maths with attitude

2nd Edition

Number & Computation Years 9 & 10

Charles Lovitt
Doug Williams

Mathematics Task Centre & Maths300
helping to create happy healthy cheerful productive inspiring classrooms

maths with attitude

2nd Edition

Number & Computation

Years 9 & 10

In this kit:

- *Hands-on problem solving tasks*
- *Detailed curriculum planning*

Access from Maths300:

- *Extensive lesson plans*
- *Software*

Doug Williams
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Black Douglas Professional Education Services
4/71 Greenhill Road
Bayswater North Vic 3153
Australia
Mobile: +61 401 177 775
Email: doug@blackdouglas.com.au

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Part 1: Preparing To Teach

Our Objective

- To support teachers, schools and systems wanting to create:

  happy, healthy, cheerful, productive, inspiring classrooms

Our Attitude

- to learning:

  learning is a personal journey stimulated by achievable challenge

- to learners:

  stimulated students are creative and love to learn

- to pedagogy:

  the art of choosing teaching strategies to involve and interest all students

- to mathematics:

  mathematics is concrete, visual and makes sense

- to learning mathematics:

  all students can learn to work like a mathematician

- to teachers:

  the teacher is the most important resource in education

- to professional development:

  teachers improve their teaching by re-enacting stories from the classrooms of their colleagues

Our Objective in Detail

What do we mean by creating:

happy, healthy, cheerful, productive, inspiring classrooms

Happy…

means the elimination of the unnecessary fear of failure that hangs over so many students in their mathematics studies. Learning experiences *can* be structured so that all students see there is something in it for them and hence make a commitment to the learning. In so many 'threatening' situations, students see the impending failure and withhold their participation.

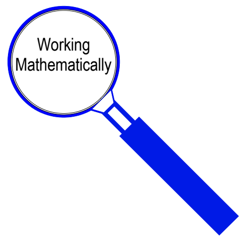
A phrase which describes the structure allowing all students to perceive something in it for them is *multiple entry points and multiple exit points*. That is, students can enter at a variety of levels, make progress and exit the problem having visibly achieved.

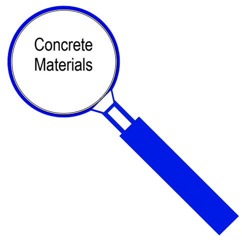
Healthy…

means *educationally healthy*. The learning environment should be a reflection of all that our community knows about how students learn. This translates into a rich array of teaching strategies that could and should be evident within the learning experience.

If we scrutinise the *exploration* through any lens, it should confirm to us that it is well structured or alert us to missed opportunities. For example, peering through a pedagogy lens we should see such features as:

- a story shell to embed the situation in a meaningful context
- significant active use of concrete materials
- a problem solving challenge which provides ownership for students
- small group work
- a strong visual component
- access to supportive software





Cheerful…

because we want 'happy' in the title twice!

Productive…

is the clear acknowledgment that students are working towards recognisable outcomes. They should know what these are and have guidelines to show they have either reached them or made progress. Teachers are accountable to these outcomes as well as to the quality of the learning environment.

Inspiring…

is about creating experiences that are uplifting or exalting; that actually *turn students on*. Experiences that make students feel great about themselves and empowered to act in meaningful ways.

Number & Computation Resources

To help you create

happy, healthy, cheerful, productive, inspiring classrooms

this kit contains

- 20 hands-on problem solving tasks from Mathematics Centre

and a Teachers' Manual which integrates the use of the tasks with

- 19 detailed lesson plans from Maths300

The kit offers **6 weeks** of Scope & Sequence planning in Number & Computation for *each* of Year 9 and Year 10. This is detailed in *Part 2: Planning Curriculum* which begins on Page 12. You are invited to map these weeks into your Year Planner.
Together, the four kits available for these levels provide 25 weeks of core curriculum in Working Mathematically (working like a mathematician).

Note: Membership of Maths300 is assumed.
The kit will be useful without it, but it will be much more useful with it.

Tasks

- Crosses
- Division Boxes
- Doctor Dart
- Fay's Nines
- 4 & 20 Blackbirds
- Ice-Cream Flavours
- Magic Cube
- Magic Hexagon
- Magic Squares
- Make A Snake
- Making Fractions 1
- Monkeys & Bananas
- Number Tiles
- Pizza Toppings
- Square Pairs
- Steps
- Take Away Tiles
- Training For Maths
- Truth Tiles 2
- Window Frames

Part 2 of this manual introduces each task. The latest information can be found at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm

Maths300 Lessons

- Billiard Ball Bounces
- Bob's Buttons
- Consecutive Sums
- Cracked Tiles
- Crosses
- Doctor Dart
- Fay's Nines
- 4 & 20 Blackbirds
- Hunting For Stars
- Ice-Cream Flavours
- Magic Cube
- Magic Squares
- Monkeys & Bananas
- Number Tiles
- Pizza Toppings
- Square Pairs
- Steps
- Tackling Times Tables
- Take Away Tiles

Lessons with Software

- Billiard Ball Bounces
- Bob's Buttons
- Cracked Tiles
- Crosses
- Doctor Dart
- Fay's Nines
- Hunting For Stars
- Ice-Cream Flavours
- Magic Squares
- Number Tiles
- Square Pairs
- Steps
- Tackling Times Tables
- Take Away Tiles

Part 2 of this manual introduces each lesson. Full details can be found at:

- http://www.maths300.com

Working Like A Mathematician

Our attitude is:

all students can learn to work like a mathematician

What does a mathematician's work actually involve? Mathematicians have provided their answer on Page 8. In particular we are indebted to Dr. Derek Holton for the clarity of his contribution to this description.

Perhaps the most important aspect of Working Mathematically is the recognition that *knowledge is created by a community and becomes part of the fabric of that community*. Recognising, and engaging in, the process by which that knowledge is generated can help students to see themselves as able to work like a mathematician. Hence Working Mathematically is the framework of **Maths With Attitude.**

Skills, Strategies & Working Mathematically

A Working Mathematically curriculum places learning mathematical skills and problem solving strategies in their true context. Skills and strategies are the tools mathematicians employ in their struggle to solve problems. Lessons on skills or lessons on strategies are not an end in themselves.

- **Our skill toolbox** can be added to in the same way as the mechanic or carpenter adds tools to their toolbox. Equally, the addition of the tools is not for the sake of collecting them, but rather for the purpose of getting on with a job. A mathematician's job is to attempt to solve problems, not to collect tools that might one day help solve a problem.
- **Our strategy toolbox** has been provided through the collective wisdom of mathematicians from the past. All mathematical problems (and indeed life problems) that have ever been solved have been solved by the application of this concise set of strategies.

About Tasks

Our attitude is:

mathematics is concrete, visual and makes sense

Tasks are from Mathematics Task Centre. They are an invitation to two students to work like a mathematician (see Page 8).

The Task Centre concept began in Australia in the late 1970s as a collection of rich tasks housed in a special room, which came to be called a Task Centre. Since that time hundreds of Australian teachers, and, more recently, teachers from other countries, have adapted and modified the concept to work in their schools. For example, the special purpose room is no longer seen as an essential component, although many schools continue to opt for this facility.

A brief history of Task Centre development, considerable support for using tasks, for example Task Cameos, and a catalogue of all currently available tasks can be found at:

- http://www.mathematicscentre.com/taskcentre

Key principles are:
- A good task is the tip of an iceberg
- Each task has three lives
- Tasks involve students in the Working Mathematically process

The Task Centre Room or the Classroom?
There are good reasons for using the tasks in a special room which the students visit regularly. There are also different good reasons for keeping the tasks in classrooms. Either system can work well if staff are committed to a core curriculum built around learning to work like a mathematician.
- A task centre room creates a focus and presence for mathematics in the school. Tasks are often housed in clear plastic 'cake storer' type boxes. Display space can be more easily managed. The visual impact can be vibrant and purposeful.
- However, tasks can be more readily integrated into the curriculum if teachers have them at their finger tips in the classrooms. In this case tasks are often housed in press-seal plastic bags which take up less space and are more readily moved from classroom to classroom.

Tip of an Iceberg
The initial problem on the card can usually be solved in 10 to 20 minutes. The investigation iceberg which lies beneath may take many lessons (even a lifetime!). Tasks are designed so that the original problem reveals just the 'tip of the iceberg'. Task Cameos and Maths300 lessons help to dig deeper into the iceberg.

We are constantly surprised by the creative steps teachers and students take that lead us further into a task. No task is ever 'finished'.

> *Most tasks have many levels of entry and exit and therefore offer an on-going invitation to revisit them, and, importantly, multiple levels of success for students.*

Three Lives of a Task
This phrase, coined by a teacher, captures the full potential and flexibility of the tasks. Teachers say they like using them in three distinct ways:
1. As on the card, which is designed for two students.
2. As a whole class lesson involving all students, as supported by outlines in the Task Cameos and in detail through the Maths300 site.
3. Extended by an Investigation Guide (project), examples of which are included in both Task Cameos and Maths300.

The first life involves just the 'tip of the iceberg' of each task, but nonetheless provides a worthwhile problem solving challenge - one which 'demands' concrete materials in its solution. This is the invitation to work like a mathematician. Most students will experience some level of success and accomplishment in a short time.

The second life involves adapting the materials to involve the whole class in the investigation, in the first instance to model the work of a mathematician, but also to develop key outcomes or specific content knowledge. This involves choosing teaching craft to interest the students in the problem and then absorb them in it.

The third life challenges students to explore the 'rest of the iceberg' independently. Investigation Guides are used to probe aspects and extensions of the task and can be introduced into either the first or second life. Typically this involves providing suggestions for the direction the investigation might take. Students submit the 'story' of their work for 'portfolio assessment'. Typically a major criteria for assessment is application of the Working Mathematically process.

About Maths300

Our attitude is:

> *teachers improve their teaching by re-enacting stories from the classrooms of their colleagues*

Maths300 is a subscription based web site. It is an attempt to collect and publish the 300 most 'interesting' maths lessons (K - 12).

- *Lessons have been successfully trialed in a range of classrooms.*
- *About one third of the lessons are supported by specially written software.*
- *Lessons are also supported by investigation sheets (with answers) and game boards where relevant.*
- *A 'living' Classroom Contributions section in each lesson includes the latest information from schools.*
- *The search engine allows teachers to find lessons by pedagogical feature, curriculum strand, content and year level.*
- *Lesson plans can be printed directly from the site.*
- *Each lesson supports teaches to model the Working Mathematically process.*

Modern internet facilities and computers allow teachers easy access to these lesson plans. Lesson plans need to be researched, reflected upon in the light of your own students and activated by collecting and organising materials as necessary.

Maths300 Software

Our attitude is:

> *stimulated students are creative and love to learn*

Pedagogically sound software is one feature likely to encourage enthusiastic learning and for that reason it has been included as an element in about one third of Maths300 lesson plans. The software is used to develop an investigation beyond its introduction and early exploration which is likely to include other pedagogical techniques such as concrete materials, physical involvement, estimation or mathematical conversation. The software is not the lesson plan. It is a feature of the lesson plan used at the teacher's discretion.

For school-wide use, the software needs to be downloaded from the site and installed in the school's network image. You will need to consult your IT Manager about these arrangements. It can also be downloaded to stand alone machines covered by the site licence, in particular a teacher's own laptop, from where it can be used with the whole class through a data projector.

Note:

- *Maths300 lessons and software may only be used by Maths300 members.*

Working Mathematically

First give me an interesting problem.

When mathematicians become interested in a problem they:

- Play with the problem to collect & organise data about it.
- Discuss & record notes and diagrams.
- Seek & see patterns or connections in the organised data.
- Make & test hypotheses based on the patterns or connections.
- Look in their strategy toolbox for problem solving strategies which could help.
- Look in their skill toolbox for mathematical skills which could help.
- Check their answer and think about what else they can learn from it.
- Publish their results.

Questions which help mathematicians learn more are:

- Can I check this another way?
- What happens if ...?
- How many solutions are there?
- How will I know when I have found them all?

When mathematicians have a problem they:

- Read & understand the problem.
- Plan a strategy to start the problem.
- Carry out their plan.
- Check the result.

A mathematician's strategy toolbox includes:

- Do I know a similar problem?
- Guess, check and improve
- Try a simpler problem
- Write an equation
- Make a list or table
- Work backwards
- Act it out
- Draw a picture or graph
- Make a model
- Look for a pattern
- Try all possibilities
- Seek an exception
- Break a problem into smaller parts
- ...

If one way doesn't work, I just start again another way.

Professional Development Purpose

Our attitude is:

the teacher is the most important resource in education

> *We had our first study group on Monday. The session will be repeated again on Thursday. I had 15 teachers attend. We looked at the task Farmyard Friends (Task 129 from the Mathematics Task Centre). We extended it out like the questions from the companion Maths300 lesson suggested, and talked for quite a while about the concept of a factorial. This is exactly the type of dialog that I feel is essential for our elementary teachers to support the development of their math background. So anytime we can use the tasks to extend the teacher's math knowledge we are ahead of the game.*
> District Math Coordinator, Denver, Colorado

Research suggests that professional development most likely to succeed:

- is requested by the teachers
- takes place as close to the teacher's own working environment as possible
- takes place over an extended period of time
- provides opportunities for reflection and feedback
- enables participants to feel a substantial degree of ownership
- involves conscious commitment by the teacher
- involves groups of teachers rather than individuals from a school
- increases the participant's mathematical knowledge in some way
- uses the services of a consultant and/or critical friend

Maths With Attitude has been designed with these principles in mind. All the materials have been tried, tested and modified by teachers from a wide range of classrooms. We hope the resources will enable teacher groups to lead themselves further along the professional development road, and support systems to improve the learning outcomes for students K - 12.

With the support of Maths300 ETuTE, professional development can be a regular component of in-house professional development. See:

- http://www.mathematicscentre.com/taskcentre/resource.htm#etute

For external assistance with professional development, contact:

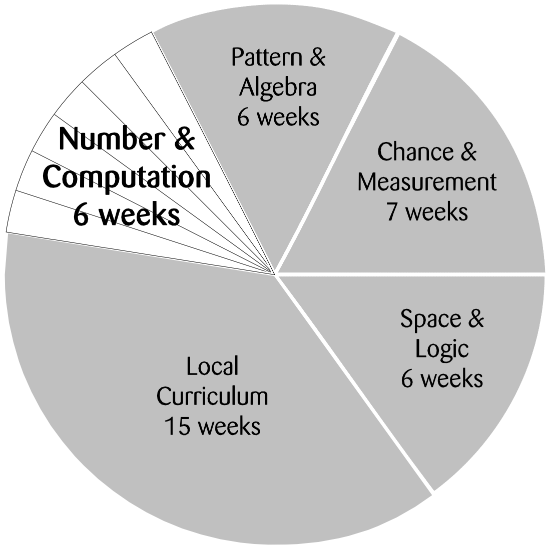
Doug Williams
Black Douglas Professional Education Services
T/F: +61 3 9720 3295
M: 0401 177 775
E: doug@blackdouglas.com.au

Part 2: Planning Curriculum

Curriculum Planners

Our attitude is:

learning is a personal journey stimulated by achievable challenge



Curriculum Planners:

- show one way these resources can be integrated into your weekly planning
- provide a starting point for those new to these materials
- offer a flexible structure for those more experienced

You are invited to map Planner weeks into your school year planner as the core of the curriculum.

Planners:

- detail each week lesson by lesson
- offer structures for using tasks and lessons
- are sequenced from lesson to lesson, week to week and year to year to 'grow' learning

Teachers and schools will map the material in their own way, but all will be making use of extensively trialed materials and pedagogy.

