when I have found them all?

Four Cube Houses

Architectural teams (students working in groups) are challenged to design as many houses as possible using four cubic modules. The resulting designs have to be drawn up, costed and justified as part of a housing development. Embedded in this story shell, the lesson is a lot of fun. However finding the complete set of houses such that the modules all touch face to face, there are no 'cantilevered' structures and designs cannot be transformed into each other by a simple rotation around a vertical axis, is quite a spatial challenge.

Costing the designs adds further interest and results in the lesson not only achieving objectives in the Space strand, but also in Measurement and Number. There are clear links to commercial mathematics at this level when the task is taken a step further to borrowing the money to pay for the house. This brings in the study of interest payments and the comparison of simple to compound interest.

Teachers looking for cross-curriculum links would do well to show students the real estate web site in the Classroom Contributions of the lesson and challenge them to create a similar one of their own based on this problem.

Haberdasher's Problem

Cut up an equilateral triangle and reassemble the pieces into a square! And do so with the least possible number of cuts. Challenges like this which can change one shape into another have always fascinated mathematicians, and this particular puzzle has an interesting history. A famous English puzzlist, Henry Ernest Dudeney, astounded everyone when he discovered how to accomplish the change from equilateral triangle to square with just four pieces.

This lesson first invites students to invent some possible construction steps, then to follow the wonderful construction discovered by Dudeney around 1902. His construction is beautiful Euclidean Geometry. Attempting the solution themselves first helps students appreciate the precision and care of the master. They also learn why the challenge is called the Haberdasher's Problem.

Land of ET

In the Land of ET the words were too long and they had too many letters in the alphabet. So the King and Queen decreed henceforth there would be only two letters - B and Y. Then they were upset that some of the words seemed to be getting even longer, eg: BBBBBYYBYBYBYBBYY, so they passed some rules for shortening the words.

- Rule 1: BBB next to each other can be added or removed.
- Rule 2: YY next to each other can be added or removed.
- Rule 3: BYB can be replaced by Y or vice versa.

After shortening, how many words are left in the language of ET?

On the surface this lesson looks quite 'different'. On one level it appears as a fantasy language puzzle, but the structure underlying the puzzle goes to the heart of number theory. ET might be thought of as 'extra-terrestrial' in terms of the fantasy - however it is actually named for the identity between the structure of the puzzle and the symmetry properties of an Equilateral Triangle. These links provide the depth and richness of the mathematics which make it a worthy investigation for students of this age.

Soft Drink Crates

This lesson is another closed problem which is 'tweaked' to illustrate the mathematical investigation process. The skill involved is simply to count odd and even numbers, but the lesson is included because, in this straightforward number context it can be used to develop and reinforce the process of working like a mathematician. The initial problem is:

> *There are eighteen soft drink cans in a box that has six columns and four rows. All the columns and all the rows have an even number of cans. How are they arranged?*

- Is there more than one arrangement?
- Are there other numbers of cans which could produce the even rows and columns situation?
- Are there numbers of cans which could produce odd rows and columns?

Where Do We Sit?

Although this lesson is written as if for infant teachers, the mathematics is identical to that of *Cars In A Garage* and so the lesson is open to all the same types of *What if...?* questions. If students are to abstract mathematical concepts from the context in which the concepts are presented, then they need to recognise that mathematics in a different context.

It may be sufficient to use *Cars In A Garage* and *Farmyard Friends* to initiate and develop work in selections and arrangements. However, *Where Do We Sit?*, which has obvious adaptations to classroom seating arrangements, offers additional experiences in this topic. It could be included in the unit as a whole class investigation along with the others, or it could be used as an assessment project.

Part 3: Value Adding

The Poster Problem Clinic

Maths With Attitude kits offer several models for building a Working Mathematically curriculum around tasks. Each kit uses a different model, so across the range of 16 kits, teachers' professional learning continues and students experience variety. The Poster Problem Clinic is an additional model. It can be used to lead students into working with tasks, or it can be used in a briefer form as a opening component of each task session.

> *I was apprehensive about using tasks when it seemed such a different way of working. I felt my children had little or no experience of problem solving and I wanted to prepare them to think more deeply. The Clinic proved a perfect way in.*

Careful thought needs to be given to management in such lessons. One approach to getting the class started on the tasks and giving it a sense of direction and purpose is to start with a whole class problem. Usually this is displayed on a poster that all can see, perhaps in a Maths Corner. Another approach is to print a copy for each person. A Poster Problem Clinic fosters class discussion and thought about problem solving strategies.

Starting the lesson this way also means that just prior to liberating the students into the task session, they are all together to allow the teacher to make any short, general observations about classroom organisation, or to celebrate any problem solving ideas that have arisen.

One teacher describes the session like this:

> *I like starting with a class problem - for just a few minutes - it focuses the class attention, and often allows me to introduce a particular strategy that is new or needs emphasis.*

It only takes a short time to introduce a poster and get some initial ideas going. The class discussion develops a way of thinking. It allows class members to hear, and learn from their peers, about problem solving strategies that work for them.

> *If we don't collectively solve the problem in 5 minutes, I will leave the problem 'hanging' and it gives a purpose to the class review session at the end.*
> *Sometimes I require everyone to work out and write down their solution to the whole class problem. The staggered finishing time for this allows me to get organised and help students get started on tasks without being besieged.*
> *I try to never interrupt the task session, but all pupils know we have a five minute review session at the end to allow them to comment on such things as an activity they particularly liked. We often close then with an agreed answer to our whole class problem.*

A Clinic in Action

The aims of the regular clinic are:

- to provide children with the opportunity to learn a variety of strategies
- to familiarise children with a process for solving problems.

