maths with attitude

2nd Edition

Pattern & Algebra Years 9 & 10

Charles Lovitt
Doug Williams

Mathematics Task Centre & Maths300

helping to create happy healthy cheerful productive inspiring classrooms

maths with attitude

2nd Edition

Pattern & Algebra

Years 9 & 10

In this kit:

- *Hands-on problem solving tasks*
- *Detailed curriculum planning*

Access from Maths300:

- *Extensive lesson plans*
- *Software*

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The **Maths With Attitude** series has been developed by The Task Centre Collective and is published by Black Douglas Professional Education Services.

Design and Desktop Publishing:
Black Douglas Professional Education Services

ISBN: 0-975-01918-X ... 978-0-975-01918-4
First published 2002
2nd Edition 2012
2016: Designed for web delivery as a computer-based file

Maths With Attitude includes reference to hands-on problem solving tasks and Maths300 material.

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Our Objective

- To support teachers, schools and systems wanting to create:

  happy, healthy, cheerful, productive, inspiring classrooms

Our Attitude

- to learning:

  learning is a personal journey stimulated by achievable challenge
- to learners:

  stimulated students are creative and love to learn
- to pedagogy:

  the art of choosing teaching strategies to involve and interest all students
- to mathematics:

  mathematics is concrete, visual and makes sense
- to learning mathematics:

  all students can learn to work like a mathematician
- to teachers:

  the teacher is the most important resource in education
- to professional development:

  teachers improve their teaching by re-enacting stories from the classrooms of their colleagues

Our Objective in Detail

What do we mean by creating:

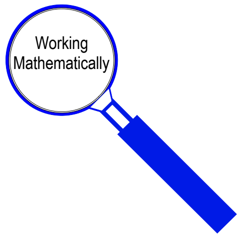
happy, healthy, cheerful, productive, inspiring classrooms

Happy…

means the elimination of the unnecessary fear of failure that hangs over so many students in their mathematics studies. Learning experiences *can* be structured so that all students see there is something in it for them and hence make a commitment to the learning. In so many 'threatening' situations, students see the impending failure and withhold their participation.

A phrase which describes the structure allowing all students to perceive something in it for them is *multiple entry points and multiple exit points*. That is, students can enter at a variety of levels, make progress and exit the problem having visibly achieved.

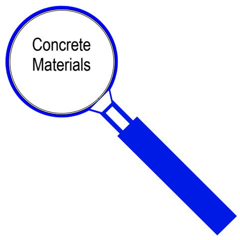
Healthy…



means *educationally healthy*. The learning environment should be a reflection of all that our community knows about how students learn. This translates into a rich array of teaching strategies that could and should be evident within the learning experience.

If we scrutinise the *exploration* through any lens, it should confirm to us that it is well structured or alert us to missed opportunities. For example, peering through a pedagogy lens we should see such features as:

- a story shell to embed the situation in a meaningful context
- significant active use of concrete materials
- a problem solving challenge which provides ownership for students
- small group work
- a strong visual component
- access to supportive software



Cheerful…

because we want 'happy' in the title twice!

Productive…

is the clear acknowledgment that students are working towards recognisable outcomes. They should know what these are and have guidelines to show they have either reached them or made progress. Teachers are accountable to these outcomes as well as to the quality of the learning environment.

Inspiring…

is about creating experiences that are uplifting or exalting; that actually *turn students on*. Experiences that make students feel great about themselves and empowered to act in meaningful ways.

Pattern & Algebra Resources

To help you create

happy, healthy, cheerful, productive, inspiring classrooms

this kit contains

- 20 hands-on problem solving tasks from Mathematics Centre

and a Teachers' Manual which integrates the use of the tasks with

- 9 detailed lesson plans from Maths300

The kit offers **6 weeks** of Scope & Sequence planning in Pattern & Algebra for *each* of Year 9 and Year 10. This is detailed in *Part 2: Planning Curriculum* which begins on Page 12. You are invited to map these weeks into your Year Planner.
Together, the four kits available for these levels provide 25 weeks of core curriculum in Working Mathematically (working like a mathematician).

Note: Membership of Maths300 is assumed.
The kit will be useful without it, but it will be much more useful with it.

Tasks

- Can Stack
- Cube Numbers
- Difference Between Two Squares
- Double Staircase
- Fold Up Houses
- How Many Squares?
- How Many Triangles?
- Intersections
- Jumping Kangaroos
- Money Money Money
- Painted Cubes
- Pyramid Puzzle
- Red To Blue
- Squares Around Squares
- Staircase
- Tetrahedron Triangles
- Tower Of Hanoi
- Triangles & Colours
- Triangles Around Triangles
- Two Squares

Part 2 of this manual introduces each task. The latest information can be found at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm

Maths300 Lessons

- Algebra Charts
- Crazy Animals
- Eric The Sheep
- Find My Pattern
- Jumping Kangaroos
- Staircases
- Painted Cubes
- Pyramid Puzzle
- Triangles & Colours

Lessons with Software

- Algebra Charts
- Crazy Animals

Part 2 of this manual introduces each lesson. Full details can be found at:

- http://www.maths300.com

Working Like A Mathematician

Our attitude is:

all students can learn to work like a mathematician

What does a mathematician's work actually involve? Mathematicians have provided their answer on Page 8. In particular we are indebted to Dr. Derek Holton for the clarity of his contribution to this description.

Perhaps the most important aspect of Working Mathematically is the recognition that *knowledge is created by a community and becomes part of the fabric of that community*. Recognising, and engaging in, the process by which that knowledge is generated can help students to see themselves as able to work like a mathematician. Hence Working Mathematically is the framework of **Maths With Attitude.**

Skills, Strategies & Working Mathematically

A Working Mathematically curriculum places learning mathematical skills and problem solving strategies in their true context. Skills and strategies are the tools mathematicians employ in their struggle to solve problems. Lessons on skills or lessons on strategies are not an end in themselves.

- **Our skill toolbox** can be added to in the same way as the mechanic or carpenter adds tools to their toolbox. Equally, the addition of the tools is not for the sake of collecting them, but rather for the purpose of getting on with a job. A mathematician's job is to attempt to solve problems, not to collect tools that might one day help solve a problem.
- **Our strategy toolbox** has been provided through the collective wisdom of mathematicians from the past. All mathematical problems (and indeed life problems) that have ever been solved have been solved by the application of this concise set of strategies.

About Tasks

Our attitude is:

mathematics is concrete, visual and makes sense

Tasks are from Mathematics Task Centre. They are an invitation to two students to work like a mathematician (see Page 8).

The Task Centre concept began in Australia in the late 1970s as a collection of rich tasks housed in a special room, which came to be called a Task Centre. Since that time hundreds of Australian teachers, and, more recently, teachers from other countries, have adapted and modified the concept to work in their schools. For example, the special purpose room is no longer seen as an essential component, although many schools continue to opt for this facility.

A brief history of Task Centre development, considerable support for using tasks, for example Task Cameos, and a catalogue of all currently available tasks can be found at:

- http://www.mathematicscentre.com/taskcentre

Key principles are:
- A good task is the tip of an iceberg
- Each task has three lives
- Tasks involve students in the Working Mathematically process

The Task Centre Room or the Classroom?

There are good reasons for using the tasks in a special room which the students visit regularly. There are also different good reasons for keeping the tasks in classrooms. Either system can work well if staff are committed to a core curriculum built around learning to work like a mathematician.
- A task centre room creates a focus and presence for mathematics in the school. Tasks are often housed in clear plastic 'cake storer' type boxes. Display space can be more easily managed. The visual impact can be vibrant and purposeful.
- However, tasks can be more readily integrated into the curriculum if teachers have them at their finger tips in the classrooms. In this case tasks are often housed in press-seal plastic bags which take up less space and are more readily moved from classroom to classroom.