Using Resources

- Your kit contains 20 hands-on problem solving tasks and reference to relevant Maths300 lessons.
- Tasks are introduced in this manual and supported by the Task Cameos at:
  http://www.mathematicscentre.com/taskcentre/iceberg.htm
- Maths300 lessons are introduced in this manual and supported by detailed lesson plans at:
  http://www.maths300.com

In your preparation, please note:

- Planners assume 4 lessons per week of about 1 hour each.
- Planners are *not* prescribing a continuous block of work.
- Weeks can be interspersed with other learning; perhaps a **Maths With Attitude** week from a different strand.
- Weeks can sometimes be interchanged within the planner.
- Lessons can sometimes be interchanged within weeks.
- The four **Maths With Attitude** kits available at each year level offer 25 weeks of a Working Mathematically core curriculum.

A Way to Begin

- Glance over the Planner for your class. Skim through the comments for each task and lesson as it is named. This will provide an overview of the kit.
- Task Comments begin after the Planners. Lesson Comments begin after Task Comments. The index will also lead you to any task or lesson comments.
- Select your preferred starting week - usually Week 1.
- Now plan in detail by researching the comments and web support. Enjoy!

Research, Reflect, Activate

Curriculum Planner

Number & Computation: Year 9

	Session 1	Session 2	Session 3	Session 4
Weeks 1 - 3	**Facts & Factors - Mixed Media Unit A:** The Mixed Media teaching model is explained on Page 17. It assumes ready access to computers for one third of the class. If these are not a fixture in the room, schools have adopted alternatives such as (a) making arrangements for students to visit computer sites within the school, or, (b) collecting an appropriate number of notebook computers (5 or 6 for a class of 30). The content focus is developing notions of proof while refreshing and applying number facts. By using problems which involve simpler skill knowledge, concentration can shift to the reasoning strategies associated with developing hypotheses and refining proof. Tasks used are **Division Boxes**, **Doctor Dart**, **Four & Twenty Blackbirds**, **Ice-Cream Flavours**, **Make A Snake**, **Making Fractions 1**, **Monkeys & Bananas**, **Pizza Toppings**, **Truth Tiles 2**, **Training For Maths**, **Window Frames**. A Mixed Media Unit includes one class lesson each week. In this unit choose from *Bob's Buttons*, *Hunting For Stars*, *Ice-Cream Flavours* or *Tackling Times Tables*.			
Week 4	**Whole Class Investigations:** The two investigations *Consecutive Sums* and *Doctor Dart* are suggested for this week but you may find one is sufficient. *Consecutive Sums* starts simply but leads as far as you wish into algebraic proof of induced results. In the process it emphasises yet again the importance of the Triangle Numbers. *Doctor Dart* is also easy to start. It is an open-ended puzzle constructed from a set of consecutive numbers. Mathematically it refreshes the importance of odd and even numbers in number theory proofs and the use of tree diagrams to explore all choices. The software builds on the tree diagram concept and student choice.			
Week 5	**Whole Class Investigations:** *Billiard Ball Bounces* & *Cracked Tiles*. Model how a mathematician works & begin investigations for Menu Maths.		**Menu Maths:** (See Page 20) Students choose from a menu of requirements to demonstrate their ability to work like a mathematician.	
Week 6	*4 & 20 Blackbirds* and *Monkeys & Bananas* which uses spreadsheets as a tool and has a fraction aspect.		Tasks, software from previous sessions, text work and assessment requirements.	

- Weeks can be interchanged.
- An activity named in **bold** refers to a hands-on task.
- An activity named in *italic* refers to a lesson from Maths300.
- Text book style Toolbox Lessons can be interwoven or set for homework.

Curriculum Planner

Number & Computation: Year 10

<table>
<tr><th></th><th>Session 1</th><th>Session 2</th><th>Session 3</th><th>Session 4</th></tr>
<tr><td>Weeks
1 - 3</td><td colspan="4">A Matter of Proof - Mixed Media Unit B: The Mixed Media teaching model is explained on Page 17. It assumes ready access to computers for one third of the class. If these are not a fixture in the room, schools have adopted alternatives such as (a) making arrangements for students to visit computer sites within the school, or, (b) collecting an appropriate number of notebook computers (5 or 6 for a class of 30).
The content focus is heavily on proof. Each of the tasks/lessons have been studied by professional mathematicians. Students have probably seen some of them in previous years but on this occasion they are revisited to encourage the highest possible levels of reasoning and proof.
Tasks used are Crosses, Fay's Nines, Magic Cube, Magic Hexagon, Magic Squares, Number Tiles, Square Pairs, Steps, Take Away Tiles.
A Mixed Media Unit includes one class lesson each week. In this unit choose from Square Pairs, Magic Squares or Magic Cube.</td></tr>
<tr><td>Week 4</td><td colspan="2">Whole Class Investigations: which relate to the properties of numbers and using those properties to develop sophisticated proof.</td><td colspan="2">Menu Maths: (See Page 20)
Students choose from a menu of requirements to demonstrate their ability to work like a mathematician.</td></tr>
<tr><td>Week 5</td><td colspan="2">Beginning these investigations as whole class lessons supports the students personal study of them in Menu Maths. Choose:</td><td colspan="2">Tasks, software from previous sessions, text work and assessment requirements.</td></tr>
<tr><td>Week 6</td><td colspan="2">Crosses, Fay's Nines, Number Tiles, Steps, or Take Away Tiles to use during the three weeks.</td><td colspan="2"></td></tr>
</table>

- Weeks can be interchanged.
- An activity named in **bold** refers to a hands-on task.
- An activity named in *italic* refers to a lesson from Maths300.
- Text book style Toolbox Lessons can be interwoven or set for homework.

Planning Notes

Enhancing Maths With Attitude

Resources to support learning to work like a mathematician are extensive and growing. There are more tasks and lessons available than have been included in this Number & Computation kit. You could use the following to enhance this kit.

Additional Tasks

- Task 22, Time Together
  A task to help students explore the passing of time. The focus is on those moments in a 12 hour cycle when the hands are 'on top of each other'. The task encourages manipulation of the clock hands, estimation and, for the more mathematically mature, precise calculation.
- Task 240, Less Than Fractions
  Number tiles (1 - 9) allow students to experiment with fractions less than 1 in a non-threatening, open-ended way. Early success is guaranteed because there are 36 possible answers and the obvious one is 1/2. But the greater challenge is to add two fractions (each tile can be used only once) and still get an answer less than one. How many solutions are there? How do you know when you have found them all?
- Task 153, Knight Protectors
  A famous chess-based puzzle requiring significant patience that results in a beautiful rotationally symmetric solution. The challenge is to place the minimum number of knights on a chess board so that every square is either occupied or attacked.

More information about these tasks may be available in the Task Cameo Library:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Additional Lessons

- Lesson 120, 1089
  With due flamboyance the teacher announces that this lesson will be special because the answer to all the questions has already been written out. It soon becomes clear that starting with (almost) any 3-digit number, followed by a subtraction and an addition, the answer is 1089. The young mathematicians in the class are challenged to recognise that this can't be magic, but rather there must be a mathematical explanation for why this happens. The challenge of the lesson is to explain why the answer is always the same and to communicate this to others.
- Lesson 130, The Mushroom Hunt
  Six people go looking for mushrooms. Each one has a basket. When they return they compare the numbers of mushrooms in their baskets. They discover two things: (a) Adding up the numbers gives a total of 63. (b) Grouping the baskets in different ways gives sub-totals equal to every number from 1 to 63. The problem is, of course, how many mushrooms are in each basket. The problem can be tackled at every level from an exploration of doubling, to an introduction to powers and indices, to the concept of binary numbers, to an investigation of multiplicative (exponential-like) growth.

- Lesson 150, Fermi Problems
  Fermi Problems have the characteristic that most people who encounter them respond that it is a problem they couldn't solve without recourse to outside information. Students see that through a series of simple steps using only common sense and their experience they can quite often come up with reasonable estimates for the answers. The lesson is structured around a common problem to illustrate the process, followed by group attempts to solve and report on a problem of their choice.
- Lesson 173, Factors
  The number 100 has 9 factors. So do 36 and 225. What do these numbers have in common? Why do they all have the same number of factors? There is a (arguably under-appreciated) rule which can tell you how many factors exist for any number. This investigation is designed to both uncover that rule and see the logic behind it.

Keep in touch with new developments which enhance **Maths With Attitude** at:

- http://www.mathematicscentre.com/taskcentre/enhance.htm

Additional Materials

As stated, our attitude is that mathematics is concrete, visual and makes sense. We assume that all classrooms will have easy access to many materials beyond what we supply. For this unit you will need:

- Balls of knitting wool
- Counters
- Cubes
- Packs of cards (optional)

Special Comments Year 9

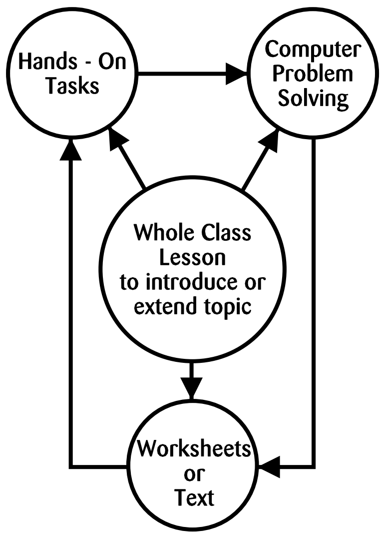
- Look ahead to Planner Weeks 1 - 3. A Mixed Media unit takes a little extra preparation in the beginning, but this is repaid during the unit because it runs for several days. Schools which have used the model find team preparation eases any burden, ie:
  - one teacher ensures the software station is prepared and runs effectively during the unit
  - one gets the tasks in order and keeps them that way during the unit
  - one selects appropriate material from the text and prepares the 'contract' sheet
  - one makes sure all teachers are resourced with whatever is necessary for the whole class lessons
- Note: In the preparation of this unit some teachers create a 'contract' sheet for students which sets out the expectations of the unit, for example:
  - Participate in two whole class lessons.
  - Keep a diary of your work on at least three tasks.
  - Complete a written report about one software-based investigation.
  - Complete Exercises ... from the text.
- Look ahead to Planner Week 6. *Monkeys & Bananas* is a very concrete lesson so you will need a good supply of cubes, or similar, to use as bananas.
- Look ahead to Planner Weeks 5 & 6. You will need to prepare your menu for Menu Maths.

Special Comments Year 10

- Lessons based on tasks with numbered tiles (eg: Crosses, Steps, Number Tiles...) do not necessarily need sets of numbered tiles. You can create the whole class investigation by having the students tear up paper to make the pieces, or by using subsets of a deck of cards.
- Look ahead to Planner Weeks 1 - 3. A Mixed Media unit takes a little extra preparation in the beginning, but this is repaid during the unit because it runs for several days. Schools which have used the model find team preparation eases any burden, ie:
  - one teacher ensures the software station is prepared and runs effectively during the unit
  - one gets the tasks in order and keeps them that way during the unit
  - one selects appropriate material from the text and prepares the 'contract' sheet
  - one makes sure all teachers are resourced with whatever is necessary for the whole class lessons
- Note: In the preparation of this unit some teachers create a 'contract' sheet for students which sets out the expectations of the unit, for example:
  - Participate in two whole class lessons.
  - Keep a diary of your work on at least three tasks.
  - Complete a written report about one software-based investigation.
  - Complete Exercises ... from the text.
- If you are using Square Pairs as a whole class lesson in the Mixed Media unit, you will need to prepare the large cards used in the lesson.
- Look ahead to Planner Weeks 4 - 6. You will need to prepare your menu for Menu Maths.

Mixed Media Unit

Mixed Media Mathematics has been created as *one* structure which allows teachers to integrate problem solving tasks into the curriculum.



The design incorporates four different modes of learning into a structure which can be readily managed by one teacher, but which is enhanced when prepared and executed by a team.

A three week Mixed Media Unit includes:

- whole class lessons
- hands-on problem solving
- problem solving software
- skill practice worksheets (or text material)
- time to reflect on learning
- assessment opportunities

If this is the first time such a structure has been used in your classroom, it is a good idea to prepare the students in a manner which 'brings them into the experiment'.

A vital element of the process is to reflect on *what* is learned and *how* it is learned *before* the final assessment of the learning. Guidance with respect to assessment is also provided in this manual. In particular, the Pupil Self-Reflection information in the Assessment section of Part 3 was designed by teachers who trialed the original Mixed Media units.

Mixed Media Unit A

Bob's Buttons, *Hunting For Stars* and *Ice-Cream Flavours* are the whole class lessons suggested for this unit. You will certainly need to use two of them over the first two weeks of the unit, but whether you use the third will depend on whether you choose an assessment focus for the third week. If so, the whole class lesson in this week could be built around the Pupil Self-Reflection information, Page 59.

Each of these lessons is well documented in Maths300. Both *Hunting For Stars* and *Bob's Buttons* involve factors, multiples and primes and both have software support. *Ice-Cream Flavours* puts multiplication to work through an introduction to the 'clever counting' of combination theory which is supported by unique software. These lessons are non-threatening, require minimal mathematical skill and suit a broad range of abilities. However, all are open-ended, offer multiple entry and exit points and require a mature level of reasoning and intelligent application of problem solving strategies consistent with expectations at this level.

Investigation Sheets to accompany each lesson are included in the lesson plan. These might be used at the Text/Worksheet station and might form part of the assessment requirements of the unit. The Investigation Sheets for *Ice-Cream Flavours* are actually designed as an assessment tool. This station can also include relevant material from your text book. Students work in pairs on the tasks and the computers. Some teachers are happy for this collegiate approach to continue at the text station as well.

Tackling Times Tables is included as a choice for this unit because of the factor problems in Option 3 of the software. Teachers may wish to develop a software station Investigation Sheet related to the hyperbolic function $xy = a$ which is embedded in the graphing facility of this option.

The tasks suggested for this unit are:

- Division Boxes
- Doctor Dart
- Four & Twenty Blackbirds
- Ice-Cream Flavours
- Make A Snake
- Making Fractions 1
- Monkeys & Bananas
- Pizza Toppings
- Truth Tiles 2
- Training For Maths
- Window Frames

The same tasks are used in all weeks of the unit because, in the main, it will take one session to complete one task fully, so no students will be able to use them all in this unit. These tasks are also available in Menu Maths (see Page 20).

It has already been mentioned that you will need to choose a new whole class lesson for the second week, and you may also need new text work. Alternatively it may be appropriate for students to continue some aspects of their software or task-based investigation during the period at the text station.

Teachers often find the third week needs to be flexible. You could continue the exploratory nature of the unit and introduce one more whole class lesson and follow up investigation from the set above. Or, if the unit is going to have a strong assessment component based on the deeper investigation, students may need time scheduled for report writing. In fact, it may be necessary to use a lesson time this week to model report writing.

In the preparation of this unit some teachers create a 'contract' sheet for students which sets out the expectations of the unit, for example:

- Participate in two whole class lessons.
- Keep a diary of your work on at least three tasks.
- Complete a written report of one software-based investigation.
- Complete Exercises ... from the text.