The following example illustrates a structure which many teachers have found successful when running a clinic.

Preparation

For each session teachers need:

- a Strategy Board as below
- a How To Solve A Problem chart as below
- to choose a suitable problem and prepare it as a poster
- to organise children into groups of two or three.

The Strategy Board can be prepared in advance as a reference for the children, or may be developed *with* the children as they explore problem solving and suggest their own versions of the strategies.

The problem can be chosen from

- a book
- the task collection
- prepared collections such as Professor Morris Puzzles which can be viewed at: http://www.mathematicscentre.com/taskcentre/resource.htm#profmorr

The example which follows is from the task collection. The teacher copied it onto a large sheet of paper and asked some children to illustrate it. *The teacher also changed the number of sheep to sixty* to make the poster a little different from the one in the task collection.

The Strategy Board and the How To Solve A Problem chart can be used in any maths activity and are frequently referred to in Maths300 lessons.

The Clinic

The poster used for this example session is:

> Eric the Sheep is lining up to be shorn before the hot summer ahead. There are sixty [60] sheep in front of him. Eric can't be bothered waiting in the queue properly, so he decides to sneak towards the front.
>
> Every time one [1] sheep is taken to be shorn, Eric <u>then</u> sneaks past two [2] sheep. How many sheep will be shorn <u>before</u> Eric?

This Poster Problem Clinic approach is also extensively explored in Maths300 Lesson 14, *The Farmer's Puzzle*.

Strategy Board

DO I KNOW A SIMILAR PROBLEM?	ACT IT OUT
GUESS, CHECK AND IMPROVE	DRAW A PICTURE OR GRAPH
TRY A SIMPLER PROBLEM	MAKE A MODEL
WRITE AN EQUATION	LOOK FOR A PATTERN
MAKE A LIST OR TABLE	TRY ALL POSSIBILITIES
WORK BACKWARDS	SEEK AN EXCEPTION
BREAK INTO SMALLER PARTS	...

How To Solve A Problem

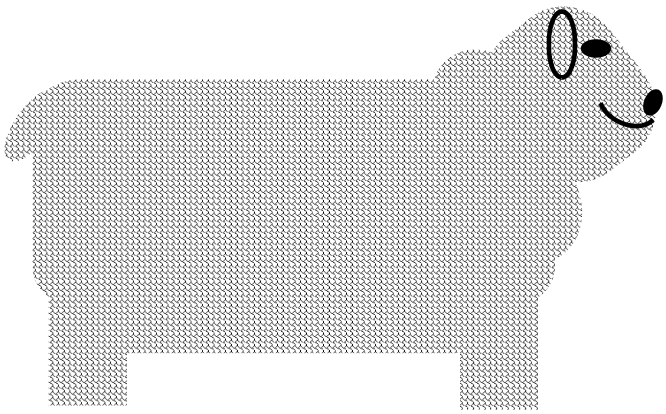
SEE & UNDERSTAND	Do I understand what the problem is asking? Discuss
PLANNING	Select a strategy from the board. Plan how you intend solving the problem.
DOING IT	Try out your idea.
CHECK IT	Did it work out? If so reflect on the activity. If not, go back to step one.

Step 1

- Tell the children that we are at Stage 1 of our four stage plan ... **See & Understand** ... Point to it! Read the problem with the class. Discuss the problem and clarify any misunderstandings.
- If children do not clearly understand what the problem is asking, they will not cope with the next stage. A good way of finding out if a child understands a problem is for her/him to retell it.
- Allow time for questions - approximately 3 to 5 minutes.

Step 2

- Tell the children that we are at Stage 2 of our four stage plan ... **Planning**. In their groups children select one or more strategies from the Strategy Board and discuss/organise how to go about solving the problem.
- Without guidance, children will often skip this step and go straight to Doing It. It is vital to emphasise that this stage is simply planning, not solving, the problem.
- After about 3 minutes, ask the children to share their plans.



Plan 1

Well we're drawing a picture and sort of making a model.

Can you give me more information please Brigid?

We're putting 60 crosses on our paper for sheep and the pen top will be Eric. Then Claire will circle one from that end, and I will pass two crosses with my pen top.

Plan 2

Our strategy is Guess and Check.

That's good Nick, but how are you going to check your guess?

Oh, we're making a model.

Go on ...

> *John's getting MAB smalls to be sheep and I'm getting a domino to be Eric and the chalk box to be the shed for shearing.*

Plan 3

> *We are doing it for 3 sheep then 4 sheep then 5 sheep and so on. Later we will look at 60.*

Great so you are going to try a simpler problem, make a table and look for a pattern.

This sharing of strategies is invaluable as it provides children who would normally feel lost in this type of activity with an opportunity to listen to their peers and make sense out of strategy selection. Note that such children are not given the answer. Rather they are assisted with understanding the power of selecting and applying strategies.

Step 3

- Tell the children that we are at Stage 3 of our four stage plan ... **Doing It**. Children collect what they need and carry out their plan.

Step 4

- Tell the children that we are at Stage 4 of our four stage plan ... **Check It**. Come together as a class for groups to share their findings. Again emphasis is on strategies.

> *We used the drawing strategy, but we changed while we were doing it because we saw a pattern.*

So Jake, you used the Look For A Pattern strategy. What was it?