Tip of an Iceberg

The initial problem on the card can usually be solved in 10 to 20 minutes. The investigation iceberg which lies beneath may take many lessons (even a lifetime!). Tasks are designed so that the original problem reveals just the 'tip of the iceberg'. Task Cameos and Maths300 lessons help to dig deeper into the iceberg.

We are constantly surprised by the creative steps teachers and students take that lead us further into a task. No task is ever 'finished'.

> *Most tasks have many levels of entry and exit and therefore offer an on-going invitation to revisit them, and, importantly, multiple levels of success for students.*

Three Lives of a Task

This phrase, coined by a teacher, captures the full potential and flexibility of the tasks. Teachers say they like using them in three distinct ways:
1. As on the card, which is designed for two students.
2. As a whole class lesson involving all students, as supported by outlines in the Task Cameos and in detail through the Maths300 site.
3. Extended by an Investigation Guide (project), examples of which are included in both Task Cameos and Maths300.

The first life involves just the 'tip of the iceberg' of each task, but nonetheless provides a worthwhile problem solving challenge - one which 'demands' concrete materials in its solution. This is the invitation to work like a mathematician. Most students will experience some level of success and accomplishment in a short time.

The second life involves adapting the materials to involve the whole class in the investigation, in the first instance to model the work of a mathematician, but also to develop key outcomes or specific content knowledge. This involves choosing teaching craft to interest the students in the problem and then absorb them in it.

The third life challenges students to explore the 'rest of the iceberg' independently. Investigation Guides are used to probe aspects and extensions of the task and can be introduced into either the first or second life. Typically this involves providing suggestions for the direction the investigation might take. Students submit the 'story' of their work for 'portfolio assessment'. Typically a major criteria for assessment is application of the Working Mathematically process.

About Maths300

Our attitude is:

> *teachers improve their teaching by re-enacting stories from the classrooms of their colleagues*

Maths300 is a subscription based web site. It is an attempt to collect and publish the 300 most 'interesting' maths lessons (K - 12).

- *Lessons have been successfully trialed in a range of classrooms.*
- *About one third of the lessons are supported by specially written software.*
- *Lessons are also supported by investigation sheets (with answers) and game boards where relevant.*
- *A 'living' Classroom Contributions section in each lesson includes the latest information from schools.*
- *The search engine allows teachers to find lessons by pedagogical feature, curriculum strand, content and year level.*
- *Lesson plans can be printed directly from the site.*
- *Each lesson supports teaches to model the Working Mathematically process.*

Modern internet facilities and computers allow teachers easy access to these lesson plans. Lesson plans need to be researched, reflected upon in the light of your own students and activated by collecting and organising materials as necessary.

Maths300 Software

Our attitude is:

> *stimulated students are creative and love to learn*

Pedagogically sound software is one feature likely to encourage enthusiastic learning and for that reason it has been included as an element in about one third of Maths300 lesson plans. The software is used to develop an investigation beyond its introduction and early exploration which is likely to include other pedagogical techniques such as concrete materials, physical involvement, estimation or mathematical conversation. The software is not the lesson plan. It is a feature of the lesson plan used at the teacher's discretion.

For school-wide use, the software needs to be downloaded from the site and installed in the school's network image. You will need to consult your IT Manager about these arrangements. It can also be downloaded to stand alone machines covered by the site licence, in particular a teacher's own laptop, from where it can be used with the whole class through a data projector.

Note:

- *Maths300 lessons and software may only be used by Maths300 members.*

Working Mathematically

First give me an interesting problem.

When mathematicians become interested in a problem they:

- Play with the problem to collect & organise data about it.
- Discuss & record notes and diagrams.
- Seek & see patterns or connections in the organised data.
- Make & test hypotheses based on the patterns or connections.
- Look in their strategy toolbox for problem solving strategies which could help.
- Look in their skill toolbox for mathematical skills which could help.
- Check their answer and think about what else they can learn from it.
- Publish their results.

Questions which help mathematicians learn more are:

- Can I check this another way?
- What happens if …?
- How many solutions are there?
- How will I know when I have found them all?

When mathematicians have a problem they:

- Read & understand the problem.
- Plan a strategy to start the problem.
- Carry out their plan.
- Check the result.

A mathematician's strategy toolbox includes:

- Do I know a similar problem?
- Guess, check and improve
- Try a simpler problem
- Write an equation
- Make a list or table
- Work backwards
- Act it out
- Draw a picture or graph
- Make a model
- Look for a pattern
- Try all possibilities
- Seek an exception
- Break a problem into smaller parts
- …

If one way doesn't work, I just start again another way.

Professional Development Purpose

Our attitude is:

the teacher is the most important resource in education

> *We had our first study group on Monday. The session will be repeated again on Thursday. I had 15 teachers attend. We looked at the task Farmyard Friends (Task 129 from the Mathematics Task Centre). We extended it out like the questions from the companion Maths300 lesson suggested, and talked for quite a while about the concept of a factorial. This is exactly the type of dialog that I feel is essential for our elementary teachers to support the development of their math background. So anytime we can use the tasks to extend the teacher's math knowledge we are ahead of the game.*
> District Math Coordinator, Denver, Colorado

Research suggests that professional development most likely to succeed:

- is requested by the teachers
- takes place as close to the teacher's own working environment as possible
- takes place over an extended period of time
- provides opportunities for reflection and feedback
- enables participants to feel a substantial degree of ownership
- involves conscious commitment by the teacher
- involves groups of teachers rather than individuals from a school
- increases the participant's mathematical knowledge in some way
- uses the services of a consultant and/or critical friend

Maths With Attitude has been designed with these principles in mind. All the materials have been tried, tested and modified by teachers from a wide range of classrooms. We hope the resources will enable teacher groups to lead themselves further along the professional development road, and support systems to improve the learning outcomes for students K - 12.

With the support of Maths300 ETuTE, professional development can be a regular component of in-house professional development. See:

- http://www.mathematicscentre.com/taskcentre/resource.htm#etute

For external assistance with professional development, contact:

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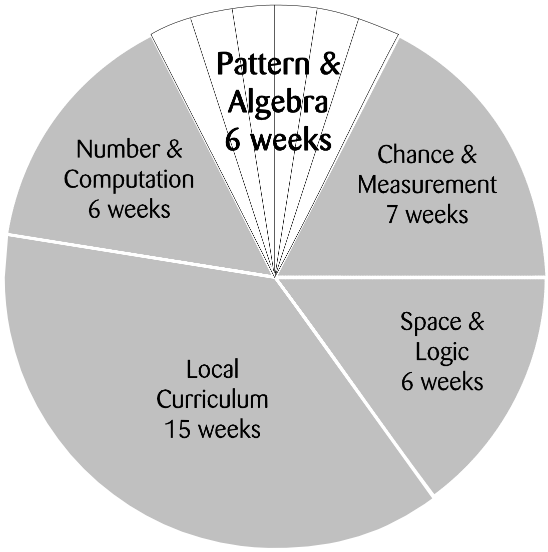
Part 2: Planning Curriculum

Curriculum Planners

Our attitude is:

learning is a personal journey stimulated by achievable challenge



Curriculum Planners:
- show one way these resources can be integrated into your weekly planning
- provide a starting point for those new to these materials
- offer a flexible structure for those more experienced

You are invited to map Planner weeks into your school year planner as the core of the curriculum.

Planners:
- detail each week lesson by lesson
- offer structures for using tasks and lessons
- are sequenced from lesson to lesson, week to week and year to year to 'grow' learning

Teachers and schools will map the material in their own way, but all will be making use of extensively trialed materials and pedagogy.