Student Publishing

It is inappropriate to simply expect students to publish a report of their investigation. We have to devote lesson time to teaching how to keep a journal while investigating and how to plan and present a report. The Recording & Publishing section of Mathematics Task Centre includes two different approaches to scaffolding this process with the class. Both include sample student work and suggest that a report can be presented in forms other than pencil and paper, for example PowerPoint. The links are titled 'Learning to Write a Maths Report' and 'Learning to Write a Maths Report 2' and can be found at:

- http://www.mathematicscentre.com/taskcentre/record.htm

Mixed Media Unit B

Magic Cube, *Magic Squares* and *Square Pairs*, are the whole class lessons suggested for this unit.

You might choose:

- *Magic Cube* because of its emphasis on logical reasoning and problem solving strategies and its links to the work of mathematicians of the past.
- *Magic Squares* because of its emphasis on breaking a problem into smaller parts and its historic connection with recreational mathematics.
- *Square Pairs* because of its challenging, physically involving introduction, the mental arithmetic and problem solving strategies involved in exploring it ever more deeply, or its link to the work of a modern day mathematician.

The tasks suggested for this unit are:

- Crosses
- Fay's Nines
- Magic Cube
- Magic Squares
- Number Tiles
- Square Pairs
- Steps
- Take Away Tiles
- Magic Hexagon

The same tasks are used in all weeks of the unit because, in the main, it will take one session to complete one task fully, so no students will be able to use them all in this unit. These tasks are also available in Menu Maths.

Menu Maths

Menus give students a chance to learn independently.

- The teacher lists the range of possible studies on a menu.
- Items relate to a particular topic or strand.
- Compulsory items, should they be necessary, are identified.
- Assessment requirements of the unit are detailed.

Students choose their own studies from the menu.

- The objective is to demonstrate that they are able to work like a mathematician.
- Mathematicians need skills to solve problems, so the students will also be demonstrating their mathematical skill knowledge.
- Should the solution of the problem require skills which a student doesn't yet have, or needs to refresh, the self-directed nature of the class frees the teacher to run tutorial style support sessions at this point of need.

Menu items include:

- Investigating the iceberg of tasks.
- Continuing investigations begun as whole class lessons.
- Students teaching themselves from their text book.

All items are related to the course content being studied at the time. Investigations begun as whole class lessons can be extended with Investigation Sheets. For some Maths300 lessons these are supplied.

Self teaching from the text may be a new concept for some, but it is consistent with:

- our attitude that students take responsibility for their own learning
- the situation of the professional mathematician who doesn't have the skills to solve a particular problem and has to set about learning them.

Self teaching allows the students to work on their own, if that is their preference, and to form and reform collegiate study groups as they perceive the need. This process is consistent with the way mathematicians move through phases of private study and consultation with colleagues. It is also consistent with the expectations of many tertiary courses which the students will begin in the next few years.

Many teachers have found that passing over the learning responsibility, and creating a classroom atmosphere which expects students to be capable of learning mathematics in this self-directed manner, results in greater student satisfaction and improved outcomes. Teachers also report they are more comfortable in this facilitator role than they were leading from the front and knowing that some students were not following.

Assessment is a clear, up front expectation which can take several forms.

- There is a range of advice about assessment practices in Part 3 of this manual (Value Adding).
- To highlight the major objective of demonstrating the ability to work like a mathematician, students are also required to present an analysis of their work against the framework of Working Mathematically (see Page 8).
- Some schools have this document displayed as a poster in maths rooms. Others have it pasted inside the students' maths journals. Others publish it in both these ways and also publish it to the school community.

Sample Menu

Menu Maths - Year 9 **Number & Computation** **Due Date: ....**

- Choose one of the Investigations and one of the Text requirements.
- You will receive more credit for activities explored in depth than for superficial attempts at several items.
- As part of each assessment requirement below you are expected to refer to the Working Mathematically process to identify how you have worked like a mathematician.

Investigations - Tasks

- ☐ Division Boxes
- ☐ Doctor Dart
- ☐ 4 & 20 Blackbirds
- ☐ Ice-Cream Flavours
- ☐ Make A Snake
- ☐ Making Fractions 1
- ☐ Monkeys & Bananas
- ☐ Pizza Toppings
- ☐ Truth Tiles 2
- ☐ Training for Maths
- ☐ Window Frames

Assessment

Hand in a **dated diary** of your investigation and a **summary** of your findings. The diary is a personal record and will not be assessed in the same way as published work. However, assume you are preparing the summary for someone who knows nothing about the problem. Diagrams, graphs etc. are encouraged to support explanation.

Investigations - Extended from Lessons

- ☐ Billiard Ball Bounces
- ☐ Consecutive Sums
- ☐ Cracked Tiles
- ☐ Doctor Dart

Assessment

Hand in a report detailing the problem, how you approached it (including approaches that didn't work), what you found out and other investigations suggested by the problem. This need not be a written document. For example, you might prepare a Power Point display. However, assume you are preparing the summary for someone who knows nothing about the problem.

Teach Yourself

- ☐ Pages ....
- ☐ Pages ....
- ☐ Pages ....

Assessment

Hand in evidence that you are able to complete the exercises in your chosen set. It is not necessary to show that you have done every problem, although you may if you wish. What you have to show is that you have developed the skills to tackle any of the problems in your set and any similar ones which might be offered to you sight unseen.

Name: .... **Class: ....**

Task Comments

- Tasks, lessons and unit plans prepare students for the more traditional skill practice lessons, which we invite you to weave into your curriculum. Teachers who have used practical, hands-on investigations as the focus of their curriculum, rather than focussing on the drill and practice diet of traditional mathematics, report success in referring to skill practice lessons as Toolbox Lessons. This links to the idea of a mathematician dipping into a toolbox to find and use skills to solve problems.

Crosses

Place the digits 1 - 9 in the shape of a X so the two arms of the cross add to the same amount. The task only asks for the first solution and then three more, but this opens the door to the mathematician's questions:

- How many solutions are there?
- How will I know I have found them all?

At this level students are expected to explore these question with a focus on reasoning and proof. As a starting point, encourage them to see that:

- The grand total of the two arms must be the sum of the digits 1 - 9 plus a repeat of the middle number.
- Since the sum of the digits 1 - 9 is 45, the middle number must be odd to make the grand total of the two arms even.
- The grand total of the two arms has to be even because it is made up of two equal parts. If both parts are Even, the grand total will be even. If both parts are Odd, the grand total will be even.

Now the adventure becomes an application of the mathematician's question:

- What happens if...?

and the strategy:

- Try every possibility.

Trying each of the odd numbers in turn in the middle position and exploring the implications of each placement for the remaining digits can lead to the complete set of solutions. However, even with this starting point, considerable reasoning is required to find all the solutions.

There is a nice connection to probability in that if a 5 is placed in the centre and the other digits placed face down at random, the chance of it being a solution turns out to be surprisingly high.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

A lesson based on the chances related to the **Crosses** task (*Chances With Crosses*) can be found on the Maths300 site.

Division Boxes

This task starts simply but through the challenge of finding all the solutions for a given number of boxes it turns into an extended investigation. The main content knowledge is tests for divisibility. However, the investigation involves many aspects of the investigative process such as using different strategies and particular tools such as tree diagrams and the (possible) use of a computer simulation.

With just one box, the number of solutions is 9. The digit 0 being the exception.

For two boxes, the first digit has to be divisible by 1 and the 2-digit number divisible by 2. This produces the following 41 solutions - and some noticeable patterns to assist the search.

- 10, 12, 14, 16, 18
- 20, 24, 26, 28
- 30, 32, 34, 36, 38
- 40, 42, 46, 48
- 50, 52, 54, 56, 58
- 60, 62, 64, 68
- 70, 72, 74, 76, 78
- 80, 82, 84, 86
- 90, 92, 94, 96, 98

Including the third box means the first digit must be divisible by 1, the first two digits by 2 and the whole three digit number by 3.

> *Students may well know one test for divisibility by 3 is that the sum of the digits must add to a multiple of 3. As the problem proceeds, knowledge of divisibility tests for other numbers will be useful. They are listed at the end of this explanation, but this need to know could be a good reason to visit the Web. Entering 'divisibility tests' into a search engine produces many useful results.*

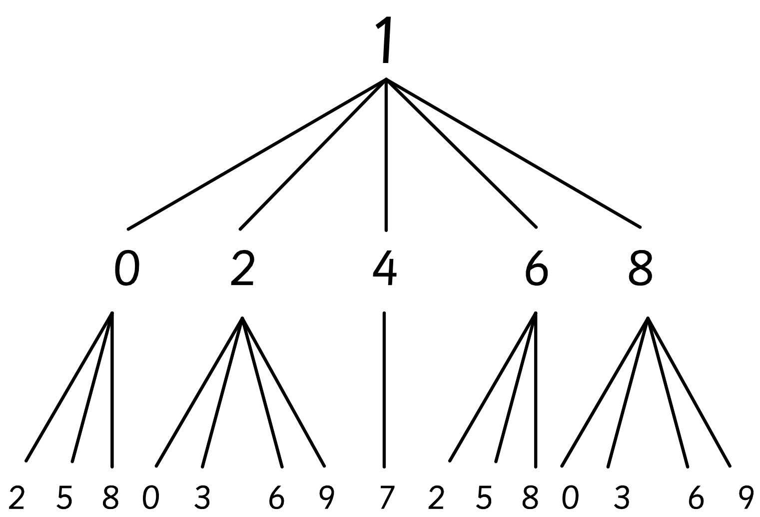
For the three box case, one strategy might be to start with the 41 solutions above and add a third digit so that the whole 3-digit number is divisible by '3'. For example:

- Starting with 10 the numbers would be 102, 105 and 108. (three solutions)
- Starting with 12, the numbers would be 120, 123, 126 and 129 (four solutions)
- Starting with 14, the numbers are 147 (only one solution)
- Starting with 16, the numbers are 162, 165, and 168 (three solutions and a similarity to the '10' group above)
- Starting with 18, the numbers are 180, 183, 186 and 189 (four solutions and a similarity to the '12' group above.

Hence when the first digit is '1' there are $3 + 4 + 1 + 3 + 4 = 15$ solutions.

This could be developed in a sequence of tree diagrams.

- 1st level: is each of the 9 possible digits for the first box (taken one at a time).
- 2nd level: has 5 branches to produce 2 digit numbers divisible by 2.
- 3rd level: has different numbers of branches appropriate to the divisibility by 3 condition.



A tree diagram is a systematic visual representation of the strategy of testing every possible combination, and may be more appropriate for learners with a visual intelligence preference.

Continuing, with 2 at the top of the tree:

- Starting with 20 the numbers are 201,204, 207.
- Starting with 24, the numbers are 240, 243, 246 and 249.
- Starting with 26, the numbers are 261, 264 and 267.
- Starting with 28, the only number is 285.

And hence a total of 11 solutions for the group starting with a 2.

The group starting with 3 produces 13 solutions:

- 306, 309, 321, 324, 327, 342, 345, 348, 360, 369, 381, 384, and 387

This method can clearly be continued right through to the 10 digits. It does seem that the 'tree' might become impossibly large, but indeed after the fourth box the possibilities start to significantly reduce.

A major benefit for students of this systematic search is firstly seeing a viable problem solving method, but also realising the need to be very systematic and rigorous in checking each possibility.

Divisibility Tests

2. *If the last digit is even, then number is divisible by 2.*
3. *If the sum of the digits is divisible by 3, then the number is also divisible by 3.*
4. *If the last two digits form a number divisible by 4, then the number is also divisible by 4.*
5. *If the last digit is a 5 or a 0, then the number is divisible by 5.*
6. *If the number is divisible by both 3 and 2, it is also divisible by 6.*
7. *Double the last digit and subtract it from the rest of the number, eg: for 301, double 1 and subtract it from 30. If the answer is divisible by 7 (including 0), then the number is too. You can keep on applying this rule until you do recognise a result which is, or is not, divisible by 7.*
8. *If the last three digits form a number divisible by 8, then the number is also divisible by 8.*
9. *If the sum of the digits is divisible by 9, then the number is also divisible by 9.*

Many of the web sources also explain why these tests work. Seeking and reporting on this information would be a worthy project at this level.

Doctor Dart

The language used on the card tends to make more sense if students stand the blocks on top of each other, rather than lie them flat as might be interpreted from the diagram. Then they look more like a door with a number pad.

The first section of the card can be answered by guess and check, but it is worth chatting with the students about some of the possibilities and restrictions.

- The final total of 60 is even.
- The second number scored must contribute an even amount to this final total because it is twice the number zapped.
- Therefore, if the first number zapped is even, the third number zapped must also be even, so that the total is even, since three times an even number is even, but three times an odd number is odd.

- However, if the first number zapped is odd, the third number zapped must also be odd, so that the total will be even.

This type of thinking raises the investigation from playing with numbers to playing with types of numbers, and applying these thoughts to the choices, then thinking ahead can lead to solving Doctor Dart's problem.

The next set of questions is designed to lead students towards exploring every possibility. Perhaps they will use a table, or perhaps they will reach into their skill toolbox and use a Tree Diagram. In answering these questions a possible reason for choosing the total 60 appears, and it also becomes clear that a second total could have been chosen which would have been equally difficult to find.

The iceberg of the task begins to appear when students notice connections between the numbers in the original problem.

- They are consecutive.
- One of them is used three times (which one in the sequence?) and the others are used only twice.
- There is a pattern in the way they are arranged.

So what happens if...?

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Fay's Nines

Use the digits 1 - 9 to make three 3-digit numbers whose sum is 999. The task only asks for the first solution and then five more, but this opens the door to the mathematician's questions:

- How many solutions are there?
- How will I know I have found them all?

At this level students are expected to explore these question with a focus on reasoning and proof. The six solutions asked for only provide a little data, so students should be encouraged to seek more.

> *As a mathematician you need to look beyond what the task card says. I want you to collect data about this task until you can see links between the solutions you find.*

Observations which allow the task to be taken further include:

- You can't do it without carrying.
- The last column has to add to 19.
- All the big numbers seem to be in the last two columns.

From here there are several strategies which can be employed, for example:

- Given the last (or ones) column adds to 19, one strategy is to find all the groups that add to 19. Systematically listed these are:
  982 973 964 874 865 ...
- These 5 groups all lead to solutions, and then there are systematic place value combinations or variations within each. Indeed there are 36 variations of each, hence the 5 'families' with 36 variations of each leads to the 180 different solutions.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Four & Twenty Blackbirds

This task involves some of the key elements of the Working Mathematically process. To begin, many students need to work on this problem by physically stacking and shifting the bird counters - playing with the problem. Others will drift to recording number patterns. Each is valid. There are many solutions to the original problem, so there is a level of success for anyone using a guess and check strategy. However, we would hope that at this level the focus is on the Challenge which raises the mathematician's questions:

- How many solutions are there?
- How do I know when I have found them all?

These questions are extensive enough in themselves, but the task also suggests that a mathematician may never be finished with a problem by asking:

- What happens if...?

and thereby opens the door to further investigation.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Ice-Cream Flavours

The task card is a perfect tip of the iceberg. All students will have success in a few minutes, but by this age, the hope is that they are beginning to dig deeper into a task on their own, rather than seeing the last question on the card as the end of the investigation.

If the students don't think of it for themselves, ask them to identify the conditions stated in the problem on the card.

- There are just three flavours.
- The ice-cream must be a triple header.
- The flavours can't be repeated in a particular ice-cream.

What happens if...

- *... you change the number of flavours?*
- *... you can have a 2-, 4-, 5-, ... header?*
- *... you can repeat flavours?*

From here the problem becomes a case of *Choose your own investigation* - in the same way that a professional mathematician can define the direction of their own investigation.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Magic Cube

Many school courses explore magic squares - but magic cubes, being 3D, are arguably much more interesting. With the hints on the card, it is relatively easy to build the magic cube required. But what to do with it? How to extend it?