> *We found that when Eric passed 10 sheep, 5 had been shorn, so 20 sheep meant 10 had been shorn ... and that means when Eric passes 40 sheep, 20 were shorn and that makes the 60 altogether.*

Great Jake. How would you work out the answer for 59 sheep or 62 sheep?

Sharing time is also a good opportunity to add in a strategy which no one may have used. For example:

> *Maybe we could've used the Number Sentence strategy, ie: 1 sheep goes to be shorn and Eric passes two sheep. That's 3 sheep, so perhaps, 60 divided into groups of 3, or 60 ÷ 3 gives the answer.*

Round off the lesson by referring to the Working Mathematically chart. There will be many opportunities to compliment the students on working like a mathematician.

Curriculum Planning Stories

Our attitude is:

> *teachers improve their teaching by re-enacting stories from the classrooms of their colleagues*

In more than a decade of using tasks and many years of using the detailed whole class lessons of Maths300, teachers have developed several models for integrating tasks and whole class lessons. Some of those stories are retold here. Others can be found at:

- http://www.mathematicscentre.com/taskcentre/plans.htm

Story 1: Threading

> *Educational research caused me a dilemma. It tells us that students construct their own learning and that this process takes time. My understanding of the history of mathematics told me that certain concepts, such as place value and fractions, took thousands of years for mathematicians to understand. The dilemma was being faced with a textbook that expected students to 'get it' in a concentrated one, two or three week block of work and then usually not revisit the topic again until the next academic year.*

A Working Mathematically curriculum reflects the need to provide time to learn in a supportive, non-threatening environment and...

> *When I was involved in a Calculating Changes PD program I realised that:*
>
> - *choosing rich and revisitable activities, which are familiar in structure but fresh in challenge each time they are used, and*
> - *threading them through the curriculum over weeks for a small amount of time in each of several lessons per week*
>
> *resulted in deeper learning, especially when partnered with purposeful discussion and recording.*

Calculating Changes:

- http://www.mathematicscentre.com/calchange

Story 2: Your turn

Some teachers are making extensive use of a partnership between the whole class lessons of Maths300 and small group work with the tasks. Setting aside a lesson for using the tasks in the way they were originally designed now seems to have more meaning, as indicated by this teacher's story:

> *When I was thinking about helping students learn to work like a mathematician, my mind drifted to my daughter learning to drive. She*

needed me to model how to do it and then she needed lots of opportunity to try it for herself.

That's when the idea clicked of using the Maths300 lessons as a model and the tasks as a chance for the students to have their turn to be a mathematician.

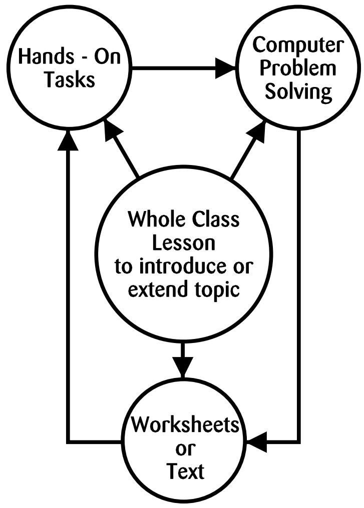
The Maths300 lessons illustrate how other teachers have modelled the process, so I felt I could do it too. Now the process is always on display on the wall or pasted inside the student's journal.

A session just using the tasks had seemed a bit like play time before this. Now I see it as an integral part of learning to work mathematically.

Story 3: Mixed Media

It was our staff discussion on Gardner's theory of Multiple Intelligences that led us into creating mixed media units. That and the access you have provided to tasks and Maths300 software.

We felt challenged to integrate these resources into our syllabus. There was really no excuse for a text book diet that favours the formal learners. We now often use four different modes of learning in the work station structure shown. It can be easily managed by one teacher, but it is better when we plan and execute it together.



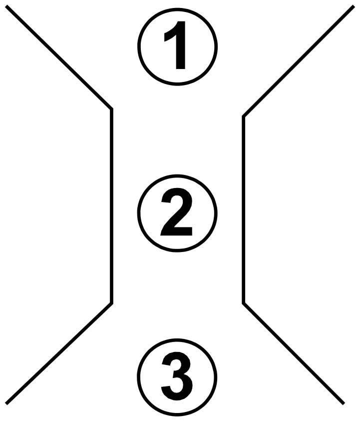
Story 4: Replacement Unit

We started meeting with the secondary school maths teachers to try to make transition between systems easier for the students. After considerable discussion we contracted a consultant who suggested that school might look too much the same across the transition when the students were hoping for something new. On the other hand our experience suggested that there needed to be some consistency in the way teachers worked.

We decided to 'bite the bullet' and try a hands-on problem solving unit in one strand. We selected two menus of twenty hands-on tasks, one for the primary and one for the secondary, that became the core of the unit. We deliberately overlapped some tasks that we knew were very rich and added some new ones for the high school.

Class lessons and investigation sheets were used to extend the tasks, within a three week model.

It is important to note that although these teachers structured a 3 week unit for the students, they strongly advised an additional *Week Zero* for staff preparation. The units came to be called Replacement Units.



Week Zero - Planning

Staff familiarise themselves with the material and jointly plan the unit. This is not a model that can be 'planned on the way to class'.

> *Getting together turned out to be great professional development for our group.*

Week 1 - Introduction

Students explore the 20 tasks listed on a printed menu:

- students explore the tip of the task, as on the card
- students move from task to task following teacher questioning that suggests there is more to the task than the tip
- in discussion with students, teachers gather informal assessment information that guides lesson planning for the following week.