Using Resources

- Your kit contains 20 hands-on problem solving tasks and reference to relevant Maths300 lessons.
- Tasks are introduced in this manual and supported by the Task Cameos at:
  http://www.mathematicscentre.com/taskcentre/iceberg.htm
- Maths300 lessons are introduced in this manual and supported by detailed lesson plans at:
  http://www.maths300.com

In your preparation, please note:
- Planners assume 4 lessons per week of about 1 hour each.
- Planners are *not* prescribing a continuous block of work.
- Weeks can be interspersed with other learning; perhaps a **Maths With Attitude** week from a different strand.
- Weeks can sometimes be interchanged within the planner.
- Lessons can sometimes be interchanged within weeks.
- The four **Maths With Attitude** kits available at each year level offer 25 weeks of a Working Mathematically core curriculum.

A Way to Begin

- Glance over the Planner for your class. Skim through the comments for each task and lesson as it is named. This will provide an overview of the kit.
- Task Comments begin after the Planners. Lesson Comments begin after Task Comments. The index will also lead you to any task or lesson comments.
- Select your preferred starting week - usually Week 1.
- Now plan in detail by researching the comments and web support. Enjoy!

Research, Reflect, Activate

Curriculum Planner
Pattern & Algebra: Year 9

<table>
<tr><th></th><th>Session 1</th><th>Session 2</th><th>Session 3</th><th>Session 4</th></tr>
<tr><td>Weeks
1 - 3</td><td colspan="4">Covering All Cases A: This unit is structured around the Mixed Media teaching model which is explained on Page 17. It assumes ready access to computers for one third of the class. If these are not a fixture in the room, schools have adopted alternatives such as (a) making arrangements for students to visit computer sites within the school, or, (b) collecting an appropriate number of notebook computers (5 or 6 for a class of 30).
The content focus is on coming to know what is true for all cases by gathering data about particular cases, then using the skills and strategies of a mathematician to establish general rules.
Tasks used are Difference Between Two Squares, Double Staircase, Intersections, Jumping Kangaroos, Money Money Money, Painted Cubes, Pyramid Puzzle, Staircase, Triangles & Colours, Two Squares.
A Mixed Media Unit includes one class lesson each week. In this unit the lessons are Algebra Charts and Find My Pattern.</td></tr>
<tr><td>Week 4</td><td colspan="2">Whole Class Investigation: Eric The Sheep. A problem involving a discontinuous relationship.</td><td colspan="2" rowspan="3">Tasks & Text: (See Page 20) Divide the class into two groups. Groups spend a full session at each station.

Tasks used are: Can Stack, Cube Numbers, Fold Up Houses, How Many Squares?, How Many Triangles?, Red To Blue, Squares Around Squares, Tetrahedron Triangles, Tower of Hanoi, Triangles Around Triangles.</td></tr>
<tr><td>Week 5</td><td colspan="2">Whole Class Investigation: Jumping Kangaroos. A problem involving a quadratic relationship.</td></tr>
<tr><td>Week 6</td><td colspan="2">Whole Class Investigation: Stair-cases. Another quadratic relationship and triangle numbers.</td></tr>
</table>

- Weeks can be interchanged.
- An activity named in **bold** refers to a hands-on task.
- An activity named in *italic* refers to a lesson from Maths300.
- Text book style Toolbox Lessons can be interwoven or set for homework.

Curriculum Planner

Pattern & Algebra: Year 10

<table>
<tr><th></th><th>Session 1</th><th>Session 2</th><th>Session 3</th><th>Session 4</th></tr>
<tr><td>Weeks
1 - 3</td><td colspan="4">Covering All Cases B: This unit is structured around the Mixed Media teaching model which is explained on Page 17. It assumes ready access to computers for one third of the class. If these are not a fixture in the room, schools have adopted alternatives such as (a) making arrangements for students to visit computer sites within the school, or, (b) collecting an appropriate number of notebook computers (5 or 6 for a class of 30).
In the main, for reasons explained on Page 19, this is the same Mixed Media unit as for Year 9. The main difference will be the textbook work you choose to integrate with it.
Tasks used are Difference Between Two Squares, Double Staircase, Intersections, Jumping Kangaroos, Money Money Money, Painted Cubes, Pyramid Puzzle, Staircase, Triangles & Colours, Two Squares.
A Mixed Media Unit includes one class lesson each week. In this unit Crazy Animals is suggested as one lesson, and the other is your choice based on the tasks, Additional Lessons on Page 15 or a favourite of your own.</td></tr>
<tr><td>Week 4</td><td colspan="2">Whole Class Investigation: Painted Cubes. A problem related to the expansion of $(a + b)^3$.</td><td colspan="2" rowspan="3">Tasks & Text: (See Page 20) Divide the class into two groups. Groups spend a full session at each station.

Tasks used are: Can Stack, Cube Numbers, Fold Up Houses, How Many Squares?, How Many Triangles?, Red To Blue, Squares Around Squares, Tetrahedron Triangles, Tower of Hanoi, Triangles Around Triangles.</td></tr>
<tr><td>Week 5</td><td colspan="2">Whole Class Investigation: Triangles & Colours. A problem involving a different cubic with several parts.</td></tr>
<tr><td>Week 6</td><td colspan="2">Whole Class Investigation: Pyramid Puzzle. A problem which brings together all aspects of the work in both Year 9 and Year 10, and in the process opens the door to Year 11 & 12 algebra and methods of proof.</td></tr>
</table>

- Weeks can be interchanged.
- An activity named in **bold** refers to a hands-on task.
- An activity named in *italic* refers to a lesson from Maths300.
- Text book style Toolbox Lessons can be interwoven or set for homework.

Planning Notes

Enhancing Maths With Attitude

Resources to support learning to work like a mathematician are extensive and growing. There are more tasks and lessons available than have been included in this Pattern & Algebra kit. You could use the following to enhance this kit.

Additional Tasks

- Task 236, Star Numbers
  Star Numbers are constructed from Square and Triangle Numbers. In this sense, they have to be made as shapes to be understood. As the size of the Star Number increases, so does the number of plugs needed to make it. Students can explore particular cases, thereby applying number skills, or can generalise for any size Star Number.
- Task 238, Growing Trisquares
  The crux of this problem is that 4 Trisquares can be joined to make a new, scaled up, Trisquare. This provides a 'template' for constructing the next size, and the next, and the next... The visual pattern can also be represented as a number pattern and this leads to graphing, equation work and scale factors.

More information about these tasks may be available in the Task Cameo Library:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Additional Lessons

- Lesson 147, Speed Graphs
  This is an applications of $y = mx$ which has personal relevance to students. Students go outside to collect data about their personal walking and jogging speeds and graph both sets of data on the same axes. The graph becomes a mathematical model of reality and every feature of it (origin, slope etc.) relates to an aspect of their physical trials
- Lesson 167, Twelve Days of Christmas
  Almost everyone knows that on the first day of Christmas my true love gave me a partridge in a pear tree and on and on for twelve days building up a musical pattern. Mathematicians seek and find patterns and teachers seek and find multiple points to encourage students to enter mathematical investigations. This lesson offers students with logical/mathematical, musical, visual/spatial, verbal, kinaesthetic and interpersonal intelligences opportunity to engage in a problem which can focus on number patterns, or extend into complex quadratic algebra.
- Lesson 171, Pick's Rule
  This easy to run lesson includes almost all Working Mathematically steps from playing with a problem to the process of proof. Students draw polygons on grid paper and count: the number of border dots (on the edge of the shape), the number of dots inside the shape, the area (number of squares) within the shape. There are underlying patterns and rules to discover that are true for all such shapes.