One way is through the 'magic' number of this cube which is 42. It seems to occur in many interesting ways throughout history. Even a web search using these digits turns up web sites devoted to such occurrences.

Lewis Carroll, the famed author of Alice in Wonderland used 42 in many ways throughout the book. Perhaps that is because Carroll was a pseudonym for the mathematician Charles Lutwidge Dodgson. For example, discounting the cover, there are 42 illustrations in Alice. A web search will yield masses of other information.

One advantage of following this path is that a real mathematician is shown to be a person 'with another life', and in this case, a life which in some ways is more well known than his mathematical career. Through Carroll you could link to Mathematics and Literature in a cross-curriculum study.

Another direction to extend the original task is simply to ask:

Is this the only solution?

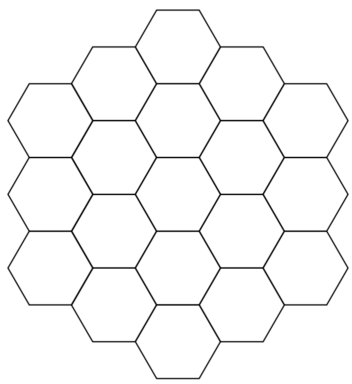
Now there are no clues to guide the construction of a new magic cube, so the students are much more in the position of the mathematicians who developed the puzzle in the first place.

In fact, there are four distinct solutions of the puzzle and twenty-four reflections and rotations of each of these. Starting point considerations in looking for a new solution are:

- Break the problem into smaller parts (the three layers).
- There must at least be 9 sets of three numbers which add to 42 to make the nine distinct rows required.
- Find ways of combining the digits 1 - 27 to make these 9 rows.
- Keep the rows fixed but shift digits within the same row so they appear in different columns.
- Look for a way to do this in each layer so the rows *and* columns now add to 42.

Another direction is a link to Algebra. Could the 42 have been predicted? If we add the numbers from 1 to 27 we get 378. And then $378 \div 9 = 42$. So what might be the magic total for a Size 4 cube, or a Size 5 cube, or a Size n cube? This leads to a very interesting algebraic investigation.

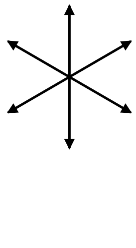
Magic Hexagon



This is probably the most difficult of the 'magic' tasks in this kit. To begin the problem, the way of thinking is the same as for the others.

- What it the grand total of the available digits?
- Looking in just one direction how many ways does this total have to be shared?

So, in this case, the digits 1 - 19 sum to 190 and they have to be shared between 5 vertical columns of the hexagon. Therefore each column has to sum to 38.



However, the difference between this and the other problems is that there are not the same number of addends in each partial sum. In some columns there are three; in some four; in some five. The three directions involved in the puzzle are also more spatially unusual.

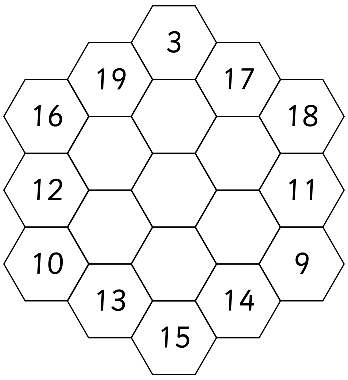
As far as is known there is only one solution to this problem - although there are several symmetry transformations of this solution. So the value of this task is the application of problem solving strategies that it requires, rather than the extensions which might be developed. For example, breaking the problem into smaller parts shows that there is an outer ring of six sets of three-digit numbers which sum to 38. Further, each set overlaps with the one clockwise and anti-clockwise from it at one number. This insight leads to applying the 'try every possibility' strategy to list all the sets of three numbers (from 1 - 19) which sum to 38.

- 38 = 1 + 37 = 1 + 19 + 18
- 38 = 2 + 36 = 2 + 19 + 17
  (1 + 18 + 18 is not allowed - digits are not repeated)
- 38 = 3 + 35 = 3 + 19 + 16 = 3 + 18 + 17
- 38 = 4 + 34 = 4 + 19 + 15 = 4 + 18 + 16
- 38 = ...

Eventually, all sets of all triples which sum to 38 will be found. Then by:

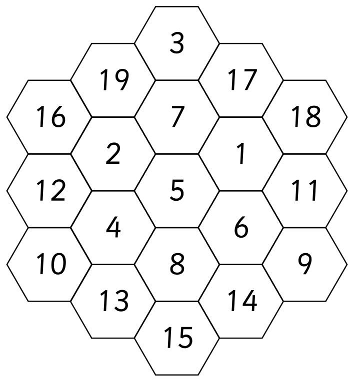
- selecting sets of six triples, and
- shuffling numbers within triples so that the 'tail' of one triple is the same as the 'head' of the next

the outer ring can be found. This is far easier, and in some ways more rewarding, to do with the numbered discs than it is by pencil and paper.



Now the problem looks a little easier. The numbers remaining are 1, 2, 4, 5, 6, 7, 8 and it might be worth trying the middle number in this set in the middle of the shape.

Eventually, with as little hint as possible, students will discover the solution.



One further investigation relates to seeing this diagram as a 'nest' of potential magic hexagons.

- The central single hexagon would be Size 1. Nothing very magic about that.
- The Size 3 (named by its longest chain of unit hexagons) is the blank section in the above diagram when the outer ring is filled in. Would the numbers 1 - 7 make a magic hexagon in the Size 3? It only takes a moment to realise that the sum of these digits is 28 which can't be divided by 3 to produce integer results.

So Size 1 and Size 3 cannot be magic, but Size 5 *is* magic.

- Is there another Size of nested hexagons which might form a magic hexagon?
- If there is, can the magic hexagon be made?
- If we can't find another possibility, can it be proven that there are none?

Investigating these questions involves working out:

- size number
- number of unit hexagons - which determines the upper limit of the consecutive numbers starting from 1

- finding the total of the sequence 1 to ...
- deciding the number of columns the total is shared between
- checking whether dividing by this number gives a whole number answer

The following pattern starts the thinking, but the general case is a substantial challenge.

Size (S)	**1**	**3**	**5**	**7**	**9**	**11**	**...**
Hexagons (H)	1	7	19	37	61	...	...
=	1	1 + 1 x 6	1 + 3 x 6	1 + 6 x 6	1 + 10 x 6		

The number of sixes used is a Triangle Number. Once this is realised there is still the challenge of linking the Size number to the appropriate Triangle Number.

Size (S)	**1**	**3**	**5**	**7**	**9**	**11**	**...**
Hexagons (H)	1	7	19	37	61	...	...
=	1	1 + 1 x 6	1 + 3 x 6	1 + 6 x 6	1 + 10 x 6		
Triangle # (T)		1 + 6T(1)	1 + 6T(2)	1 + 6T(3)	1 + 6T(4)		
Connect H & S?							

Once H and S are connected, the upper limit of the list of consecutive numbers can be found, and the sum of those numbers can then be derived. But how is the Size related to the number of columns between which this total is shared?

Size (S)	**1**	**3**	**5**	**7**	**9**	**11**	**...**
Columns (C)	1	3	5	7	9	...	...

Easy enough, but now, if a general formula is being developed to link the Column Share to the Size, it has to be tested in each case to see if a whole number results. If it does, there is still the question of whether the magic hexagon of that size can actually be made.

There are possibilities in this extension for:

- pattern work with numbers in a table
- symbolic algebra to explore the general case
- spreadsheets to explore the possibilities empirically

Magic Squares

This is an old favourite task which the students are sure to have met previously. Often though the task is presented with the Magic Total given. Part of this problem is to use the same sort of thinking as in Magic Cube and Magic Hexagon to decide the Magic Sum. Digits 1 - 9 total 45 and this is to be shared evenly between three columns. Similarly for the 4 x 4 case.

The problem now becomes a challenge to apply strategies.

- Break the problem into smaller parts by first getting the rows to work.
- Keep the numbers in these rows, but shift within rows to get the columns to work.
- Shift whole columns or rows to get the diagonals to work.

There are eight ‘different’ solutions, which all turn out to be rotations and reflections of each other. Realising this adds a spatial symmetry aspect to the task.

The Contra Challenge would be to arrange the digits so that NO row, column, or diagonal had the same total, ie: they were all different.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Make A Snake

The visual/tactile nature of the task is important, but students may find the threading and unthreading of the string a little frustrating. If so, suggest the beads are simply stood on the table - the top and bottom of each bead is flat. If you have Poly Plug the yellow/blue plugs are perfect to represent the growing snakes.

Find more information about Poly Plug at:

- http://www.mathematicscentre.com/taskcentre/polyplug.htm

Mathematically, the task leads to a pattern involving powers.

		As	Bs	Tot.
Birth	A	1	0	**1**
Season 1	ABA	2	1	**3**
Season 2	ABA/ B/ ABA	4	3	**7**
Season 3	ABA/ B/ ABA/ B/ ABA/ B/ ABA	8	7	**15**
Season 4	ABA/ B/ ABA/ B/ ABA/ B/ ABA/ B/ ABA/ B/ ABA/ B/ ABA/ B/ ABA	16	15	**31**

So, after 10 seasons the number of As is $2^{10} = 1024$ and the number of Bs is one less than the number of As. At this age it is appropriate to ask for an explanation of why this pattern works. Students would need to express that:

- each season the number of As is double by the rule
- the rule introduces a new B for each old A, so if the number of As doubles, so will the number of Bs, but it is one less because there isn't a B colour to start with.

Students could also be expected to graph the data (on paper or with a spreadsheet or calculator) and to make predictions based on the reverse questions. For example, if a biologist found a skin with ... As, how old (in seasons) was the snake which shed it?

The second part of the card asks students to make up their own pattern for shedding and investigate the results. Some examples to suggest are:

- Each A is replaced by an AB and each B is replaced by a BA.

- Each A is replaced by a BAB AND when two bands of the same colour occur together they become just one (wider) band.
- Each A is replaced by ABA and each B is replaced by BAB.

There are many other possibilities. Teachers may also wish to challenge students to find a rule which produces a given result, eg: a rule which results in a total of As and Bs which trebles.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Making Fractions 1

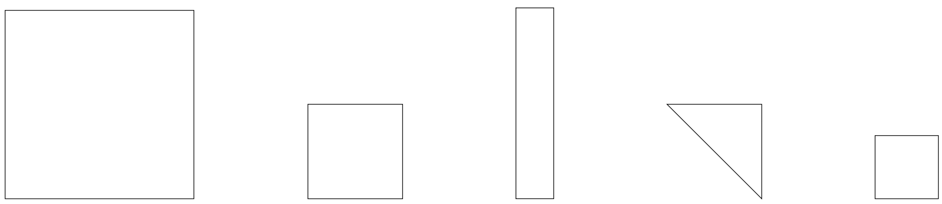
Students may have met **Making Fractions 2** and **Making Fractions 3** in their younger days, in fact, both are included in the equivalent **Maths With Attitude** kit for Years 7 & 8. However, **Making Fractions 1** is harder than these because the relationships between some of the pieces is not whole. That is, instead of reasoning such as *One of these is four of those*, it now involves reasoning like *Two of these is three of those, so one of them must be...*

The task is provided as an opportunity to refresh and extend fraction understanding in a practical manner. Students appreciate being able to 'measure' and compare to make their decisions, rather than merely manipulating symbols. One key concept embodied by the task is that any piece, not just the largest, can be the whole.

Note that the intention for the top cells of each column on the card is drawings, or a code such as LS for larger square. The pieces are not intended to fit in these cells. However, some students do stand them on their edge outside the cells.

An additional spatial challenge is provided by trying to fit the pieces back into the frame at pack up time. This should be encouraged because it is a check that all the pieces are there.

In order, the sizes of the pieces are:



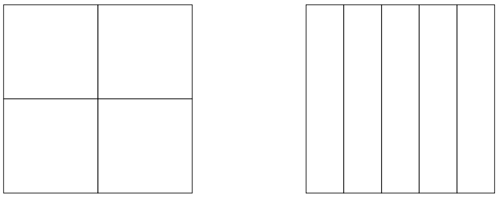
The first row of the table is then straightforward:

Row 1	1	$^1/_4$	$^1/_5$	$^1/_8$	$^1/_9$

but things get a little trickier after that:

Row 1	1	$^1/_4$	$^1/_5$	$^1/_8$	$^1/_9$
Row 2	4	1	$^?/_?$		

For instance, how does the rectangle relate to the second square? The pieces allow the following comparison to be made, but there is still room for discussion in deciding the value of the rectangle, given a single square is worth 1.



And a mathematician would want to ask *Can I check this another way?*. In fact, in how many different ways can we reason that the rectangle is worth $^4/_5$ if the single square is worth 1?

Continuing the reasoning by reference to the pieces (and who would want to do it without them?) the table can be completed. In the process there are many times when the size of one fraction has to be compared to the size of another, eg; which is bigger, $^1/_5$ or $^1/_4$.

> *Darren invented a rule to answer these questions. He multiplied the bottom line of one fraction by the top line of the other (cross multiplied) and whichever side the larger product came out on, that was the larger fraction.*

So using Darren's Rule, $^1/_5$ compares to $^1/_4$ in the same way as 4 compares to 5, so $^1/_4$ is larger.

Row 1	1	$^1/_4$	$^1/_5$	$^1/_8$	$^1/_9$
Row 2	4	1	$^4/_5$	$^1/_2$	$^4/_9$
Row 3	5	$^5/_4$	1	$^5/_8$	$^5/_9$
Row 4	8	2	$^8/_5$	1	$^8/_9$
Row 5	9	$^9/_4$	$^9/_5$	$^9/_8$	1

Students are asked to look for patterns in the table and hopefully they will see the reflective symmetry of reciprocals across the diagonal of 1s.

The last part of the card introduces decimals by changing the value of 1 to $1.

> *For some students, using money eg: If this piece costs $1, how much should the other pieces cost?, made the task more accessible.*

Some of the values will now be in fractions of cents and the students will need to decide whether they would be left in that form on the grounds of selling large quantities, or rounded off. This is an important consideration in exchange rates, for example, which run to six decimal places, even though dollars and cents only run to two.

If rounding off is used, the table becomes:

$1	25¢	20¢	13¢	11¢
$4	$1	80¢	50¢	44¢
$5	$1.25	$1	63¢	56¢
$8	$2	$1.60	$1	89¢
$9	$2.25	$1.80	$1.13	$1

How are reciprocals to be interpreted now?

Extension

Although the card doesn't suggest it, there is considerable opportunity here for the students to create fraction equations. They must first choose one piece as the whole, then find the parts of it that are represented by the other blocks. This is equivalent to focusing on just one row of the table. Encourage students to record as many equations as they can for that one row and ask them to justify some. Their responses should provide some assessment information.

Monkeys & Bananas

The task has a relatively complex, yet engaging, story shell and one closed question. Students can work symbolically if they wish, but the hint of fractions in the problem means that, at some point, most choose the concrete materials for security. There is also room to approach the problem through algebraic reasoning or spreadsheets. The following approach highlights thinking which uses fractions in a conceptual way. However, this is not the only approach and some students avoid thinking about the fractions altogether by thinking in terms of multiplication; an approach that is equally valid.

Working backwards and trying possibilities: In the morning, the number of bananas must be a multiple of 3, so try 3 as the first possibility.