> *We gave the kids an 'encouragement talk' first about joining us in an experiment in ways of learning maths and then gave out the tasks. The response was intelligent and there was quite a buzz in the room.*

Week 2 - Formalisation

> *It was good for both us and the students that the lessons in this week were a bit more traditional. However, they weren't text book based. We used whole class lessons based on the tasks they had been exploring and taught the Working Mathematically process, content and report writing.*
>
> *Assessment was via standard teacher-designed tests, quizzes and homework.*

Week 3 - Investigations

> *We were most delighted with Week 3. Each student chose one task from the menu and carried out an in-depth investigation into the iceberg guided by an investigation sheet. They had to publish a report of their investigation and we were quite surprised at the outcomes. It was clear that the first two weeks had lifted the image of mathematics from 'boring repetition' to a higher level of intellectual activity.*

Story 5: Curriculum shift

> *I think our school was like many others. The syllabus pattern was 10 units of three weeks each through the year. We had drifted into that through a text book driven curriculum and we knew the students weren't responding.*

Our consultant suggested that there was sameness about the intellectual demands of this approach which gave the impression that maths was the pursuit of skills. We agreed to select two deeper investigations to add to each unit. It took some time and considerable commitment, but we know that we have now made a curriculum shift. We are more satisfied and so are the students.

The principles guiding this shift were:

- Agree

  The 20 particular investigations for the year are agreed to by all teachers. If, for example, Cube Nets *is decided as one of these, then all the teachers are committed to present this within its unit.*

- Publish

  The investigations are written into the published syllabus. Students and parents are made aware of their existence and expect them to occur.

- Commit

  Once agreed, teachers are required to present the chosen investigations. They are not a negotiable 'extra'.

- Value

  The investigations each illustrate an explicit form of the Working Mathematically process. This is promoted to students, constantly referenced and valued.

- Assess

  The process provides students with scaffolding for their written reports and is also known by them as the criteria for assessment. *(See next page.)*

- Report

  The assessment component features within the school reporting structure.

A Final Comment

Including investigations has become policy.

Why? Because to not do so is to offer a diminished learning experience.

The investigative process ranks equally with skill development and needs to be planned for, delivered, assessed and reported.

Perhaps most of all we are grateful to our consultant because he was prepared to begin where we were. We never felt as if we had to throw out the baby and the bath water.

Assessment

Our attitude is:

stimulated students are creative and love to learn

Regardless of the way you use your **Maths With Attitude** resource, a variety of procedures can be employed to assess this learning.

Where these assessment procedures are applied to task sessions and involve written responses from students, teachers will need to be careful that the writing does not become too onerous. Students who get bogged down in doing the writing may lose interest in doing the tasks.

In addition to the ideas below, useful references are:

- http://www.mathematicscentre.com/taskcentre/assess.htm
- http://www.mathematicscentre.com/taskcentre/report.htm

The first offers several methods of assessment with examples and the second is a detailed lesson plan to support students to prepare a Maths Report.

Journal Writing

Journal writing is a way of determining whether the task or lesson has been understood by the student. The pupil can comment on such things as:

- What I learned in this task.
- What strategies I/we tried (refer to the Strategy Board).
- What went wrong.
- How I/we fixed it.
- Jottings - ie: any special thoughts or observations

Some teachers may prefer to have the page folded vertically, so that children's reflective thoughts can be recorded adjacent to critical working.

Assessment Form

An assessment form uses questions to help students reflect upon specific issues related to a specific task.

Anecdotal Records

Some teachers keep ongoing records about how students are tackling the tasks. These include jottings on whether students were showing initiative, whether they were working co-operatively, whether they could explain ideas clearly, whether they showed perseverance.

Checklists

A simple approach is to create a checklist based on the Working Mathematically process. Teachers might fill it in following questioning of individuals, or the students may fill it in and add comments appropriately.

Pupil Self-Reflection

Many theorists value and promote metacognition, the notion that learning is more permanent if pupils deliberately and consciously analyse their own learning. The deliberate teaching strategy of oral questioning and the way pupils record their work is an attempt to manifest this philosophy in action. The alternative is the tempting 'butterfly' approach which is to madly do as many activities as possible, mostly superficially, in the mistaken belief that quantity equates to quality.

> *I had to work quite hard to overcome previously entrenched habits of just getting the answer, any answer, and moving on to the next task.*

Thinking about *what* was learned *how* it was learned consolidates and adds to the learning.

When it follows an extensive whole class investigation, a reflection lesson such as this helps to shift entrenched approaches to mathematics learning. It is also an important component of the assessment process. On the one hand it gives you a lot of real data to assist your assessment. On the other it prepares the students for any formal assessment which you may choose to round off a unit.

Introduction

Ask students to recall what was done during the unit or lesson by asking a few individuals to say what *they* did, eg:

> *What did you do or learn that was new?*
> *What can you now do/understand that is new?*
> *What do you know now that you didn't know 1 (2, 3, ...) lesson ago?*

Continuing Discussion

Get a few ideas from the first students you ask, then:

- organise 5 -10 minute buzz groups of three or four students to chat together with one person to act as a recorder. These groups address the same questions as above.
- have a reporting session, with the recorder from each group telling the class about the group's ideas.

Student comments could be recorded on the board, perhaps in three groups.