Keep in touch with new developments which enhance **Maths With Attitude** at:

- http://www.mathematicscentre.com/taskcentre/enhance.htm

Additional Materials

As stated, our attitude is that mathematics is concrete, visual and makes sense. We assume that all classrooms will have easy access to many materials beyond what we supply. For this unit you will need:

- Counters
- Linking Cubes
- Packs of cards (optional)
- Popsticks in a range of colours (obtainable at most craft shops)
- Pyramid Puzzles (see below)
- Wooden MAB cubes in sizes other than 10 (optional)

Special Comments Year 9

- *Mixed Media Unit*, Planner Weeks 1 - 3. A Mixed Media unit takes a little extra preparation in the beginning, but this is repaid during the unit because it runs for several days. Schools which have used the model find team preparation eases any burden, ie:
  - one teacher ensures the software station is prepared and runs effectively during the unit
  - one gets the tasks in order and keeps them that way during the unit
  - one selects appropriate material from the text and prepares the 'contract' sheet
  - one makes sure all teachers are resourced with whatever is necessary for the whole class lessons
- Note: In the preparation of this unit some teachers create a 'contract' sheet for students which sets out the expectations of the unit, for example:
  - Participate in two whole class lessons.
  - Keep a diary of your work on at least three tasks.
  - Complete a written report about one software-based investigation.
  - Complete Exercises ... from the text.
- *Staircases* Planner Week 6. This lesson requires linking cubes to build the staircases. A less satisfactory alternative, if you have enough table or floor space, is to use playing cards. Treat the cards as objects. Place them face down to make the staircases.

Special Comments Year 10

- *Mixed Media Unit*, Planner Weeks 1 - 3. A Mixed Media unit takes a little extra preparation in the beginning, but this is repaid during the unit because it runs for several days. Schools which have used the model find team preparation eases any burden, ie:
  - one teacher ensures the software station is prepared and runs effectively during the unit
  - one gets the tasks in order and keeps them that way during the unit
  - one selects appropriate material from the text and prepares the 'contract' sheet
  - one makes sure all teachers are resourced with whatever is necessary for the whole class lessons
- Note: In the preparation of this unit some teachers create a 'contract' sheet for students which sets out the expectations of the unit, for example:
  - Participate in two whole class lessons.

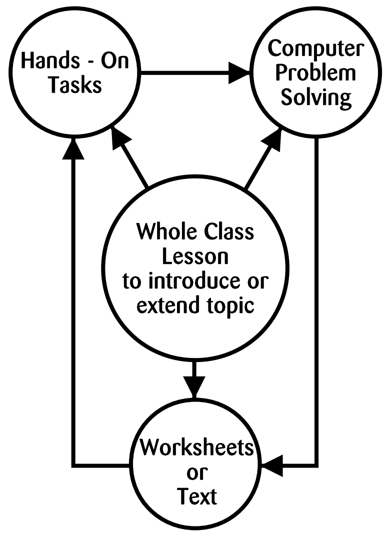
- Keep a diary of your work on at least three tasks.
- Complete a written report about one software-based investigation.
- Complete Exercises ... from the text.

- The suggested lesson for the Mixed Media unit is *Crazy Animals*, which requires cards printed from the Lesson Plan. You could carry out the relevant part of the lesson without them but it is so much more meaningful for many students to use the materials. If you print them from the lesson, then we suggest you do this onto thin card and laminate to create a class set which can be used for many years.
- *Painted Cubes*, Planner Week 4. You will need plenty of linking cubes (1cm or 2cm size) to make various size cubes, or sets of wooden MAB cubes of different sizes.
- *Triangles & Colours*, Planner Week 5. You will need a pack of about 1000 coloured pop sticks. These are obtained from most craft shops at a reasonable price.
- *Pyramid Puzzle*, Planner Week 6. You will need multiple copies of the Pyramid Puzzle and plenty of linking cubes. In this lesson the 2cm ones are best. Information about copies of Pyramid Puzzle can be obtained from the Resources section of Mathematics Centre web site:

  http://www.mathematicscentre.com/mathematicscentre

  Although you may be able to develop much of the mathematics of this lesson without multiple copies of the puzzle, experience has shown that, even in Year 12, students benefit from seeing and touching the patterns in the pyramid which are the genesis of the lesson.

Mixed Media Unit



Mixed Media Mathematics has been created as *one* structure which allows teachers to integrate problem solving tasks into the curriculum.

The design incorporates four different modes of learning into a structure which can be readily managed by one teacher, but which is enhanced when prepared and executed by a team.

A three week Mixed Media Unit includes:

- whole class lessons
- hands-on problem solving
- problem solving software
- skill practice worksheets (or text material)
- time to reflect on learning
- assessment opportunities

If this is the first time such a structure has been used in your classroom, it is a good idea to prepare the students in a manner which 'brings them into the experiment'.

A vital element of the process is to reflect on *what* is learned and *how* it is learned *before* the final assessment of the learning. Guidance with respect to assessment is also provided in this manual. In particular, the Pupil Self-Reflection information in the Assessment section of Part 3 was designed by teachers who trialed the original Mixed Media units.

Covering All Cases A

Algebra Charts is suggested as the first lesson in the unit. It is easy to start with the help of the Investigation Sheets provided in the lesson. It focuses on manipulating symbolic algebra within a problem solving context. Every student can have success at some level with these sheets and the software offers a non-threatening environment for continuing to develop facility with these tools of a mathematician. The software becomes the focus of the Software Station in the other lessons of each week.

The second lesson suggested as a whole class investigation is *Find My Pattern*. This lessons offers an opportunity to revise the value of graphically representing algebraic relationships as a tool for analysing and extending problems.

If you need a whole class lesson for the third week, you could base it on one of the tasks. Alternatively, if you choose an assessment focus for the third week, the whole class lesson could be built around the Pupil Self-Reflection information (see Page 57).

For the Worksheet/Text station, you might choose to write Investigation Sheets for each task, or you can require completion of relevant material from your textbook. Students work in pairs on the tasks and the computers. Some teachers are happy for this collegiate approach to continue at the text station as well.

It has already been mentioned that you will need to choose a new whole class lesson for the second week. You may also need new text work. Alternatively it may be appropriate for students to continue some aspects of their software or task-based investigation during the period at the text station.

The tasks suggested for this unit are:

- Difference Between Two Squares
- Double Staircase
- Intersections
- Jumping Kangaroos
- Money Money Money
- Painted Cubes
- Pyramid Puzzle
- Staircase
- Triangles & Colours
- Two Squares

The same tasks are used in all weeks of the unit because, in the main, it will take one session to complete one task fully, so no students will be able to use them all in this unit.

As all aspects of this kit are explored through Years 9 & 10, **Jumping Kangaroos**, **Painted Cubes**, **Pyramid Puzzle**, **Staircase** and **Triangles & Colours** are used as whole class lessons. It is important that students have had a chance to first try some of these tasks for themselves. When introducing a class lesson built on one of the tasks, teachers find it fruitful to 'tuck the task under their arm' and begin the lesson along the lines of:

> *Many of you have had the opportunity to work on this task before. Tell me what you can remember about it. ... Now let's see what we can learn from it by working together in the way mathematicians often do.*

Teachers often find the third week needs to be flexible. You might want to continue the exploratory nature of the unit and thus introduce one more whole class lesson and follow up investigation. However, if the unit is going to have a strong assessment component based on the deeper investigation of a task, students may need time scheduled for report writing. In fact, it may be necessary to use the whole class lesson this week to model report writing.

Student Publishing

It is inappropriate to simply expect students to publish a report of their investigation. We have to devote lesson time to teaching how to keep a journal while investigating and how to plan and present a report. The Recording & Publishing section of Mathematics Task Centre includes two different approaches to scaffolding this process with the class. Both include sample student work and suggest that a report can be presented in forms other than pencil and paper, for example PowerPoint. The links are titled 'Learning to Write a Maths Report' and 'Learning to Write a Maths Report 2' and can be found at:

- http://www.mathematicscentre.com/taskcentre/record.htm

Covering All Cases B

In essence this is the same unit as the one above. Many aspects are rich enough to use again, but you would expect student growth from the previous year. *Algebra Charts* is extremely rich software and students benefit by continuing to grow their ability to manipulate symbolic algebra expressions. There are ten tasks, and in Year 9 students could only have had time to use three or four of them. There are plenty more to explore.