- If 3, then the third monkey, after eating 1, was looking at a pile of bananas that could be divided into three equal parts. One of those parts she hid. So two of the parts became the 3 that might have been left in the morning.
- But if two parts is 3, then one part is 1.5 and the pile she was looking at must be 4.5 in total.
- This isn't allowed because the monkeys were dealing with whole bananas.
- The next possible 'morning number' is 6.
- If 6, then the third monkey, after eating 1, was looking at a pile of bananas two equal parts of which was 6.
- But if two parts is 6, then one part is 3 and the pile must be 9 in total.
- This is allowed so, counting the third monkey's night snack, the pile she saw when waking up must have been 10.
- But this third monkey number (10) came from the second monkey's night time adventure.

- If 10, then the second monkey, after eating 1, was looking at a pile of bananas two equal parts of which was 10.
- But if two parts is 10, then one part is 5 and the pile must be 15 in total.
- This is allowed so, counting the second monkey's night snack, the pile he saw when waking up must have been 16.
- But this second monkey number (16) came from the first monkey's night time adventure.
- If 16, then the first monkey, after eating 1, was looking at a pile of bananas two equal parts of which was 16.
- But if two parts is 16, then one part is 8 and the pile must be 24 in total.
- This is allowed so, counting the first monkey's night snack, the pile she saw when waking up must have been 25.
- Therefore the monkeys must have collected 25 bananas during the day.

This answers the closed question on the card, but young mathematicians should be encouraged to ask questions which extend the problem.

In this case, we might ask:

- Suppose the 'morning number' was 9. Could we find a solution?
- Are there other 'morning numbers' which provide solutions?
- Can we investigate the number of hidden bananas?
- Suppose there were four, five, six... monkeys who behaved in the same way. Do the key numbers (fourth monkey number, third monkey number...) form a pattern?
- Suppose the three monkeys hid a fourth of the pile each time after eating one and were still able to share equally in the morning. Does this problem have a solution?
- Suppose there were four, five, six... monkeys who behaved in the same way. Do the key numbers (fourth monkey number, third monkey number...) form a pattern?

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Number Tiles

Students have possibly met this task before, and finding a few solutions won't be too hard. The reason for including it at this age is that it begs the questions:

- *How many solutions are there?*
- *How do we know when we have found them all?*

The highest possible level of proof can be expected at this age and when hypotheses develop about connections between the solutions, symbolic algebra can be used to test those hypotheses. For example, one hypothesis which shows up in the presence of sufficient data is that it appears there must be ‘carrying’ somewhere in the solution. Can the students show that must be true by using deductive reasoning based on the form of the problem, ie:

$$\begin{array}{rl} & abc \\ + & \underline{def} \\ & \underline{ghi} \end{array}$$

If we assume there is no carrying, then the following equations must be true:

- c + f = i
- b + e = h
- a + d = g

Substitute these into the fact that the 9 digits add to 45, ie:

- a + b + c + d + e + f + g + h + i = 45

and the students see a logical contradiction, hence disproving the assumption of no carrying.

Similarly if we assume carrying in particular places and create new equations, the results beautifully show other observations such as the digits in the answer line seem to always add to 18.

Determining and solving the set of equations buried in this representation and recognising the need for the additional equation (sum of the digits) which is unstated in the problem, is as challenging as any text book work on simultaneous equations.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Pizza Toppings

Like **Ice-Cream Flavours** this task card is a perfect tip of the iceberg. In fact, it is really a step up in difficulty because now there are four 'flavours', and the order in which they are used doesn't matter. The question *How do you know when you have found all of them?*, almost 'requires' students to organise their data. This involves plenty of discussion, and recording the results for someone else to understand begs the use of symbolic notation.

Powers of 2, Pascal's Triangle and Binomial Coefficients are part of the iceberg of this task.

Square Pairs

This task is the genuine work of mathematicians. The lesson notes on Maths300 include reference to an article by Johnston Anderson & Andy Walker, University of Nottingham, who have not only explored this problem as part of their work, but have noted that it has aspects which are appropriate for young mathematicians of this level. Not the least of these is the concept and construction of proof. The task offers an open-ended starting point for the problem.

Steps

As with some other tasks in this kit, students may have experienced this one before. That is good. A rich problem can be revisited as the catalyst for deeper learning as the students' mathematical maturity develops. At this level we do want students to find some particular solutions to this problem, but the deeper challenge is to find all the solutions and to prove that all the solutions have been found. The card hints at a way to do this, by suggesting that the students focus on the possible line totals. There are extensive notes (which include a link to algebra) in the Maths300 lesson.

Take Away Tiles

This task is a member of a family completed with Fay's Nines, Number Tiles and Steps. Each can be tackled at the level of finding some solutions, or at the level of finding all solutions, or at the level of proving that all solutions have been found. The task has connections to place value, and the connections between addition and subtraction. The search for all 96 solutions benefits greatly from application of various strategies. There are extensive notes in the lesson which is also supported by companion software.

Training For Maths

This apparently simple problem is rich in pattern and combination theory. However the patterns will only be revealed to careful 'train makers'. Without *all* the possible solutions to each size of train, patterns are less likely to be discovered.

The answer to the first problem on the card is thirteen possible 6-unit trains, which are:

- -111111 — 1 train with 0 x Size 2 carriages.
- -21111, -12111, -11211, -11121, -11112 — 5 trains with 1 x Size 2 carriage.
- -2211, -2121, -2112, -1221, -1212, -1122 — 6 trains with 2 x Size 2 carriages.
- -222 — 1 train with 3 x Size 2 carriages.

(The - indicates the link to the engine.)

The second problem on the card is an open exploration. If students organise their search into different size trains and then record the number of trains with each possible number of Size 2 carriages, their data will begin to grow something like this:

Train Lengths / No. of 2s	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9
2				1	3	6	10	15	21	28
3						1	4	10	20	35
4								1	5	15
5										1
6										

As the students make more trains and gather more data, encourage them to move away from actually making to using the patterns in the table to hypothesise extensions to the table. The mathematicians' question:

- Can I check this another way?

can be applied to check these hypotheses as soon as the students have enough data to see that there are several patterns in the table. For example:

- Familiar patterns such as the natural numbers (Row 1) and the triangle numbers (Row 2).
- Starting with 1 in Row 1, the next term in each row is formed by adding on the number one cell up and one left.
- The terms of the diagonals which fall downwards to the right are the rows of Pascal's Triangle, ie: 1-1, 1-2-1, 1-3-3-1, 1-4-6-4-1, etc.

Now any cell can be checked in at least three ways.

So that the connection with train making isn't lost in playing with patterns, students should also be challenged, by reference to the trains, to show why some of the cells must be the number shown. To do so may involve making a list of all possibilities, but it could also involve reasoning such as the following which justifies the number 15 for four double carriages in a 10 carriage train.

- The two single carriages could act as a double, in which case they could be in positions (1, 2) / (3, 4) / (5, 6) / (7, 8) / (9, 10) ... 5 possibilities.
- When acting as a double they could not be in positions such as (2, 3) because there would only be one carriage space to the left and the other actual Size 2 carriages couldn't fit in that space.
- If acting 'individually', one carriage could be fixed in Position 1. The possible positions are then (1, 4) / (1, 6) / (1, 8) / (1, 10) ... 4 possibilities.
- These are the only possibilities with Size 2 carriages between.
- If acting 'individually', one carriage could be fixed in Position 3. Why not Position 2? The possible positions are then (3, 6) / (3, 8) / (3, 10) ... 3 possibilities.
- If acting 'individually', one carriage could be fixed in Position 5. The possible positions are then (5, 8) / (5, 10) ... 2 possibilities.
- If acting 'individually', one carriage could be fixed in Position 7. The possible positions are then (7, 10) ... 1 possibility.
- A total of 15 possibilities as predicted.

A different (but related) way to investigate this problem is to look at the number of carriages in the train without distinguishing between single or double carriages. So the solutions for the 6-unit trains are:

- -111111 — 1 train with 6 carriages.
- -21111, -12111, -11211, -11121, -11112 — 5 trains with 5 carriages.
- -2211, -2121, -2112, -1221, -1212, -1122 — 6 trains with 4 carriages.
- -222 — 1 train with 3 carriages.

Extending this search produces a data table which shows the rows of Pascal's Triangle more obviously (see next page):

Train Lengths Carriages	1	2	3	4	5	6	7	8	9	10
1	1	1								
2		1	2	1						
3			1	3	3	1				
4				1	4	6	4	1		
5					1	5	10	10	5	1
6						1	6	15	20	15
7							1	7	21	35

Again, there is a connection between any number (other than a 1) and numbers in the row above. Each number is the sum of the number one above and one to the left, and the number one above and two to the left. But why??

Consider, for example, the 6 in the 4 carriage row.

- It is addition of the two 3s in the rows above.
- These 3s represent the number of 4 and 5 length trains with 3 carriages, ie:
  4 length trains with 3 carriages: -211, -121, -112
  5 length trains with 3 carriages: - 221, -212, -122

In how many ways can 6 length trains with 4 carriages be made from these?

- 6 length trains with 4 carriages can be made from 4 length trains with 3 carriages only by adding one Size 2 carriage, ie: -*2*211, -*2*121, -*2*112.
- 6 length trains with 4 carriages can be made from 5 length trains with 3 carriages only by adding one Size 1 carriage, ie: -*1*221, -*1*212, -*1*122.

Since there are no other trains in the row above which can be converted into 6 length trains with 4 carriages, we now know why the 6/4 train is the sum of the 4/3 and 5/3 trains in the way described previously.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Truth Tiles 2

This problem seems trivial at first, but hidden depths appear on closer inspection. Once one solution is found, we already have four solutions, since some numbers may be transposed. For example, 3 + 6 - 4 = 5 also gives:

- 3 + 6 - 5 = 4
- 4 + 5 - 3 = 6
- 4 + 5 - 6 = 3

If you also swap the order of the first two numbers in each case there are double the number of solutions. It is at this stage that you begin to realise that these are all versions of:

- 3 + 6 = 4 + 5

Equivalence of two operations and the commutative law does the rest.

It is the search for all solutions which unlocks the iceberg of this task. Students have to collect some initial solutions, see patterns in these, use those patterns to form hypotheses, then apply problem solving strategies to prove or disprove their theories (hypotheses) and finally communicate their findings.

The following comments from teachers indicate the interest the task has developed in some classes.

Concrete materials

> *I used to use this task as a pencil and paper item, but the simple addition of having tiles to move around (or ripped up pieces of paper) made the task more accessible to my students.*

Mixed ability

> *I also like the task because it is so open-ended. All my students can find some solutions and achieve a level of success, but the option of adding more tiles increases the depth of the task significantly and can challenge my very capable students.*

Student generated investigations

> *One student noticed that 3, 4, 5, 6, 7 was made up of 3 odd numbers and 2 evens. Would it make a difference if the sequence was 2, 3, 4, 5, 6 being 3 evens and 2 odds? This led to some fascinating arguments. He was initially convinced that it did make a difference and that different combinations were now possible - but finally proved that it made no difference and that there were still 24 solutions. The important aspect of this was that the student saw the possibility and created his own investigation.*
>
> *Another student noticed that if you read any solution backwards it still worked. For example, 3 + 6 - 5 = 4. Writing this backwards gives 4 + 5 - 6 = 3. Why this observation always works became a small investigation.*
>
> *Yet another student noticed subtractions such as 6 - 5 = 4 - 3 with both sides having the same difference. How many ways can this be done? Is it a totally different investigation or are they similar? (It turns out they are identical, but it took some time for the student to realise this and to understand why!)*

Returning to the task card, there are either 12 solutions, or 24, depending on the definition of 'different'. If students feel that 3 + 6 - 5 = 4 and 6 + 3 - 5 = 4 are different, then this doubles the number of solutions. The 12 basic solutions are:

- 3 + 6 - 4 = 5
- 3 + 6 - 5 = 4
- 4 + 5 - 3 = 6
- 4 + 5 - 6 = 3
- 3 + 7 - 4 = 6
- 3 + 7 - 6 = 4
- 4 + 6 - 3 = 7
- 4 + 6 - 7 = 3
- 4 + 7 - 5 = 6
- 4 + 7 - 6 = 5
- 5 + 6 - 4 = 7
- 5 + 6 - 7 = 4

Several strategies can be used to find these and often students start with a simple idea such as testing every possible combination, and then adapt as they notice patterns.

Strategy 1

Put every digit in turn in the first spot and then try all the other numbers there in turn.

For example, start with 7 first.

- 7 + 3 - 4 = 6 and 7 + 3 - 6 = 4
- 7 + 4 - 5 = 6 and 7 + 4 - 6 = 5
- 7 + 5 gives no solutions nor does 7 + 6

Now put 6 in the first position and so on.

This strategy is testing every possible combination, but the pattern of pairs becomes obvious after a while leading to efficiencies in the search.

Strategy 2

Notice that the answers form groups. For example, 5 + 6 - 4 = 7 are based on a total of 11. The smallest group total is 9 (3 + 6 - 4 = 5), and the highest is 11 (4 + 7 - 5 = 6).

There are 3 groups in all:

Group Total	Group	Solutions
9	3, 6 with 4, 5	3 + 6 - 4 = 5 3 + 6 - 5 = 4 4 + 5 - 3 = 6 4 + 5 - 6 = 3 6 + 3 - 4 = 5 6 + 3 - 5 = 4 5 + 4 - 3 = 6 5 + 4 - 6 = 3
10	3, 7 with 4, 6	...
11	4, 7 with 5, 6	...

and each group has 8 solutions. So 3 lots of 8 = 24 altogether (but should they all be considered different).

This strategy is more sophisticated than Strategy 1. It shows a more visible understanding of the patterns and the commutative property. It also has the potential to be generalised.

Strategy 3

Leave out the 7 to start with and just use the four smallest numbers 3, 4, 5, 6. This gives four solutions. Then include the 7 which means leaving out one of the others in turn.

This reasoning leads to:

- 4, 5, 6, 7 (no 3): 4 solutions
- 3, 5, 6, 7 (no 4): no solutions
- 3, 4, 6, 7 (no 5): 4 solutions
- 3, 4, 5, 7 (no 6): no solutions

So, 12 solutions altogether.

> *This strategy shows the power of breaking a large problem into smaller more manageable parts. It also has the potential to be generalised, but does not provide evidence as to why some groups give no solutions. However it clearly shows a systematic method which tests every possible combination.*

Strategy 4

Transform the problem to:

□ + □ = □ + □

and recognise the consecutive numbers in the problem can be represented as:
x, $x + 1$, $x + 2$, $x + 3$, $x + 4$
where $x = 3$.

Then it may be clearer to see that some pairings have the same value and some don't. For example:

- $x + \underline{x+1} = 2x + 1$, which cannot also be made with any two remaining ones.
- $x + \underline{x+2} = 2x + 2$, which cannot also be made with any two remaining ones.
- $x + \underline{x+3} = 2x + 3$, which can also be made with $x + 1$ and $x + 2$.
- $x + \underline{x+4} = 2x + 4$, which can also be made with $x + 1$ and $x + 3$.

Continuing with this reasoning will find all solutions.

> *This is the most sophisticated strategy, and, in fact, subsumes all the others. It is completely generalisable and shows that there will be 12 (or 24) solutions regardless of the lowest number in the set. The underlying structure of the puzzle - ie: its dependence on equivalent expressions and consecutive numbers - is revealed. So too is its connection with the task Window Frames and the lesson Consecutive Sums.*

Extensions

1. What happens if the tiles run from 3 to 8:
   How many solutions are there?
   How do you know when you have found them all?
2. What happens if the tiles run from 3 to 9:
   How many solutions are there?
   How do you know when you have found them all?
3. Are there any patterns in the number of solutions for 3 to 6 ...to 7...to 8 ...to 9 ...