<u>Ideas & Facts</u> <u>Maths Skills</u> <u>Process (learning) Skills</u>

If you need more questions to probe deeper and encourage more thought about process, try the following:

> *What new things did you do that were part of* how *you learned?*
> *Who uses this kind of knowledge and skill in their work?*

Student Recording

Hand out the REFLECTION sheet (next page) and ask students to write their own reflection about what they did, based on the ideas shared by the class. Collect these for interest and, possibly, assessment information.

REFLECTION

me looking at me learning

NAME: CLASS:

Working With Parents

Balancing Problem Solving with Basic Skill Practice

Many schools find that parents respond well to an evening where they have an opportunity to work with the tasks and perhaps work a task together as a 'whole class'. Resourced by the materials in this kit, teachers often feel quite confident to run these practical sessions. Comments from parents like:

> *I wish I had learnt maths like this.*

are very supportive. Letting students 'host' the evening is an additional benefit to the home/school relationship.

The 4½ Minute Talk

Charles Lovitt has considerable experience working with parents and has developed a crisp, parent-friendly talk which he shares below. Many others have used it verbatim with great success.

Why the Four and a Half Minute Talk?

When talking with parents about Problem Solving or the meaning of the term Working Mathematically, I have often found myself in the position, after having promoted inquiry based or investigative learning, of the parents saying:

> *Well - that's all very well - BUT ...*

at which stage they often express their concern for basic (meaning arithmetic) skill development.

The weakness of my previous attempts has been that I have been unable to reassure parents that problem solving does not mean sacrificing our belief in the virtues of such basic skill development.

One of the unfortunate perceptions about problem solving is that if a student is engaged in it, then somehow they are not doing, or it may be at the expense of, important skill based work.

This Four and a Half Minute Talk to parents is an attempt to express my belief that basic skill practice and problem solving development can be closely intertwined and not seen as in some way mutually exclusive.

(I'm still somewhat uncomfortable using the expression 'basic skills' in the above way as I am certain that some thinking, reasoning, strategy and communication skills are also 'basic'.)

Another aspect of the following 'talk' is that, as teachers put more emphasis on including investigative problem solving into their courses, a question arises about the source of suitable tasks.

This talk argues that we can learn to create them for ourselves by 'tweaking' the closed tasks that heavily populate out existing text exercises, and hence not be dependent on external suppliers. (Even better if students begin to create such opportunities for themselves.)

The Talk

In preparation, write the following graphic on the board:

CLOSED	OPEN	EXTENDED INVESTIGATION
		How many solutions exist? How do you know you have found them all?

I would like to show you what teachers are beginning to do to achieve some of the thinking and reasoning and communication skills we hope students will develop. I would like to show you three examples.

Example One: *6 + 5 = ?*

I write this question under the 'closed' label on the diagram:

CLOSED	OPEN	EXTENDED INVESTIGATION
6 + 5 ___		How many solutions exist? How do you know you have found them all?

And I ask:

What is the answer to this question?

I then explain that:

We often ask students many closed questions such as 6 + 5 = ?

The only response the students can tell us is "The answer is 11." … and as a reward for getting it correct we ask another twenty questions just like it.

What some teachers are doing is trying to *tweak* the question and ask it a different way, for example:

I have two counting numbers that add to 11. What might the numbers be?

[Counting numbers = positive whole numbers including zero]

I write this under the 'open' label on the diagram:

CLOSED	OPEN	EXTENDED INVESTIGATION
6 + 5 ___	? + ? 11	How many solutions exist? How do you know you have found them all?

What is the answer to the question now?

At this stage it becomes apparent there are several solutions:

The question is now a bit more open than it was before, allowing students to tell you things like 8 + 3, or 10 + 1, or 11 + 0 etc.

Let's see what happens if the teacher 'tweaks' it even further with the investigative challenge *or* extended investigation question:

How many solutions are there altogether?

and more importantly, and with greater emphasis on the second question:

How could you convince someone else that you have found them all?

Now the original question is definitely different - it still involves the skills of addition but now also involves thinking, reasoning and problem solving skills, strategy development and particularly communication skills.

Young students will soon tell you the answer is 'six different ones', but they must also confront the communication and reasoning challenge of convincing you that there are only six and no more.

Example Two: *Finding Averages*

Again, as I go through this example, I write it into the diagram on the board in the relevant sections.

The CLOSED question is: *11, 12, 13 - find the average*

Tweaking this makes it an OPEN question and it becomes:

I have three counting numbers whose average is 12. What might the numbers be?

Students will often say:

10, 12, 14 … or 9, 12, 15 … or even 12, 12, 12

After realising there are many answers, you can tweak it some more and turn it into an EXTENDED INVESTIGATION:

How many solutions exist? … AND …

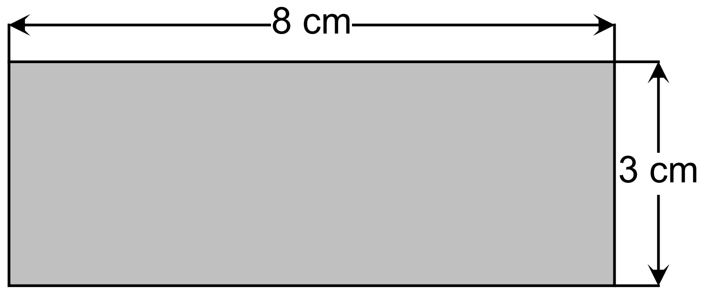
How do you know you have found them all?