What does change is the text material you choose to integrate, and the whole class lessons you decide to use.

- *Algebra Charts* could be introduced in the same way as in Year 9, but you would probably want to generate your own Investigation Sheets from the software.
- *Crazy Animals* is suggested too. The algebra component of this lesson is very strong. How many animals are there? How do you know when you have found them all? Option 5 of the software offers problems based in these questions which lead students to making a generalisation which covers all cases.
- Now look at the 0-part, 1-part, 2-part and 3-part animals in the set (assuming three basic animals are used). The results lead to an expansion of $(n + 1)^3$, one of several ways of deriving the equivalent of this result which students will come across if the full range of material for Year 10 is used. The lesson plan for this aspect of *Crazy Animals* is in the Whole & Parts Appendix of the lesson.

If you need another class lesson, depending on your choice for Week 3 as described above, you could base it on one of the tasks. Alternatively, if you choose an assessment focus for the third week, the whole class lesson could be built around the Pupil Self-Reflection information (see Page 57).

Tasks & Text

The tasks used in this component of the kit are:

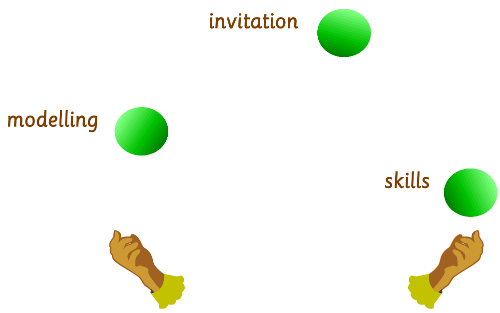
- Can Stack
- Cube Numbers
- Fold Up Houses
- How Many Squares?
- How Many Triangles?
- Red To Blue
- Squares Around Squares
- Tetrahedron Triangles
- Tower of Hanoi
- Triangles Around Triangles

An Investigation Guide to support the deeper investigation of each task is supplied in the Appendix to this manual.

The class is divided into two groups. Time has been planned so that each group spends one session per week on a task and its deeper investigation, and one session per week on appropriate text material of your choice. Working in two groups like this means there is sufficient material for everyone to use and there are fewer tasks with which a teacher needs to be familiar.

The tasks are playing the role of offering students genuine problems in the same sense as indicated on the Working Mathematically sheet (Page 8) where professional mathematicians have responded to the question "What do you do?" with:

First give me an interesting problem.



With the help of the Investigation Guide, it will soon become clear that most of these tasks are worthy of more than one session of study. So, you might expect some students to use the same task through all three weeks of Tasks & Text. In fact, exploring, and reporting on, one task in depth can be an assessment requirement of these weeks. This depth is also the reason why the same tasks can be used in both Years 9 & 10.

In the other two sessions of the week, the whole class investigations suggested allow you to model how a mathematician goes about an investigation. The combination of these four sessions offers a week by week balanced program of the components necessary to learn to work like a mathematician.

Task Comments

- Tasks, lessons and unit plans prepare students for the more traditional skill practice lessons, which we invite you to weave into your curriculum. Teachers who have used practical, hands-on investigations as the focus of their curriculum, rather than focussing on the drill and practice diet of traditional mathematics, report success in referring to skill practice lessons as Toolbox Lessons. This links to the idea of a mathematician dipping into a toolbox to find and use skills to solve problems.

Can Stack

The story shell which is the context for this task reflects the part-time work many students of around this age find in supermarkets. The 'boss' in the story actually leaves the worker with an unclear instruction about how to stack the cans in a more stable way. The investigation of the (at least) two ways to build based on the instructions leads to a significant investigation. There is an Investigation Guide for this task in Appendix 1.

Answers to Investigation Guide

1- 2. Harry's sequence is the sum of the natural numbers. The expectation is that students will explain these in words, so their answers will vary. The sequence of cans used:

1, 3, 6, 10, ...

is the triangle numbers, another sequence likely to be familiar to students.

3. There are several ways to develop the formula for summing the natural numbers and students have probably had experience with this challenge previously. Lesson 12, *Gauss Beats The Teacher*, on the Maths300 site is built around this challenge. In Harry's case, the formula would be like this:

$$C = {}^{L}/_{2}(L + 1)$$

In this context L and C must be natural numbers (or perhaps whole numbers if you accept that zero layers needs zero cans), because it doesn't make sense to stack parts of a can. However, there is room for discussion related to incomplete layers.

The point of the request for explanation here is to informally raise the concepts of domain and range which become more relevant in later years of maths education.

4. The points graphed will lie on a parabolic curve. Should the points be joined to show the curve? Technically no, because of restrictions in the domain, but if this is discussed first, then perhaps the points could be joined with a dotted line.

5. The following insights are expected to form the core of student answers:

One Direction

Each can stands on two others, except for those in the second last layer. The last layer has twice the expected number of cans so that each can above stands on four below. The building pattern is:

1 + 2 + 3 + 4 + 5 + ... and at the *n*th layer, an extra *n* cans are added.

which results in:

$$C = {}^{L}/_{2}(L + 1) + L$$

Alternatively, students might think of this pattern as:

$$C = {}^{(L-1)}/_{2}[(L - 1) + 1] + 2L$$

and therein lies an opportunity to explore equivalent algebraic expressions.

Two Directions

Only the top can stands on two cans beneath. All others stand on four. The single top can is added last and since it is not part of the visual pattern, it becomes one extra added at the end of the number pattern investigation. Excluding this Layer 1 can for the moment, it can be seen that the building pattern for Layer 2 onwards is:
(2 x 1) + (3 x 2) + (4 x 3) + (5 x 4) + ...

Again the visual understanding of the problem provides an approach to finding the number in any layer, ie: $L(L - 1)$ for $L > 1$. This approach also provides a way of finding the total of cans needed to build to the 10th layer, as asked on the card, but generalising this to the *n*th layer is a more difficult challenge. To achieve it, students would need to realise that the series above is actually:
2 + 6+ 12+ 20 + ... = 2(1 + 3 + 6 + 10 + ...) = 2 x sum of triangle numbers and that the last term of the series above is $L(L - 1)$.

Summing the triangle numbers is a challenge in itself and this is dealt with in full detail in Lesson 138, *Pyramid Puzzle & Other Algebra Excursions*. At this stage it might be best to 'give' students the formula for this sum and ask them to test it for the data they have, with the promise that, before they finish Year 10, you will explore the investigation which leads to its development.

$$\text{Sum } (T_n) = {}^{n}/_{6}(n + 1)(n + 2)$$

With this information, the number of cans in any layer can be calculated, but don't forget to add 1 for the can on the top.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Cube Numbers

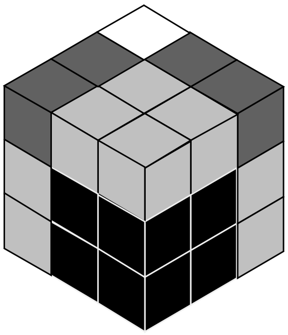
This task is testimony to the maxim that 'algebra makes sense'. Cube numbers come from making cubes; just as square numbers come from making squares and triangle numbers come from making triangles.

(Natural Numbers don't come from making 'naturals' do they? Perhaps they do. The word Natural has a common root with the word Nature; and like nature itself, it has been said that: *God gave the natural numbers; all others are the invention of man*.)

Further, the Investigation Guide in Appendix 1 leads students on a visual exploration of formulas related to cube numbers which are often learned by rote. The isometric paper supplied in Appendix 2 could assist students in drawing their own diagrams.