4. Convert the problem to ☐ x ☐ ÷ ☐ = ☐ and the numbers to 8, 16, 32, 64.
   How many solutions are there?
   How do you know when you have found them all?
5. How about if the numbers are 8, 16, 32, 64, 128?
6. Write these numbers as powers of 2 and investigate the connections between this form of the problem and the original form.

Find more information about this task in the Task Cameo Library at:
- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Window Frames

The interconnectedness of the natural numbers is highlighted in this task. All other natural numbers are formed from the number 1. The task invites a search for pattern and generalisation among this infinite set. The plastic window is a device which enables students to focus on a small part of the set, then explore whether any discoveries also occur in other equivalent parts. If they do, then students are challenged to form a generalisation, or hypothesis.

The next step is to prove the hypothesis, which may be with verbal or written reasoning or with algebraic symbols. For example, question 2 on the card:

12	13	14

leads to the hypothesis that the sum of the end numbers is twice the middle number.

Verbal reasoning which could prove this hypothesis is:
- The left end is one less than the middle. The right end is one more than the middle. When these are added, the one less plus the one more has zero total effect and the middle number occurs twice.

Algebraic reasoning which could prove this hypothesis is:

a	a+1	a+2

- Sum of the ends = $a + a + 2 = 2a + 2 = 2\ (a + 1)$ = twice the middle number

As with many other tasks and lessons at this level, the emphasis is on proof. Each window guides the formation of generalisations which then have to be proved. Students might also explore broader patterns in the chart, or convert each number to its digital root, then see what patterns can be found.

Note: Digital sum is the sum of the digits of a number. If this is not a single digit, the digits are added again (and again...) until a single digit does result. This single digit is the digital root of the original number. For example:

Digital sum of 74 = 11, the digits of which are added again to give 2 as the digital root. This means that 74 and 11 have the same digital root.

Another possible exploration of the number chart is to divide each number by a chosen number, say 3 and write only the remainder in the cell. What patterns result?

Students might also be challenged with extension problems like:

- Suppose cubes were used to build towers on the chart which were as big as each number (ie: the number 87 was replaced with a cube tower 87 cubes high), how many cubes would be needed?
- Place your pencil down along any line on the grid. Start at the smallest number and plot it on a graph to show how high it would be in cubes. Repeat for each number along your pencil. What is the equation to the line which joins the tops of the cube towers?

Students can also be asked to apply their windows to a times tables chart to look for and prove generalisations.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Lesson Comments

- These comments introduce you to each Maths300 lesson. The complete plan is easily accessed through the lesson library available to members at:
  http://www.maths300.com
  where they are listed alphabetically by lesson name.

Billiard Ball Bounces

Students have a rectangular 'billiard ball' table of size, say, 10 by 6. There are 'pockets' in all 4 corners, but not in the centres of the longer sides. A ball is hit from the lower left corner at 45 degrees.

- *How many bounces will it make until it goes into a pocket?*
- *And into which pocket will it go?*

Drawing the pattern on grid paper answers these specific questions, but:

- *What if the size of the table is changed to 10 by 7, or 8 by 5, or 100 by 47?*
- *Is it possible to predict in advance the number of bounces and final pocket?*

Exploring this question converts the activity into an extended investigation involving many number patterns. A computer simulation allows theories to be developed and tested. The lesson also nicely connects geometry patterns, symmetry, number theory (factors and ratios) and also algebra (the generalisation of the patterns). This is one of several lessons where geometric patterns lead to algebraic investigations. Others in this kit are:

- Lesson 51, *Hunting For Stars*
- Lesson 54, *Cracked Tiles*

Bob's Buttons

Students may have met this lesson in earlier years. They may even have been involved physically in a game of making groups based on the total number in the class today. This 'Clumps' game leads into the double condition problem which is the central focus of the lesson. On a signal from the teacher the children form into groups of a stated size and the number of groups and the number of ungrouped children is recorded. This is played several times then the teacher poses the problem which involves working backwards from two pieces of groups/remainder data to find the original number in the class.

The problem is easily understandable when students are physically involved in setting it up, so, even at this level, some teachers think it important to begin in this way. Once the initial problem is solved, there is an extensive set of investigations which can be developed, and this depth is supported by two Investigation Sheets. As illustrated by an item in the Classroom Contributions, although the problem contains content which is not much more difficult than dividing to find remainders, it can be extended to a level of proof which challenges even the most experienced students at this level.

Consecutive Sums

Understanding the structure of the number system is more than being able to use it. Do students see it as an infinite set of disconnected entities or are they starting to perceive that relationships between numbers are a consequence of basic laws which underpin the system? This lesson provides challenges which highlight the interconnections and lead into symbolic proof. At this level, the concept of proof is the main thrust of the lesson. For most students, the arithmetic demands of the problem do not get in the way of the reasoning required to 'prove' their discoveries.

Cracked Tiles

This is another investigation linking number, space and algebra. Again, the emphasis at this level is on being able to hypothesise about, and prove results for, every case. The fantasy story concerns a builder who has just laid square tiles to cover a rectangular floor. The electrician arrives and declares that a cable must be laid diagonally across the floor so some of the tiles will have to be cracked and torn up. Of course the builder wants to know how many tiles, and the deeper investigation is:

> Given any size rectangular room, can you predict the number of tiles which have to be replaced?

Crosses

This is one of a set of number puzzles, which also includes *Fay's Nines*, *Magic Squares*, *Number Tiles*, *Steps* and *Take Away Tiles*, which are easy to state, easy to start and contain heaps of maths. The only equipment needed is paper to tear up and the demands of the mathematical skill in each is sufficiently gentle to allow the focus to be on the concept, process and strategies of proof.

In *Crosses* the digits 1 to 9 are arranged in a cross (plus sign) so that the sum of the digits on one arm equals the sum of the digits on the other.

- How many solutions are there?
- How do I know when I have found them all?

Software contributes the strategy of try every possible case as one way to approach this latter question.

Doctor Dart

In this puzzle the focus is not so much on how many solutions there are, as it is on why, given all the possible solutions, the Evil Professor might have chosen this particular solution as the one Doctor Dart needed to find in order to save the world. The initial problem is usually solved by the class in 10 - 15 minutes. Then, the door which Doctor Dart metaphorically opens leads to a universe of extensions which are supported by the companion software. One content specific outcome of the lesson is the power of a tree diagram as a mathematical tool in situations where there are choices. The lesson is also underpinned by consecutive numbers and this can lead to an algebra link.

Fay's Nines

A simply stated problem unfolding a world of mathematics adaptable to a wide range of students. The problem is simply to make three 3 digit numbers using the digits 1 through 9 and make them add to 999.

There are many solutions. As a puzzle, finding one or even several solutions provides plenty of basic skill practice. However, the challenge of finding and justifying all solutions (180 in all) lifts the puzzle into an extended investigation and shifts it from arithmetic to concepts of strategy and proof. The problem follows a classic investigation pathway. As solutions are found, patterns can be noticed, these lead to theories and conjectures and strategies to unlock all of the solutions. It is an ideal problem for exposing the Working Mathematically process and it's solution can be supported by the companion software.

Four & Twenty Blackbirds

The story shell surrounding the lesson is that after the pie was open, the Queen decided that she liked the blackbirds' singing so much that she built feeding platforms for them in the Royal Garden. The platforms were built at the four corners and the four mid-points of the rectangular Royal Garden. Each morning when the Queen came to listen to the singing, she counted 24 blackbirds. She also noticed that they always arranged themselves with a total of 9 along each edge of the courtyard. The lesson involves working out all the possible ways the blackbirds could be arranged, and in the process illustrates all the aspects of Working Mathematically. It is supported by an Investigation Guide and a piece of software contributed by a teacher.

Hunting For Stars

Eight students are sitting in a circle. The first holds the loose end of a large ball of wool and they pass the ball on to every third person (an 'Add 3' rule) around the circle. The trail of wool makes an interesting geometric pattern.

- *But what if the rule is changed to 'Add 4', or 'Add 2'. What happens to the pattern?*
- *And what if we change the number of students in the circle to 6, or 11, or any other number?*

Exploring these questions converts the activity into an extended investigation involving many number patterns. The openness of the investigation invites many levels of challenge - however the main one is the generalisation:

- *Predict the pattern for any number of students (N) and any rule (Add Y).*

A computer simulation allows theories about the shapes and the relationship between N and Y to be tested.

Ice-Cream Flavours

Mrs. Smith owns an ice-cream shop and as the summer rush ends she realises there are too many containers of three flavours - strawberry, banana and vanilla. She decides to run a special on triple headers for people who buy one of each flavour. Students first explore the number of different triple headers that could be ordered, then, by asking *What happens if...?* investigate several other challenges.

The overall challenge in the lesson is:

Given

- *any number of flavours*
- *any number of scoops*
- *and whether or not repeats are allowed*

predict the number of ice-creams that could be ordered.

The software helps to explore this challenge and the underlying Multiplication Principle is exposed. There is strong focus in this lesson on gathering assessment information using a variety of assessment tools.

Magic Cube

Magic Cubes may be less well known than Magic Squares, but they are arguably much more interesting. There is a long history of exploration of these curiosities since the versions first published in the 1880s. This lesson introduces Magic Cubes through the Size 3 cube, and in particular invites students to see, and hopefully appreciate, the exquisite precision of the language used in the historic general instructions for any size magic cube. Students often respond with a mixture of incredulity and wonder when they have explored the puzzle sufficiently to find meaning in these classic words. In one school we know of the teacher challenged the students to test the clarity of the instructions by following them through to make an 8 x 8 Magic Cube. The team work needed, and the completed cube displayed for all to see, did much to enhance the students' self images and classroom esteem.

Magic Squares

Perhaps students have seen and worked with magic squares before, however, there are plenty of extensions in the lesson which encourage a revisit. Tear a piece of paper into nine tiles and number them 1 through 9. Now arrange them in a 3x3 square so that all rows, columns and the main diagonals sum to the same number. Throughout thousands of years of human history, these Magic Squares have popped up in the recordings of several cultures. The earliest known examples appear to be from China over 3000 years ago. The lesson highlights particular strategies used by mathematicians and is supported by software which assists students to apply the strategy of breaking a problem into more manageable parts. Then, just when it seems the problem may be exhausted, the Classroom Contribution suggests exploring the Contra Challenge, which is to arrange the nine tiles so that none of the rows, columns or main diagonals have the same total.

Monkeys & Bananas

The story shell concerns three monkeys who collect a pile of bananas, then resolve to share them in the morning. However, one by one they get up in the night and not only take a share of what is there, but have a midnight snack as well. In the morning we know they were still able to equally share what was left, but how many bananas were in the original pile?

In the first phase of this problem students frequently 'automatically' come to use the working backwards strategy. That will lead to the solution of the original puzzle, but the search for further solutions and the *What if...?* questions are what lead to deeper algebraic thinking, spreadsheet applications and the hint of exponential growth. At the deeper end of the investigation the algebraic demands are worthy of the most experienced students at this level.

Number Tiles

This is another of the family mentioned earlier. In this case the students use the digits 1 to 9 to make two 3-digit numbers, the addition of which makes the third. The problem contains every aspect of the work of a mathematician and leads to significant use of symbolic proof in a context that students can follow. The lesson is also supported by companion software.

Pizza Toppings

Students are tempted into the problem through the context of making pizzas. The clever counting that is required to find all the combinations of ingredients is non-trivial and extensive. In fact the task has a very deep iceberg which leads to Powers of 2, Pascal's Triangle and Binomial Coefficients. However, the depth of the problem allows students to leave it at several levels and still experience success. As one teacher commented:

> *The context of pizzas, and the concrete materials which allow students to actually 'make' all the possibilities, made the task more enjoyable and accessible to students. I have seen this combination theory mathematics presented in many ways but this was an excellent starting point. It nicely leads on to some challenging extensions.*

Square Pairs

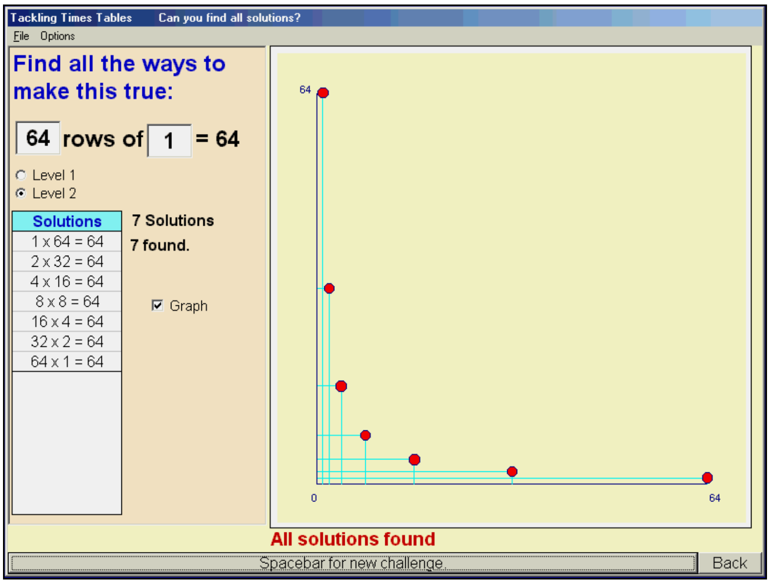
Eighteen students each have a number from 1 to 18. They are asked to pair up so that the sum of each and every pair is a square number. Can it be done? If so, what other lists of even numbers can be 'square paired'? Are there any lists which can't be square paired? The investigation involves all aspects of the work of a mathematician and is supported by software. The final phase of a mathematician's work is to publish results, and one of the advantages of this lesson is its reference to the work of two modern mathematicians who have made this problem their own. This link confirms our attitude that school mathematics curriculum can reflect the work of professional mathematicians, rather than, in the main, practising a small subset of the skills of a mathematician.

Steps

The initial puzzle involves placing the nine digits into a frame so that each straight line of three boxes has the same total. The frame is something like a staircase, so, like the edge of a step, there are cells which are in both the vertical and the horizontal line. Despite its apparent simplicity (a feature which allows a wide range of students to access the problem), it invites application of the Working Mathematically process to find all the solutions and show there are no more. This leads into all sorts of discussions about the relationship between the numbers. Software supports the solution of the problem.

Tackling Times Tables

Hopefully students at this level don't need the conceptual part of this lesson which establishes multiplication through an array model. However, it is there if needed to support particular students. The main reason for making it part of this kit is the link the software makes between times tables and the algebra of an hyperbola. Students would be used to questions like:



> *The answer to the times table is 64, what are the possible (whole number) questions?*

The software opens the door to exploring the fact that all solutions to this question (in fact, whether or not they are whole number solutions) are of the form:

$$xy = 64$$

and that is an example of the equation to an hyperbola.

Take Away Tiles

This digit tile problem uses the digits from 0 to 9. The challenge is to create a 4-digit number from which a 3-digit number can be subtracted in such a way that the three unused tiles can be arranged to form the answer to the subtraction. The only condition is not to have any leading zeros. As students shuffle the tiles to find a solution they are engaged in considerable mental arithmetic. However, the challenge of finding (and proving) all of the solutions turns the puzzle into a substantial investigation. The companion software allows the problem to be solved electronically, but its major attribute is to highlight particular strategies.