Now the question is of a quite different nature. It still involves the arithmetic skill, but has something else as well - and that something else is the thinking, reasoning and communication skills necessary to find all of the combinations and convince someone else that you have done so.

By the time a student announces, with confidence, there are 127 different ways (which there are) that student will have engaged in all of these aspects, ie: the skill of calculating averages, (and some combination number theory) as well as significant strategy and reasoning experiences.

Example Three: *Finding the Area of a Rectangle*

A typical CLOSED question is:



Find the area. Find the perimeter.

The OPEN question is:

A rectangle has 24 squares inside:

What might its length and width be?

What might its perimeter be?

The EXTENDED INVESTIGATION version is:

Given they are whole number lengths, how many different rectangles are there? … AND …

How do you know you have found them all?

In summary, mathematics teachers are trying to convert *some* (not all) of the many closed questions that populate our courses and 'push' them towards the investigation direction. In doing so, we keep the skills we obviously value, but also activate the thinking, reasoning and justification skills we hope students will also develop.

This sequence of three examples hopefully shows two major features:

- That skills and problem solving can 'live alongside each other' and be developed concurrently.
- That the process of creating open-ended investigations can be done by anyone - just go to any source of closed questions and try 'tweaking' them as above. If it only worked for one question per page it would still provide a very large supply of investigations.

In terms of the effect of the talk on parents, I have usually found them to be reassured that we are not compromising important skill development (and nor do we want to). The only debate then becomes whether the additional skills of thinking, reasoning and communication are also desirable.

I've also been told that parents appreciate it because of the essential simplicity of the examples - no complicated theoretical jargon.

A Working Mathematically Curriculum

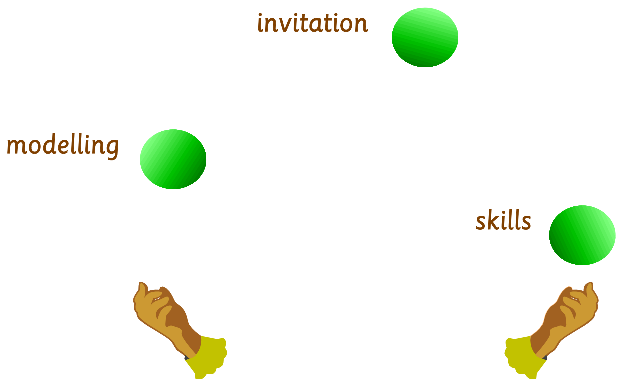
An Investigative Approach to Learning

The aim of a Working Mathematically curriculum is to help students learn to work like a mathematician. This process is detailed earlier (Page 8) in a one page document which becomes central to such a curriculum.

The change of emphasis brings a change of direction which *implies and requires* a balance between:

- the process of being a mathematician, and
- the development of skills needed to be a *successful* mathematician.

This journey is not two paths. It is one path made of two interwoven threads in the same way as DNA, the building block of life, is one compound made of two interwoven coils. To achieve a Working Mathematically curriculum teachers need to balance three components.



The task component of **Maths With Attitude** offers each pair of students an invitation to work like a mathematician.

The Maths300 component of **Maths With Attitude** assists teachers to model working like a mathematician.

Content skills are developed in context. They *are* important, but it is the application of skills within the process of Working Mathematically that has developed, and is developing, the human community's mathematical knowledge.

A focus for the Working Mathematically teacher is to help students develop mathematical skills in the context of problem posing and solving.

> *We are all 'born' with the same size mathematical toolbox, in the same way as I can own the same size toolbox as my motor mechanic. However, my motor mechanic has many more tools in her box than I and she has had more experience than I using them in context. Someone has helped her learn to use those tools while crawling under a car.*

Afzal Ahmed, Professor of Mathematics at Chichester, UK, once quipped:

> *If teachers of mathematics had to teach soccer, they would start off with a lesson on kicking the ball, follow it with lessons on trapping the ball and end with a lesson on heading the ball. At no time would they play a game of football.*

Such is not the case when teaching a Working Mathematically curriculum.

Elements of a Working Mathematically Curriculum

Working Mathematically is a K - 12 experience offering a balanced curriculum structured around the components below.

Hands-on Problem Solving Play

Mathematicians don't know the answer to a problem when they start it. If they did, it wouldn't be a problem. They have to play around with it. Each task invites students to play with mathematics 'like a mathematician'.

Skill Development

A mathematician needs skills to solve problems. Many teachers find it makes sense to students to place skill practice in the context of *Toolbox Lessons* which *help us better use the Working Mathematically Process* (Page 8).

Focus on Process

This is what mathematicians do; engage in the problem solving process.

Strategy Development

Mathematicians also make use of a strategy toolbox. These strategies are embedded in Maths300 lessons, but may also have a separate focus. Poster Problem Clinics are a useful way to approach this component.

Concept Development

A few major concepts in mathematics took centuries for the human race to develop and apply. Examples are place value, fractions and probability. In the past students have been expected to understand such concepts after having 'done' them for a two week slot. Typically they were not visited again until the next year. A Working Mathematically curriculum identifies these concepts and regularly 'threads' them through the curriculum.

Planning to Work Mathematically

The class, school or system that shifts towards a Working Mathematically curriculum will no longer use a curriculum document that looks like a list of content skills. The document would be clear in:

- choosing genuine problems to initiate investigation
- choosing a range of best practice teaching strategies to interest a wider range of students
- practising skills for the purpose of problem solving

Some teachers have found the planning template on the next page assists them to keep this framework at the forefront of their planning. It can be used to plan single lessons, or units built of several lessons. There are examples from schools in the Curriculum & Planning section of Maths300 and a Word document version of the template.