Answers to Investigation Guide

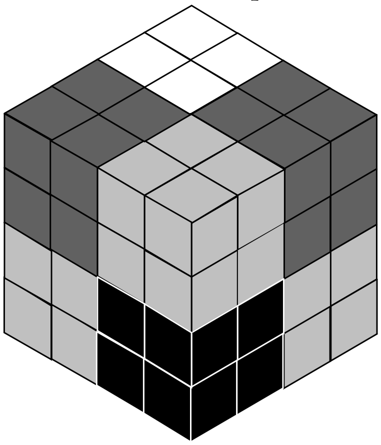
1 - 2. There is an a^2 like the light grey one on each of the three covered faces of the black cube (a^3).



On top of two of these sits a dark grey 'a'. The third 'a' stands vertically (hidden at the back) with a single white cube on top.

3. $1001^3 = (1000 + 1)^3$.
   One would presume students would find it easier to calculate this using the simplest form. Note the opportunity here to link with toolbox lessons from the text which practise exercises related to these formulas.
4. Working backwards, as in the first approach on the Guide gives:
   $a^3 + 2 \times a^2 + 2[a(a + 2)] + 2(a + 2)^2 = (a + 2)^3$

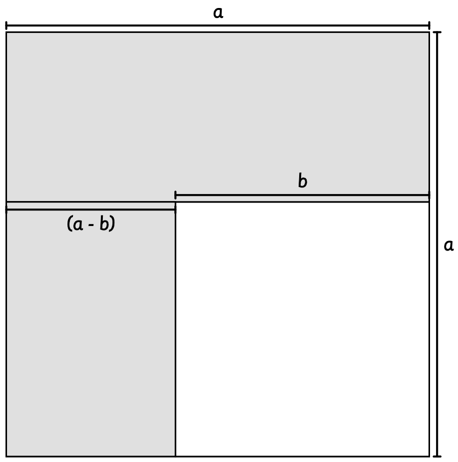
   The deconstruction equivalent to the simplest form would be:

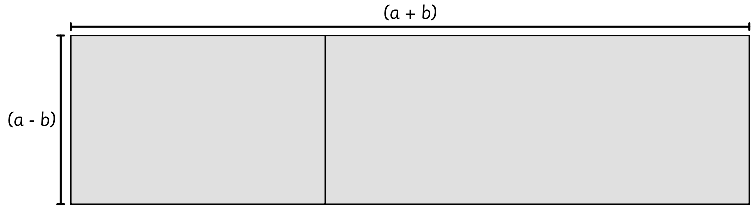
   

   The process is the same as the description above, although in this case the picture is slightly complicated by the a^3 chosen for the drawing being the same size as the number added. Light grey 'squares' represent a^2. Dark grey 'rectangles' represent 'a'.
   When this diagram and the one above are looked at a second time, what may have seemed like filling spaces at the end with white cubes, now begins to look like filling the space with one white cube.
5. With a little imagination, and more cubes, the standard expansion:
   $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
   can be discovered by the students.

While the students are working with cube numbers, you might also like them to graph the sequence: (1, 1), (2, 8), (3, 27), (4, 64) ... to see a visual representation of a cubic function. Also, while there are cubes of different sizes on the table, you might raise the challenge of summing the cube numbers, a challenge which is also dealt with in Lesson 138, *Pyramid Puzzle & Other Algebra Excursions*.

Difference Between Two Squares





It is one thing to see pictures in a book which demonstrate that: $a^2 - b^2 = (a - b)(a + b)$

For some students, it is a totally different thing to experience the equivalence for yourself in the way this task allows.

In fact, the task is in the collection because many years ago a neighbour's child came to one of our consultants looking for help with Year 9 maths. Her text book had pictures of the construction, but she 'didn't get it'. It wasn't until they had cut out squares from a couple of pieces of paper that she remarked:

Oh, is that all it is?

The task develops a visual image to support what is otherwise often presented as a rule, then, with this clearer understanding as a guide, encourages practise of the new mathematical skill in a way that is likely to 'make more sense' of the text book exercises.

It is quite powerful to use grid paper to show the construction several times with specific numbers. For example, get a square of size 7 and cut a square of size 3 out of the corner. The remaining area is clearly:

$7^2 - 3^2 = 49 – 9 = 40$

Now do the construction and *turn it into a rectangle* of size:

$(7+3) \times (7-3) = 10 \times 4 = 40$

Doing this for several specific examples such as:

$8^2 - 5^2$ ... $10^2 - 3^2$ ... $9^2 - 2^2$ ... $6^2 - 5^2$

shows the general process. The important aspect of the visual image is that the construction *always creates a rectangle* - and the length (a + b) and width (a - b) of the rectangle are factors.

This same visual approach is encouraged in the task Cube Numbers to help develop a rule for the sum and difference of two cubes.

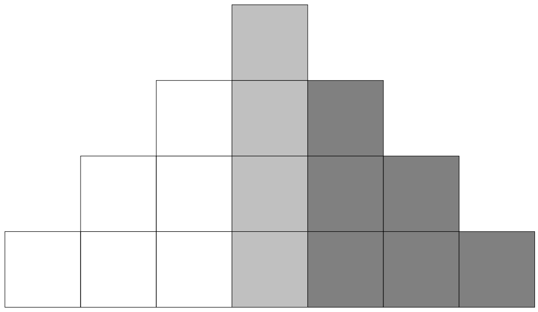
Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Double Staircase

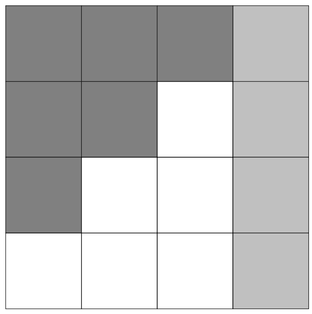
In this task students have the opportunity to discover that a visual pattern produces a number pattern which can be described in words or symbols. In this way the task contributes to the understanding that algebra makes sense.

Encourage the students to 'see' the visual construction first. The words and symbols flow from that.
In Question 1, perhaps the students will see a central column with an identical single staircase on each side.



That is, they 'see' the construction (1 + 2 + 3) ... + 4 + ... (3 + 2 + 1)

The number of blocks in the central column (4 in the diagram above) is the same as the number of steps (S) in the double staircase. The number of steps in each single staircase (3 above) is one less than the steps in the double staircase. So, if we could easily find the blocks in a single staircase, we could use that to sum the blocks in the double staircase.



The single staircase is the sum of the natural numbers, and if you perceive the students have not had experience with that challenge, then Lesson 12, *Gauss Beats The Teacher*, could be appropriate. The total of the natural numbers T_N is given by:

$T_N = {}^{N(N+1)}/_2$

so the total of blocks in the double staircase (T_D) is given by:

$T_D = 2\ [{}^{(S-1)(S-1+1)}/_2] + S = S(S - 1) + S = S^2$

Of course, this simplification must make sense in the context too...

In the diagram to the right, the two single staircases put together make a rectangle (4 x 3). This is S(S - 1). Then add the central column (S) on the side, and the whole thing is always a square (S^2).

Trying several specific cases with concrete blocks can show the general process. For example, if S = 5, then the double staircase total can be 'seen' as:

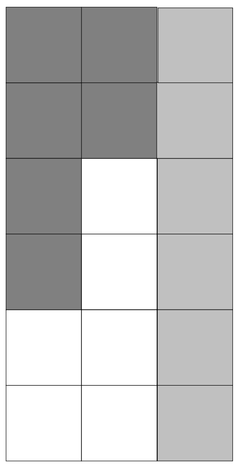
(1 + 2 + 3 + 4) ... + 5 + ... (4 + 3 + 2 + 1)

Similarly if S = 6, the pattern is:

(1 + 2 + 3 + 4 + 5) ... + 6 + ... (5 + 4 + 3 + 2 + 1)

And if S = 10, the pattern is:

(1 + 2 + ... + 9) ... + 10 + ... (9 + 8 + ... 2 + 1)



The focus of the task is on concrete experience and visual reasoning to develop words, which are then re-expressed symbolically. A table of values is not necessarily the only pathway into a formula.
Continuing this type of reasoning in Question 2 reveals that one way of seeing the pattern is: $T = 2S \times S = 2S^2$.