Part 3:
Value
Adding

The Poster Problem Clinic

Maths With Attitude kits offer several models for building a Working Mathematically curriculum around tasks. Each kit uses a different model, so across the range of 16 kits, teachers' professional learning continues and students experience variety. The Poster Problem Clinic is an additional model. It can be used to lead students into working with tasks, or it can be used in a briefer form as a opening component of each task session.

> *I was apprehensive about using tasks when it seemed such a different way of working. I felt my children had little or no experience of problem solving and I wanted to prepare them to think more deeply. The Clinic proved a perfect way in.*

Careful thought needs to be given to management in such lessons. One approach to getting the class started on the tasks and giving it a sense of direction and purpose is to start with a whole class problem. Usually this is displayed on a poster that all can see, perhaps in a Maths Corner. Another approach is to print a copy for each person. A Poster Problem Clinic fosters class discussion and thought about problem solving strategies.

Starting the lesson this way also means that just prior to liberating the students into the task session, they are all together to allow the teacher to make any short, general observations about classroom organisation, or to celebrate any problem solving ideas that have arisen.

One teacher describes the session like this:

> *I like starting with a class problem - for just a few minutes - it focuses the class attention, and often allows me to introduce a particular strategy that is new or needs emphasis.*

It only takes a short time to introduce a poster and get some initial ideas going. The class discussion develops a way of thinking. It allows class members to hear, and learn from their peers, about problem solving strategies that work for them.

> *If we don't collectively solve the problem in 5 minutes, I will leave the problem 'hanging' and it gives a purpose to the class review session at the end.*
> *Sometimes I require everyone to work out and write down their solution to the whole class problem. The staggered finishing time for this allows me to get organised and help students get started on tasks without being besieged.*
> *I try to never interrupt the task session, but all pupils know we have a five minute review session at the end to allow them to comment on such things as an activity they particularly liked. We often close then with an agreed answer to our whole class problem.*

A Clinic in Action

The aims of the regular clinic are:

- to provide children with the opportunity to learn a variety of strategies
- to familiarise children with a process for solving problems.

The following example illustrates a structure which many teachers have found successful when running a clinic.

Preparation

For each session teachers need:

- a Strategy Board as below
- a How To Solve A Problem chart as below
- to choose a suitable problem and prepare it as a poster
- to organise children into groups of two or three.

The Strategy Board can be prepared in advance as a reference for the children, or may be developed *with* the children as they explore problem solving and suggest their own versions of the strategies.

The problem can be chosen from

- a book
- the task collection
- prepared collections such as Professor Morris Puzzles which can be viewed at: http://www.mathematicscentre.com/taskcentre/resource.htm#profmorr

The example which follows is from the task collection. The teacher copied it onto a large sheet of paper and asked some children to illustrate it. *The teacher also changed the number of sheep to sixty* to make the poster a little different from the one in the task collection.

The Strategy Board and the How To Solve A Problem chart can be used in any maths activity and are frequently referred to in Maths300 lessons.

The Clinic

The poster used for this example session is:

> Eric the Sheep is lining up to be shorn before the hot summer ahead. There are sixty [60] sheep in front of him. Eric can't be bothered waiting in the queue properly, so he decides to sneak towards the front.
>
> Every time one [1] sheep is taken to be shorn, Eric <u>then</u> sneaks past two [2] sheep. How many sheep will be shorn <u>before</u> Eric?

This Poster Problem Clinic approach is also extensively explored in Maths300 Lesson 14, *The Farmer's Puzzle*.

Strategy Board

DO I KNOW A SIMILAR PROBLEM?	ACT IT OUT
GUESS, CHECK AND IMPROVE	DRAW A PICTURE OR GRAPH
TRY A SIMPLER PROBLEM	MAKE A MODEL
WRITE AN EQUATION	LOOK FOR A PATTERN
MAKE A LIST OR TABLE	TRY ALL POSSIBILITIES
WORK BACKWARDS	SEEK AN EXCEPTION
BREAK INTO SMALLER PARTS	...

How To Solve A Problem

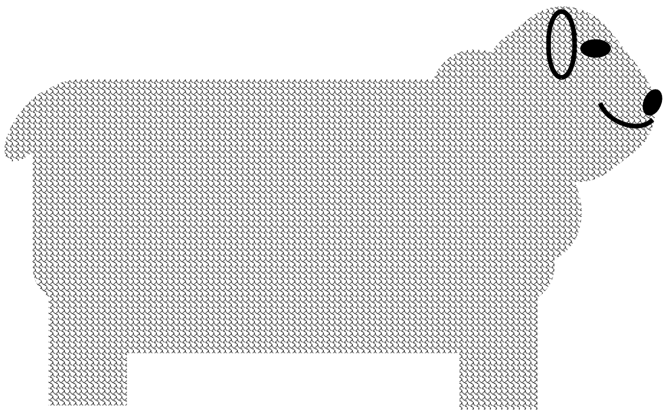
SEE & UNDERSTAND	Do I understand what the problem is asking? Discuss
PLANNING	Select a strategy from the board. Plan how you intend solving the problem.
DOING IT	Try out your idea.
CHECK IT	Did it work out? If so reflect on the activity. If not, go back to step one.

Step 1

- Tell the children that we are at Stage 1 of our four stage plan ... **See & Understand** ... Point to it! Read the problem with the class. Discuss the problem and clarify any misunderstandings.
- If children do not clearly understand what the problem is asking, they will not cope with the next stage. A good way of finding out if a child understands a problem is for her/him to retell it.
- Allow time for questions - approximately 3 to 5 minutes.

Step 2

- Tell the children that we are at Stage 2 of our four stage plan ... **Planning**. In their groups children select one or more strategies from the Strategy Board and discuss/organise how to go about solving the problem.
- Without guidance, children will often skip this step and go straight to Doing It. It is vital to emphasise that this stage is simply planning, not solving, the problem.
- After about 3 minutes, ask the children to share their plans.



Plan 1

Well we're drawing a picture and sort of making a model.

Can you give me more information please Brigid?

We're putting 60 crosses on our paper for sheep and the pen top will be Eric. Then Claire will circle one from that end, and I will pass two crosses with my pen top.

Plan 2

Our strategy is Guess and Check.

That's good Nick, but how are you going to check your guess?

Oh, we're making a model.

Go on ...

> *John's getting MAB smalls to be sheep and I'm getting a domino to be Eric and the chalk box to be the shed for shearing.*

Plan 3

> *We are doing it for 3 sheep then 4 sheep then 5 sheep and so on. Later we will look at 60.*

Great so you are going to try a simpler problem, make a table and look for a pattern.

This sharing of strategies is invaluable as it provides children who would normally feel lost in this type of activity with an opportunity to listen to their peers and make sense out of strategy selection. Note that such children are not given the answer. Rather they are assisted with understanding the power of selecting and applying strategies.

Step 3

- Tell the children that we are at Stage 3 of our four stage plan ... **Doing It**. Children collect what they need and carry out their plan.

Step 4

- Tell the children that we are at Stage 4 of our four stage plan ... **Check It**. Come together as a class for groups to share their findings. Again emphasis is on strategies.

> *We used the drawing strategy, but we changed while we were doing it because we saw a pattern.*

So Jake, you used the Look For A Pattern strategy. What was it?

> *We found that when Eric passed 10 sheep, 5 had been shorn, so 20 sheep meant 10 had been shorn ... and that means when Eric passes 40 sheep, 20 were shorn and that makes the 60 altogether.*

Great Jake. How would you work out the answer for 59 sheep or 62 sheep?

Sharing time is also a good opportunity to add in a strategy which no one may have used. For example:

> *Maybe we could've used the Number Sentence strategy, ie: 1 sheep goes to be shorn and Eric passes two sheep. That's 3 sheep, so perhaps, 60 divided into groups of 3, or 60 ÷ 3 gives the answer.*

Round off the lesson by referring to the Working Mathematically chart. There will be many opportunities to compliment the students on working like a mathematician.

Curriculum Planning Stories

Our attitude is:

> *teachers improve their teaching by re-enacting stories from the classrooms of their colleagues*

In more than a decade of using tasks and many years of using the detailed whole class lessons of Maths300, teachers have developed several models for integrating tasks and whole class lessons. Some of those stories are retold here. Others can be found at:

- http://www.mathematicscentre.com/taskcentre/plans.htm

Story 1: Threading

> *Educational research caused me a dilemma. It tells us that students construct their own learning and that this process takes time. My understanding of the history of mathematics told me that certain concepts, such as place value and fractions, took thousands of years for mathematicians to understand. The dilemma was being faced with a textbook that expected students to 'get it' in a concentrated one, two or three week block of work and then usually not revisit the topic again until the next academic year.*

A Working Mathematically curriculum reflects the need to provide time to learn in a supportive, non-threatening environment and...

> *When I was involved in a Calculating Changes PD program I realised that:*
>
> - *choosing rich and revisitable activities, which are familiar in structure but fresh in challenge each time they are used, and*
> - *threading them through the curriculum over weeks for a small amount of time in each of several lessons per week*
>
> *resulted in deeper learning, especially when partnered with purposeful discussion and recording.*

Calculating Changes:

- http://www.mathematicscentre.com/calchange

Story 2: Your turn

Some teachers are making extensive use of a partnership between the whole class lessons of Maths300 and small group work with the tasks. Setting aside a lesson for using the tasks in the way they were originally designed now seems to have more meaning, as indicated by this teacher's story:

> *When I was thinking about helping students learn to work like a mathematician, my mind drifted to my daughter learning to drive. She*

needed me to model how to do it and then she needed lots of opportunity to try it for herself.

That's when the idea clicked of using the Maths300 lessons as a model and the tasks as a chance for the students to have their turn to be a mathematician.

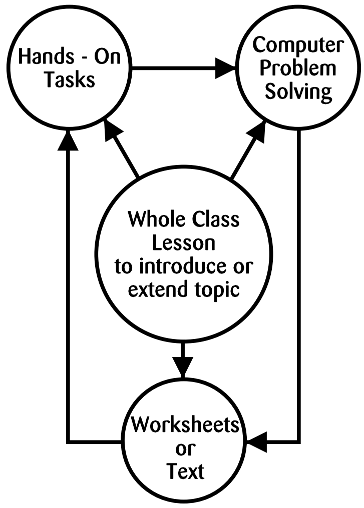
The Maths300 lessons illustrate how other teachers have modelled the process, so I felt I could do it too. Now the process is always on display on the wall or pasted inside the student's journal.

A session just using the tasks had seemed a bit like play time before this. Now I see it as an integral part of learning to work mathematically.

Story 3: Mixed Media

It was our staff discussion on Gardner's theory of Multiple Intelligences that led us into creating mixed media units. That and the access you have provided to tasks and Maths300 software.

We felt challenged to integrate these resources into our syllabus. There was really no excuse for a text book diet that favours the formal learners. We now often use four different modes of learning in the work station structure shown. It can be easily managed by one teacher, but it is better when we plan and execute it together.



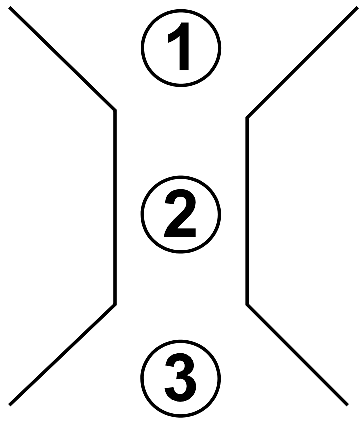
Story 4: Replacement Unit

We started meeting with the secondary school maths teachers to try to make transition between systems easier for the students. After considerable discussion we contracted a consultant who suggested that school might look too much the same across the transition when the students were hoping for something new. On the other hand our experience suggested that there needed to be some consistency in the way teachers worked.

We decided to 'bite the bullet' and try a hands-on problem solving unit in one strand. We selected two menus of twenty hands-on tasks, one for the primary and one for the secondary, that became the core of the unit. We deliberately overlapped some tasks that we knew were very rich and added some new ones for the high school.

Class lessons and investigation sheets were used to extend the tasks, within a three week model.

It is important to note that although these teachers structured a 3 week unit for the students, they strongly advised an additional *Week Zero* for staff preparation. The units came to be called Replacement Units.



Week Zero - Planning

Staff familiarise themselves with the material and jointly plan the unit. This is not a model that can be 'planned on the way to class'.

> Getting together turned out to be great professional development for our group.

Week 1 - Introduction

Students explore the 20 tasks listed on a printed menu:

- students explore the tip of the task, as on the card
- students move from task to task following teacher questioning that suggests there is more to the task than the tip
- in discussion with students, teachers gather informal assessment information that guides lesson planning for the following week.

> We gave the kids an 'encouragement talk' first about joining us in an experiment in ways of learning maths and then gave out the tasks. The response was intelligent and there was quite a buzz in the room.

Week 2 - Formalisation

> It was good for both us and the students that the lessons in this week were a bit more traditional. However, they weren't text book based. We used whole class lessons based on the tasks they had been exploring and taught the Working Mathematically process, content and report writing.
>
> Assessment was via standard teacher-designed tests, quizzes and homework.

Week 3 - Investigations

> We were most delighted with Week 3. Each student chose one task from the menu and carried out an in-depth investigation into the iceberg guided by an investigation sheet. They had to publish a report of their investigation and we were quite surprised at the outcomes. It was clear that the first two weeks had lifted the image of mathematics from 'boring repetition' to a higher level of intellectual activity.

Story 5: Curriculum shift

> I think our school was like many others. The syllabus pattern was 10 units of three weeks each through the year. We had drifted into that through a text book driven curriculum and we knew the students weren't responding.

Our consultant suggested that there was sameness about the intellectual demands of this approach which gave the impression that maths was the pursuit of skills. We agreed to select two deeper investigations to add to each unit. It took some time and considerable commitment, but we know that we have now made a curriculum shift. We are more satisfied and so are the students.

The principles guiding this shift were:

- Agree

  The 20 particular investigations for the year are agreed to by all teachers. If, for example, Cube Nets *is decided as one of these, then all the teachers are committed to present this within its unit.*

- Publish

  The investigations are written into the published syllabus. Students and parents are made aware of their existence and expect them to occur.

- Commit

  Once agreed, teachers are required to present the chosen investigations. They are not a negotiable 'extra'.

- Value

  The investigations each illustrate an explicit form of the Working Mathematically process. This is promoted to students, constantly referenced and valued.

- Assess

  The process provides students with scaffolding for their written reports and is also known by them as the criteria for assessment. *(See next page.)*

- Report

  The assessment component features within the school reporting structure.

A Final Comment

Including investigations has become policy.

Why? Because to not do so is to offer a diminished learning experience.

The investigative process ranks equally with skill development and needs to be planned for, delivered, assessed and reported.

Perhaps most of all we are grateful to our consultant because he was prepared to begin where we were. We never felt as if we had to throw out the baby and the bath water.

Assessment

Our attitude is:

stimulated students are creative and love to learn

Regardless of the way you use your **Maths With Attitude** resource, a variety of procedures can be employed to assess this learning.

Where these assessment procedures are applied to task sessions and involve written responses from students, teachers will need to be careful that the writing does not become too onerous. Students who get bogged down in doing the writing may lose interest in doing the tasks.

In addition to the ideas below, useful references are:

- http://www.mathematicscentre.com/taskcentre/assess.htm
- http://www.mathematicscentre.com/taskcentre/report.htm

The first offers several methods of assessment with examples and the second is a detailed lesson plan to support students to prepare a Maths Report.