Unit Planning Page

Class

Topic

Pedagogy	Problem Solving	Skills
	In this topic how will I engage my students in the Working Mathematically process?	
How do I create an environment where students know what they are doing and why they have accepted the challenge?		Does the challenge identify skills to practise? Are there other skills to practise in preparation for future problem solving?

Notes

As a general guide:

- Find a problem(s) to solve related to the topic.
- Choose the best teaching craft likely to engage the learners.
- Where possible link skill practice to the problem solving process.

More on Professional Development

For many teachers there will be new ideas within **Maths With Attitude**, such as unit structures, views of how students learn, teaching strategies, classroom organisation, assessment techniques and use of concrete materials. It is anticipated (and expected) that as teachers explore the material in their classrooms they will meet, experiment with and reflect upon these ideas with a view to long term implications for the school program and for their own personal teaching.

Being explored 'on-the-job' so to speak, in the teacher's own classroom, makes the professional development more meaningful and practical for the teacher. This is also a practical and economic alternative for a local authority.

Strategic Use by Systems

From Years 3 - 10, **Maths With Attitude** is designed as a professional development vehicle by schools or clusters or systems because it carries a variety of sound educational messages. They might choose **Maths With Attitude** because:

- It can be used to highlight how investigative approaches to mathematics can be built into balanced unit plans without compromising skill development and without being relegated to the margins of a syllabus as something to be done only after 'the real' content has been covered.
- It can be used to focus on how a balance of concept, skill and application work can all be achieved within the one manageable unit structure.
- It can be used to show how a variety of assessment practices can be used concurrently to build a picture of student progress.
- It can be used to focus on transition between primary and secondary school by moving towards harmony and consistency of approach.
- It can be used to raise and continue debate about the pedagogy (art of teaching) that supports deeper mathematical learning for a wider range of students.

Teachers in Years K - 2 are similarly encouraged in professional growth through **Working Mathematically with Infants**, which derives from Calculating Changes, a network of teachers enhancing children's number skills from Years K - 6.

In supporting its teachers by supplying these resources in conjunction with targeted professional development over time, a system can fuel and encourage classroom-based debate on improving outcomes. There is evidence that by exploring alternative teaching strategies and encouraging curriculum shift towards Working Mathematically, learners improve and teachers are more satisfied. For more detail visit Research & Stories at:

- http://www.mathematicscentre.com/taskcentre/do.htm

We would be happy to discuss professional development with system leaders.

Web Reference

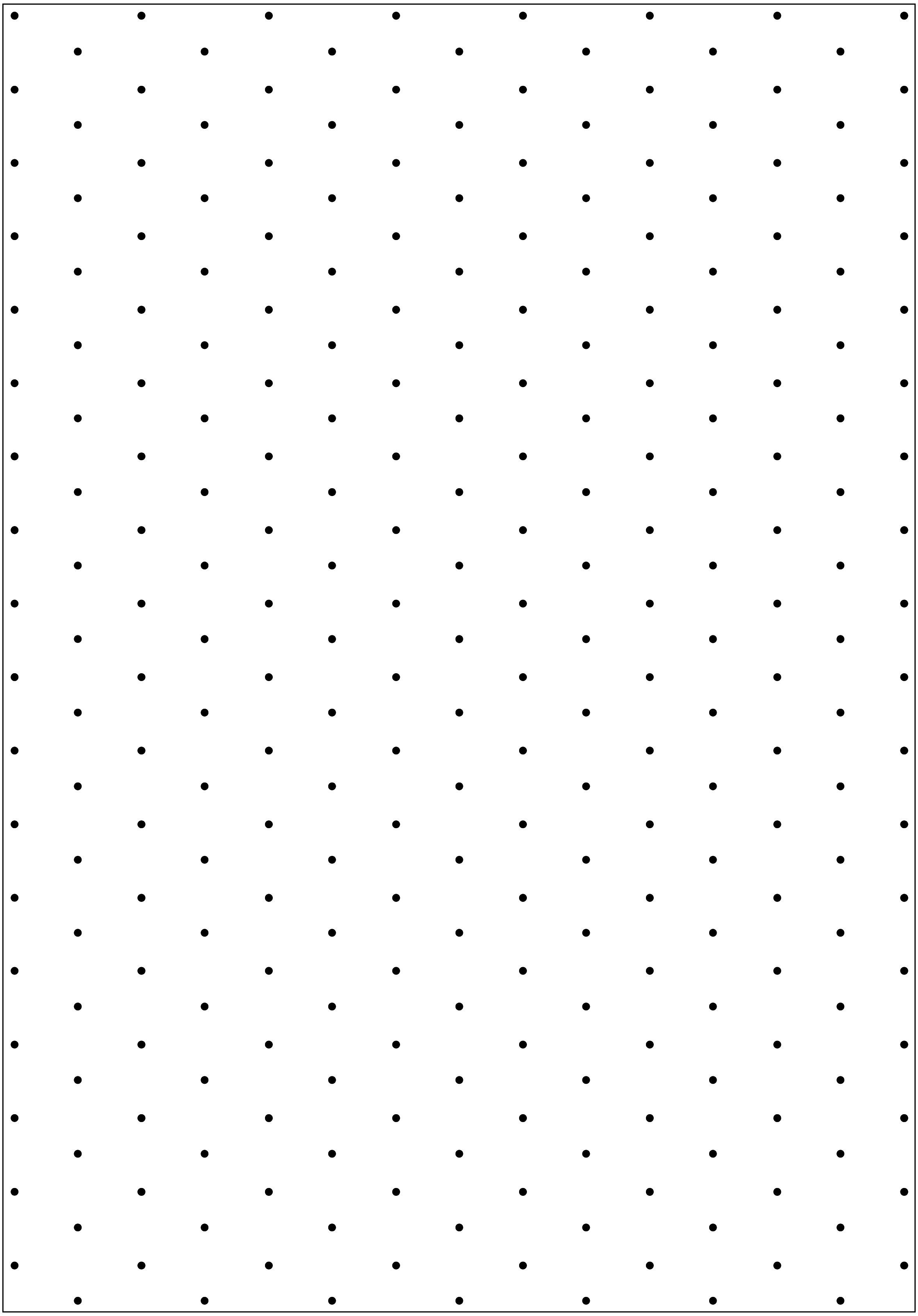
The starting point for all aspects of learning to work like a mathematician, including Calculating Changes, and the teaching craft which encourages it is:

- http://www.mathematicscentre.com

Appendix: Recording Sheets

Pattern Cube Recording Sheet

Reproducible Page



Names: .... Class: ....

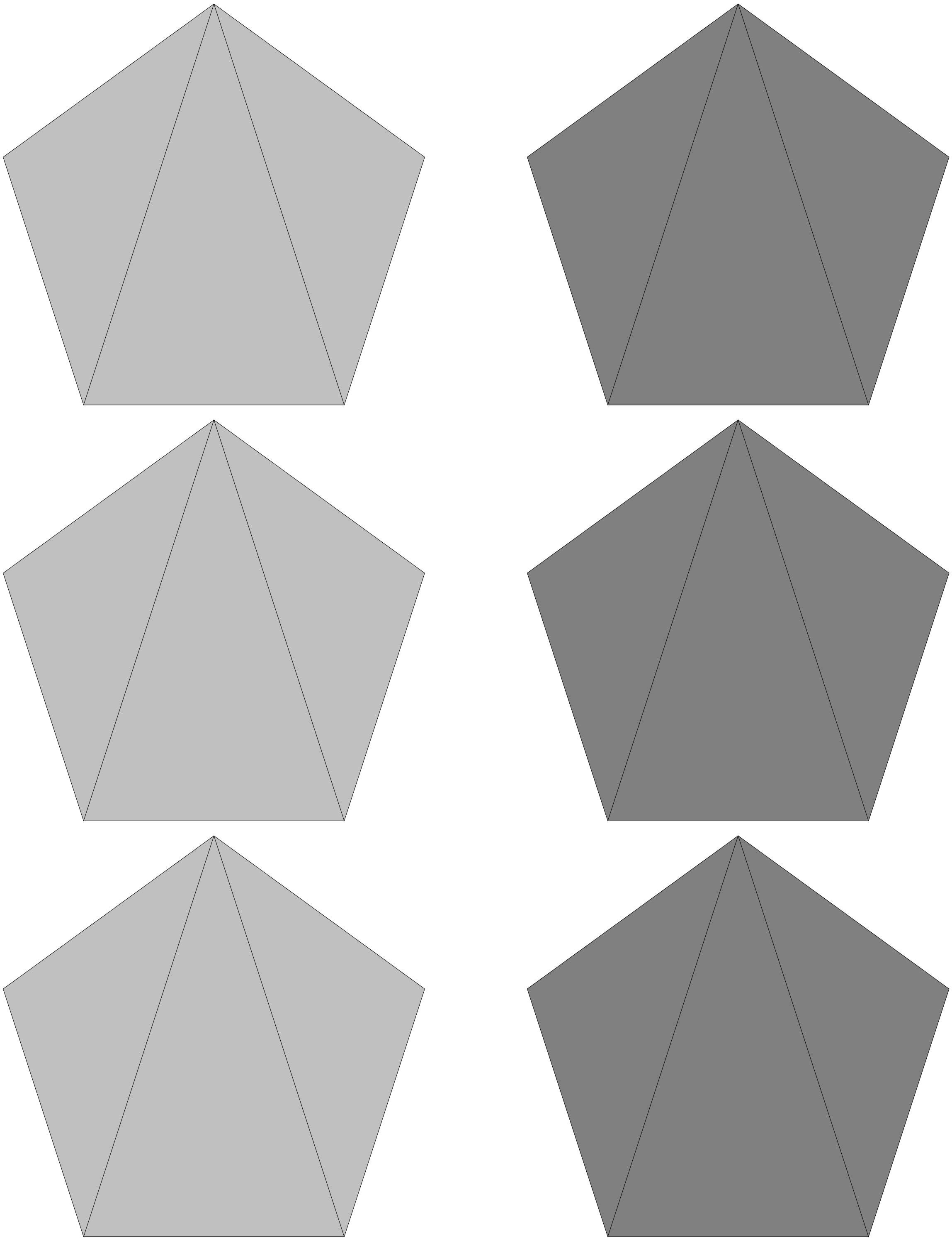
Mirror Patterns 3 Recording Sheet

Reproducible Page

Names: .... Class: ....

Pentagon Triangles

Reproducible Page



Cut out one sheet of tiles per person, then encourage students to work in pairs.

Symmetric Shapes Recording Sheet

Reproducible Page

Names: .... Class: ....