There are extensive notes, and an Investigation Sheet in Lesson 115 *Staircases*, which encourages the development of this visual intelligence as a way into formal algebraic symbolisation. The lesson also explores substitution and solution of equations based on formulas in context, and graphing the formulas. All the contexts are based on quadratic equations.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Fold Up Houses

This task involves a spatial challenge which leads to a number pattern which can be expressed in words and symbols. It neatly connects experience with nets to an algebraic context. Further, because there are two types of shape involved and the numbers of each vary, there is opportunity to introduce the idea of simultaneous linear equations. There is an Investigation Guide for this task in Appendix 1. A particular thrust of this sheet is the bigger picture of a generalisation which can be painted by combining the use of words, symbols, tables of values and graphs.

Answers to Investigation Guide

These answers are an indication of what students might write. Equivalent answers are acceptable.

1. Squares: Multiply the size number of the house by three and add two.
   Triangles: Multiply the size number of the house by two, add twice the size less two, then add two more.
2. $S = 3N + 2$
   $T = 2N + (2N - 2) + 2 = 4N$
3. Students might draw up a traditional table like this one, or they might use a spreadsheet.

Size	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Squares	5	8	11	14	17	20	23	26	29	32
Triangles	4	8	12	16	20	24	28	32	36	40

4. Students would need to graph (not necessarily on the same axis) the functions in 8.2. Again, a spreadsheet could be used as a tool.
5. The table shows that the Size 2 house is built from equal numbers of squares and triangles.
   If graphed on the same axes in 4, the students would need to note the importance of the intersection point to explain this simultaneous solution.
   Symbolically the expected response would be something like:

The squares and triangles have to be equal so we want:

$$S = T$$
$$3N + 2 = 4N$$
$$N = 2$$

6. In essence students are being asked to **simultaneously** solve the inequalities:
   3N + 2 less than or equal to 100
   4N less than or equal to 100
   They might do this graphically, with a table, or symbolically. The result will be that the largest house built from 100 squares and 100 triangles would be Size 25 and there would be 23 squares and 0 triangles left.
   If solved graphically, students might also comment on the rate at which the use of triangles grows compared with the squares.
7. It is possible that some students might ask why these buildings are called Hulk Houses. It may be that it is just a marketing name (Why call a drink Coca-Cola?); it may be that some sizes of the house look like hulks of ships; it may be that these houses have the strength of the Incredible Hulk.

8.1 Squares: Multiply the size number of the HH by five and add four.
Triangles: Multiply the size number of the HH by two, add twice the size less two, then add ten more.

8.2 $S = 5N + 4$
$T = 2N + (2N - 2) + 10 = 4N + 8 = 4(N + 2)$

8.3 Students might draw up a traditional table like this one, or they might use a spreadsheet.

Size	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Squares	9	14	19	24	29	34	39	44	49	54
Triangles	12	16	20	24	28	32	36	40	44	48

8.4 Students would need to graph (not necessarily on the same axis) the functions in 8.2. Again, a spreadsheet could be used as a tool.

8.5 The table shows that the Size 4 HH is built from equal numbers of squares and triangles.
If graphed on the same axes in 8.4, the students would need to note the importance of the intersection point to explain this simultaneous solution.
Symbolically the expected response would be something like:

The squares and triangles have to be equal so we want:

$$S = T$$
$$5N + 4 = 4(N + 2)$$
$$5N + 4 = 4N + 8$$
$$N = 4$$

8.6 In essence students are being asked to **simultaneously** solve the inequalities:
5N + 4 less than or equal to 100
4(N + 2) less than or equal to 100
They might do this graphically, with a table, or symbolically. The result will be that the largest HH built from 100 squares and 100 triangles would be Size 19 and there would be 1 square and 16 triangles left.

9. Mathematicians need to consider their solutions in the practical circumstances of the problem. In line with the story shell, this question is intended to bring something of that sort of reality into the context. What is actually written

doesn't matter as long as it is justifiable. Answers might be enhanced by sketches and might be along the lines of:

The slope of the roof encourages snow to slide off and it would fall beyond the walls of the building due to the angles.

Windows could be placed in the angled parts of the walls so that light would enter, but direct sunlight would not, therefore slowing the heating effect inside the house. It is like the shape has a built-in verandah.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

How Many Squares?

This task and the Investigation Guide which accompanies it leads the pursuit of algebra in two directions. On the one hand the total of the squares is governed by a recursive formula:

$$T_S = T_{S-1} + S^2$$

which opens the door to a whole branch of mathematics. On the other hand, the formula can be seen as the sum of the squares since $T_1 = 1$. But how do you sum the square numbers? In this case the formula is given:

$$T_S = 1 + 4 + 9 + \ldots + S^2 = [S(S + 1)(2S + 1)] \div 6$$

so the focus can shift to investigating the nature of the cubic function that it represents. However, Lesson 138, *Pyramid Puzzle & Other Algebra Excursions* gives plenty of support for developing a class lesson to background this formula and give it visual meaning. The Year 10 Planner suggests this lesson as one which brings together two years of algebra study and opens doors to further directions, for example, the strategy of 'proof by induction'.

Answers to Investigation Guide

1. This question extends and records the work on the card. It is somehow satisfying to see that the count of squares is governed by square numbers.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
1 x 1	1	4	9	16	25	36	49	64
2 x 2		1	4	9	16	25	36	49
3 x 3			1	4	9	16	25	36
4 x 4				1	4	9	16	25
5 x 5					1	4	9	16
6 x 6						1	4	9
7 x 7							1	4
8 x 8								1
Total	1	5	14	30	55	91	140	204

2. The diagrams on the Investigation Guide show two ways in which odd numbers and squares come into play in this problem. The first is that each new size of square can be formed from the previous one by adding on an odd number of 1 x 1 squares. The second is that doing this produces a sequence

of odd numbers to count the new positions to which the various size squares (1 x 1 through to S x S) can be shifted.
For example, if the students go on to make the Size 4 square from the Size 3 they have to add 7 unit squares in an L shape. Doing this creates:

- 1 new 4 x4 square
- 3 new positions to shift the top left 3 x 3 square
- 5 new positions to shift the top left 2 x 2 square
- 7 new positions to shift the top left 1 x 1 square

The sum of these odd numbers is a square number and this is the S^2 in the recursive formula $T_S = T_{S-1} + S^2$.

3. The recursive formula lends itself to the spreadsheet tool. Formats will vary.

4 - 5. The thrust of these questions is the mathematician's question:

Can I check this another way?

Students have now solved and checked the problem by combining a table with a visual method, with a formula they can derive and extend as far as necessary on a spreadsheet, and with a formula 'given' by authority.

6. An introduction to a cubic graph. Should the points be joined? Domain? Range?

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

How Many Triangles?

The task card has two parts. The first is a section similar to **How Many Squares?** which involves counting triangles of different sizes within the largest equilateral triangle. The second combines the counting with the spatial problem of creating tetrahedrons from nets and counting the number of unit triangles on the surface. In tackling the first section students discover that unit triangles in the largest equilateral triangle can be counted by square numbers, so the formula required in the last part of the card is:

Number of unit triangles = 4 x (Size of tetrahedron)²

The Investigation Guide for this task (see Appendix 1) extends the first section and helps students work towards a generalisation for counting triangles of all sizes within the largest equilateral triangle. This is not as easy as **How Many Squares?** because there are two types of triangles to count - point up and point down.

Answers to Investigation Guide

1. This question extends and records the work on the card. Patterns are not as obvious as with the **How Many Squares?** task.