Journal Writing

Journal writing is a way of determining whether the task or lesson has been understood by the student. The pupil can comment on such things as:

- What I learned in this task.
- What strategies I/we tried (refer to the Strategy Board).
- What went wrong.
- How I/we fixed it.
- Jottings - ie: any special thoughts or observations

Some teachers may prefer to have the page folded vertically, so that children's reflective thoughts can be recorded adjacent to critical working.

Assessment Form

An assessment form uses questions to help students reflect upon specific issues related to a specific task.

Anecdotal Records

Some teachers keep ongoing records about how students are tackling the tasks. These include jottings on whether students were showing initiative, whether they were working co-operatively, whether they could explain ideas clearly, whether they showed perseverance.

Checklists

A simple approach is to create a checklist based on the Working Mathematically process. Teachers might fill it in following questioning of individuals, or the students may fill it in and add comments appropriately.

Pupil Self-Reflection

Many theorists value and promote metacognition, the notion that learning is more permanent if pupils deliberately and consciously analyse their own learning. The deliberate teaching strategy of oral questioning and the way pupils record their work is an attempt to manifest this philosophy in action. The alternative is the tempting 'butterfly' approach which is to madly do as many activities as possible, mostly superficially, in the mistaken belief that quantity equates to quality.

> *I had to work quite hard to overcome previously entrenched habits of just getting the answer, any answer, and moving on to the next task.*

Thinking about *what* was learned *how* it was learned consolidates and adds to the learning.

When it follows an extensive whole class investigation, a reflection lesson such as this helps to shift entrenched approaches to mathematics learning. It is also an important component of the assessment process. On the one hand it gives you a lot of real data to assist your assessment. On the other it prepares the students for any formal assessment which you may choose to round off a unit.

Introduction

Ask students to recall what was done during the unit or lesson by asking a few individuals to say what *they* did, eg:

> *What did you do or learn that was new?*
> *What can you now do/understand that is new?*
> *What do you know now that you didn't know 1 (2, 3, ...) lesson ago?*

Continuing Discussion

Get a few ideas from the first students you ask, then:

- organise 5 -10 minute buzz groups of three or four students to chat together with one person to act as a recorder. These groups address the same questions as above.
- have a reporting session, with the recorder from each group telling the class about the group's ideas.

Student comments could be recorded on the board, perhaps in three groups.

Ideas & Facts	Maths Skills	Process (learning) Skills

If you need more questions to probe deeper and encourage more thought about process, try the following:

> *What new things did you do that were part of* how *you learned?*
> *Who uses this kind of knowledge and skill in their work?*

Student Recording

Hand out the REFLECTION sheet (next page) and ask students to write their own reflection about what they did, based on the ideas shared by the class. Collect these for interest and, possibly, assessment information.

REFLECTION

me looking at me learning

NAME: CLASS:

Working With Parents

Balancing Problem Solving with Basic Skill Practice

Many schools find that parents respond well to an evening where they have an opportunity to work with the tasks and perhaps work a task together as a 'whole class'. Resourced by the materials in this kit, teachers often feel quite confident to run these practical sessions. Comments from parents like:

> *I wish I had learnt maths like this.*

are very supportive. Letting students 'host' the evening is an additional benefit to the home/school relationship.

The 4½ Minute Talk

Charles Lovitt has considerable experience working with parents and has developed a crisp, parent-friendly talk which he shares below. Many others have used it verbatim with great success.

Why the Four and a Half Minute Talk?

When talking with parents about Problem Solving or the meaning of the term Working Mathematically, I have often found myself in the position, after having promoted inquiry based or investigative learning, of the parents saying:

> *Well - that's all very well - BUT ...*

at which stage they often express their concern for basic (meaning arithmetic) skill development.

The weakness of my previous attempts has been that I have been unable to reassure parents that problem solving does not mean sacrificing our belief in the virtues of such basic skill development.

One of the unfortunate perceptions about problem solving is that if a student is engaged in it, then somehow they are not doing, or it may be at the expense of, important skill based work.

This Four and a Half Minute Talk to parents is an attempt to express my belief that basic skill practice and problem solving development can be closely intertwined and not seen as in some way mutually exclusive.

(I'm still somewhat uncomfortable using the expression 'basic skills' in the above way as I am certain that some thinking, reasoning, strategy and communication skills are also 'basic'.)

Another aspect of the following ‘talk’ is that, as teachers put more emphasis on including investigative problem solving into their courses, a question arises about the source of suitable tasks.

This talk argues that we can learn to create them for ourselves by 'tweaking' the closed tasks that heavily populate out existing text exercises, and hence not be dependent on external suppliers. (Even better if students begin to create such opportunities for themselves.)

The Talk

In preparation, write the following graphic on the board:

CLOSED	OPEN	EXTENDED INVESTIGATION
		How many solutions exist? How do you know you have found them all?

I would like to show you what teachers are beginning to do to achieve some of the thinking and reasoning and communication skills we hope students will develop. I would like to show you three examples.

Example One: *6 + 5 = ?*

I write this question under the ‘closed’ label on the diagram:

CLOSED	OPEN	EXTENDED INVESTIGATION
6 + 5 ___		How many solutions exist? How do you know you have found them all?

And I ask:

What is the answer to this question?

I then explain that:

We often ask students many closed questions such as 6 + 5 = ?

The only response the students can tell us is "The answer is 11." … and as a reward for getting it correct we ask another twenty questions just like it.

What some teachers are doing is trying to *tweak* the question and ask it a different way, for example:

I have two counting numbers that add to 11. What might the numbers be?

[Counting numbers = positive whole numbers including zero]

I write this under the 'open' label on the diagram:

CLOSED	**OPEN**	**EXTENDED INVESTIGATION**
6 + 5 ___	? + ? 11	How many solutions exist? How do you know you have found them all?

What is the answer to the question now?

At this stage it becomes apparent there are several solutions:

The question is now a bit more open than it was before, allowing students to tell you things like 8 + 3, or 10 + 1, or 11 + 0 etc.

Let's see what happens if the teacher 'tweaks' it even further with the investigative challenge *or* extended investigation question:

How many solutions are there altogether?

and more importantly, and with greater emphasis on the second question:

How could you convince someone else that you have found them all?

Now the original question is definitely different - it still involves the skills of addition but now also involves thinking, reasoning and problem solving skills, strategy development and particularly communication skills.

Young students will soon tell you the answer is 'six different ones', but they must also confront the communication and reasoning challenge of convincing you that there are only six and no more.

Example Two: *Finding Averages*

Again, as I go through this example, I write it into the diagram on the board in the relevant sections.

The CLOSED question is: *11, 12, 13 - find the average*

Tweaking this makes it an OPEN question and it becomes:

I have three counting numbers whose average is 12. What might the numbers be?

Students will often say:

10, 12, 14 … or 9, 12, 15 … or even 12, 12, 12

After realising there are many answers, you can tweak it some more and turn it into an EXTENDED INVESTIGATION:

How many solutions exist? … AND …

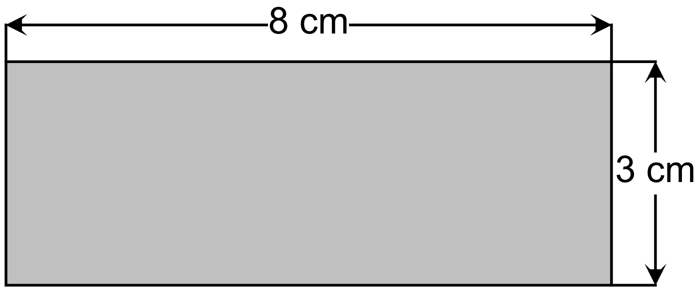
How do you know you have found them all?

Now the question is of a quite different nature. It still involves the arithmetic skill, but has something else as well - and that something else is the thinking, reasoning and communication skills necessary to find all of the combinations and convince someone else that you have done so.

By the time a student announces, with confidence, there are 127 different ways (which there are) that student will have engaged in all of these aspects, ie: the skill of calculating averages, (and some combination number theory) as well as significant strategy and reasoning experiences.

Example Three: *Finding the Area of a Rectangle*

A typical CLOSED question is:



Find the area. Find the perimeter.

The OPEN question is:

A rectangle has 24 squares inside:

What might its length and width be?

What might its perimeter be?

The EXTENDED INVESTIGATION version is:

Given they are whole number lengths, how many different rectangles are there? … AND …

How do you know you have found them all?

In summary, mathematics teachers are trying to convert *some* (not all) of the many closed questions that populate our courses and 'push' them towards the investigation direction. In doing so, we keep the skills we obviously value, but also activate the thinking, reasoning and justification skills we hope students will also develop.

This sequence of three examples hopefully shows two major features:

- That skills and problem solving can 'live alongside each other' and be developed concurrently.
- That the process of creating open-ended investigations can be done by anyone - just go to any source of closed questions and try 'tweaking' them as above. If it only worked for one question per page it would still provide a very large supply of investigations.

In terms of the effect of the talk on parents, I have usually found them to be reassured that we are not compromising important skill development (and nor do we want to). The only debate then becomes whether the additional skills of thinking, reasoning and communication are also desirable.

I've also been told that parents appreciate it because of the essential simplicity of the examples - no complicated theoretical jargon.

A Working Mathematically Curriculum

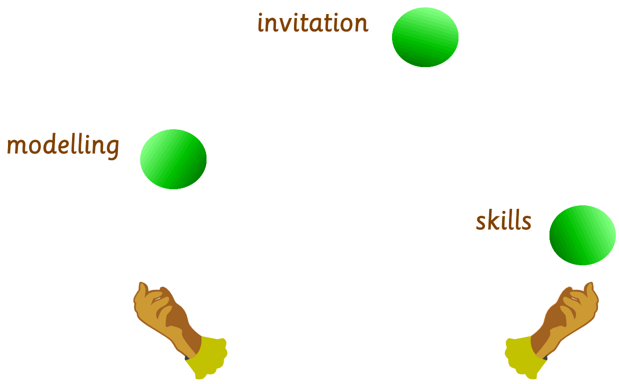
An Investigative Approach to Learning

The aim of a Working Mathematically curriculum is to help students learn to work like a mathematician. This process is detailed earlier (Page 8) in a one page document which becomes central to such a curriculum.

The change of emphasis brings a change of direction which *implies and requires* a balance between:

- the process of being a mathematician, and
- the development of skills needed to be a *successful* mathematician.

This journey is not two paths. It is one path made of two interwoven threads in the same way as DNA, the building block of life, is one compound made of two interwoven coils. To achieve a Working Mathematically curriculum teachers need to balance three components.



The task component of **Maths With Attitude** offers each pair of students an invitation to work like a mathematician.

The Maths300 component of **Maths With Attitude** assists teachers to model working like a mathematician.

Content skills are developed in context. They *are* important, but it is the application of skills within the process of Working Mathematically that has developed, and is developing, the human community's mathematical knowledge.

A focus for the Working Mathematically teacher is to help students develop mathematical skills in the context of problem posing and solving.

> *We are all 'born' with the same size mathematical toolbox, in the same way as I can own the same size toolbox as my motor mechanic. However, my motor mechanic has many more tools in her box than I and she has had more experience than I using them in context. Someone has helped her learn to use those tools while crawling under a car.*

Afzal Ahmed, Professor of Mathematics at Chichester, UK, once quipped:

> *If teachers of mathematics had to teach soccer, they would start off with a lesson on kicking the ball, follow it with lessons on trapping the ball and end with a lesson on heading the ball. At no time would they play a game of football.*

Such is not the case when teaching a Working Mathematically curriculum.

Elements of a Working Mathematically Curriculum

Working Mathematically is a K - 12 experience offering a balanced curriculum structured around the components below.

Hands-on Problem Solving Play

Mathematicians don't know the answer to a problem when they start it. If they did, it wouldn't be a problem. They have to play around with it. Each task invites students to play with mathematics 'like a mathematician'.

Skill Development

A mathematician needs skills to solve problems. Many teachers find it makes sense to students to place skill practice in the context of *Toolbox Lessons* which *help us better use the Working Mathematically Process* (Page 8).

Focus on Process

This is what mathematicians do; engage in the problem solving process.

Strategy Development

Mathematicians also make use of a strategy toolbox. These strategies are embedded in Maths300 lessons, but may also have a separate focus. Poster Problem Clinics are a useful way to approach this component.

Concept Development

A few major concepts in mathematics took centuries for the human race to develop and apply. Examples are place value, fractions and probability. In the past students have been expected to understand such concepts after having 'done' them for a two week slot. Typically they were not visited again until the next year. A Working Mathematically curriculum identifies these concepts and regularly 'threads' them through the curriculum.

Planning to Work Mathematically

The class, school or system that shifts towards a Working Mathematically curriculum will no longer use a curriculum document that looks like a list of content skills. The document would be clear in:

- choosing genuine problems to initiate investigation
- choosing a range of best practice teaching strategies to interest a wider range of students
- practising skills for the purpose of problem solving

Some teachers have found the planning template on the next page assists them to keep this framework at the forefront of their planning. It can be used to plan single lessons, or units built of several lessons. There are examples from schools in the Curriculum & Planning section of Maths300 and a Word document version of the template.

Unit Planning Page

Class

Topic

Pedagogy	Problem Solving	Skills
	In this topic how will I engage my students in the Working Mathematically process?	
How do I create an environment where students know what they are doing and why they have accepted the challenge?		Does the challenge identify skills to practise? Are there other skills to practise in preparation for future problem solving?

Notes

As a general guide:

- Find a problem(s) to solve related to the topic.
- Choose the best teaching craft likely to engage the learners.
- Where possible link skill practice to the problem solving process.

More on Professional Development

For many teachers there will be new ideas within **Maths With Attitude**, such as unit structures, views of how students learn, teaching strategies, classroom organisation, assessment techniques and use of concrete materials. It is anticipated (and expected) that as teachers explore the material in their classrooms they will meet, experiment with and reflect upon these ideas with a view to long term implications for the school program and for their own personal teaching.

Being explored 'on-the-job' so to speak, in the teacher's own classroom, makes the professional development more meaningful and practical for the teacher. This is also a practical and economic alternative for a local authority.

Strategic Use by Systems

From Years 3 - 10, **Maths With Attitude** is designed as a professional development vehicle by schools or clusters or systems because it carries a variety of sound educational messages. They might choose **Maths With Attitude** because:

- It can be used to highlight how investigative approaches to mathematics can be built into balanced unit plans without compromising skill development and without being relegated to the margins of a syllabus as something to be done only after 'the real' content has been covered.
- It can be used to focus on how a balance of concept, skill and application work can all be achieved within the one manageable unit structure.
- It can be used to show how a variety of assessment practices can be used concurrently to build a picture of student progress.
- It can be used to focus on transition between primary and secondary school by moving towards harmony and consistency of approach.
- It can be used to raise and continue debate about the pedagogy (art of teaching) that supports deeper mathematical learning for a wider range of students.

Teachers in Years K - 2 are similarly encouraged in professional growth through **Working Mathematically with Infants**, which derives from Calculating Changes, a network of teachers enhancing children's number skills from Years K - 6.

In supporting its teachers by supplying these resources in conjunction with targeted professional development over time, a system can fuel and encourage classroom-based debate on improving outcomes. There is evidence that by exploring alternative teaching strategies and encouraging curriculum shift towards Working Mathematically, learners improve and teachers are more satisfied. For more detail visit Research & Stories at:

- http://www.mathematicscentre.com/taskcentre/do.htm

We would be happy to discuss professional development with system leaders.

Web Reference

The starting point for all aspects of learning to work like a mathematician, including Calculating Changes, and the teaching craft which encourages it is:

- http://www.mathematicscentre.com