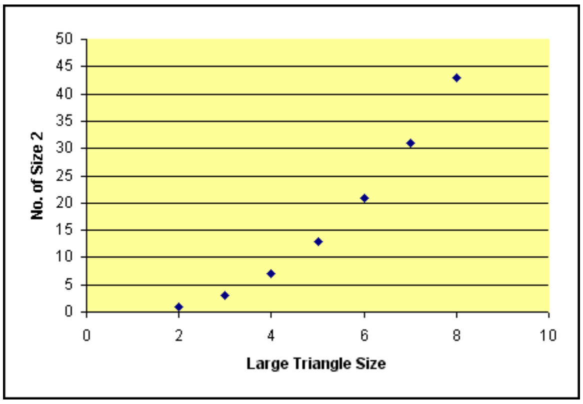
	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
Size 1 (S_1)	1	4	9	16	25	36	49	64
Size 2 (S_2)		1	3	7	13	21	31	43
Size 3 (S_3)			1	3	6	11	18	27
Size 4 (S_4)				1	3	6	10	16
Size 5 (S_5)					1	3	6	10
Size 6 (S_6)						1	3	6
Size 7 (S_7)							1	3
Size 8 (S_8)								1
Total	1	5	14	30	55	91	140	204

Looking for example at the Size 5 row there is a tantalising hint that the Triangle Numbers are involved in these counts, but given that the Size 4 row, for example, starts in the same tantalising way and then breaks the pattern with a 16 rather than a 15, it is clear that if the Triangle Numbers are involved, then it is not in a straightforward way.

2. The Size 2 triangle line in the table is:

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
Size 2		1	3	7	13	21	31	43

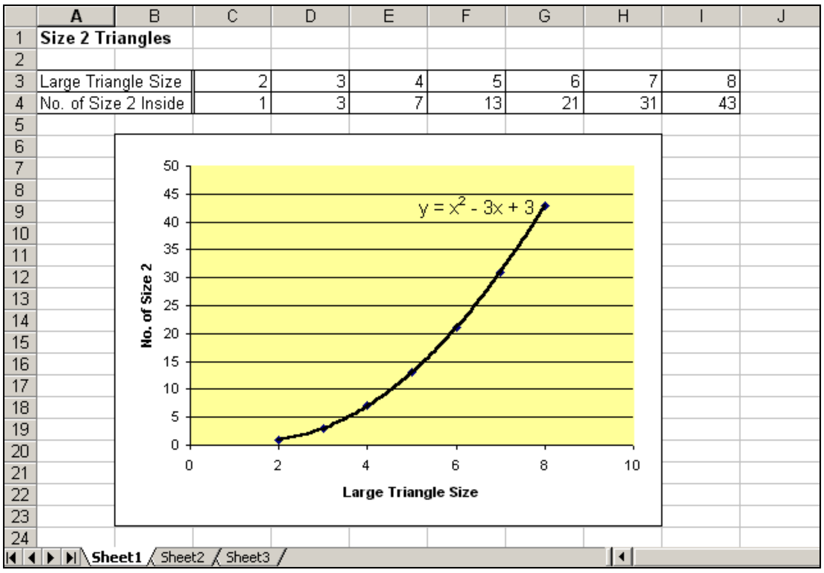
If this data is to be linked to the size of the largest equilateral triangle (E_N), then each of these ordered pairs
(2, 1), (3, 3), (4, 7), (5, 13), (6, 21), (7, 31), (8, 43)...
must be governed by the same rule. Graphing these points produces a shape that could be parabolic, as shown, which can guide the search for a rule. Using a difference table can confirm that the relationship is indeed quadratic:



Size 2	1		3		7		13		21		31		43
1st Diff.		2		4		6		8		10		12	
2nd Diff.			2		2		2		2		2		...

but it is still a tough call to discover by 'fiddling' with the numbers that, for E_N: $S_2 = N^2 - 3N + 3$

The Investigation Guide suggests using an Excel spreadsheet might help because it has a facility called Add Trendline which, when a Polynomial trend of order 2 is selected, gives the actual equation as shown.



3. Investigating the point up and down contributions to the count for Size 2 gives:

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
Up		1	3	6	10	15	21	28
Down		0	0	1	3	6	10	15

This gives the same totals as above, but reveals how the Triangle Numbers are involved in the count of triangles.

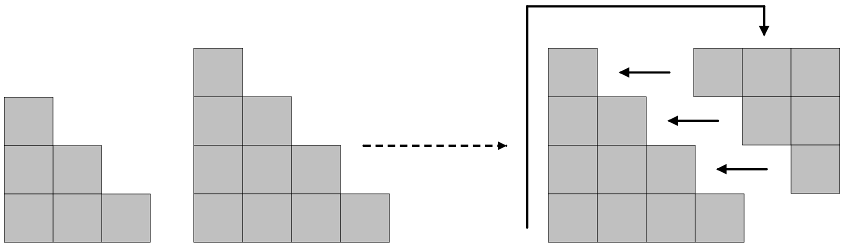
It's like you have two strips of paper with the same bunch of numbers and one slides past the other.

There is something satisfying about that.

4 - 6. This revelation is the basis of investigating the other questions on the Guide and the results obtained this way can be checked against the formula above. Interestingly, the process also shows that the Square Numbers which count the Size 1 triangles are actually the sum of two sequences of Triangle Numbers, one of which is displaced by a term. That is, for Size 1:

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
Up	1	3	6	10	15	21	28	36
Down		1	3	6	10	15	21	28

This fact is confirmed visually in Lesson 115 *Staircases*, with constructions like:



Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Intersections

Students will soon realise that the most intersections (M) for any number of sticks (S) will be a pattern of triangle numbers. The challenge is to explain why this is so.

- A straight line can only intersect once with another straight line.
- To achieve the most intersections each new stick must intersect with all the current sticks without passing through any current intersections.
- These new intersections are added to the number already present.

So the pattern grows from the term before, and since the first case is 1 stick and 0 intersections (1, 0), the continuing terms must be: (1, 0), (2, 1), (3, 3), (4, 6)...

The general rule for finding the Nth Triangle Number is: $T_N = {}^{N(N+1)}/_2$
but in this case there is a little twist because it takes the presence of 1 stick to get things started. So:

$$M = {}^{(S-1)[(S-1)+1]}/_2 = {}^{S(S-1)}/_2$$

Jumping Kangaroos

This is a well known task with a variety of story shells - frogs, mountain goats etc. Most students find this problem quite engaging, and the background equation is a quadratic. However, many students who have never heard of quadratics can work out the pattern and make the equivalent generalisation. In fact there are several ways of generalising the situation and each can be interpreted uniquely in terms of the rules of the problem.

A clue to working out the challenge of solving the case for three kangaroos each side is to check the options at each move and look ahead to the consequences of each. The 'critical' moves (in other words the moves that many people get wrong) are moves 3, 6 and 10 which involve bringing up one of the end kangaroos to establish the alternating pattern of positions.

The problem also resolves when the mathematician's strategy of *Try a simpler case* is applied. There is more than one way to predict the number of moves given the number of kangaroos on each side. The problem has many extensions.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Money Money Money

This task is included because it involves simultaneous linear equations. Challenge 1 offers an introduction which only requires logical thought. For example, one aspect of this is:

- Only one person receives a $100 note.
- All people receive five notes.
- So it must be either C or D who gets the $100.

Students then have to look at the possibilities for making either $35 or $20 from four more notes. The solution is:

A: $90 made up from $50, $20, $10, 2 x $5

B: $100 made up from 5 x $20

C: $135 made up from 2 x $50, $20, $10, $5

D: $120 made up from $100, 4 x $5

Challenge 2 introduces conditions which could be (but don't have to be) written as equations.

B = 2A ... C = 2B ... D = B - 30

With these facts in mind, and the total of all monies being $870,a key clue is that A has only two notes. The choices for those two are limited so one approach is to build a table exploring all possibilities:

A (2 notes)	B (3 notes)	C (4 notes)	D (5 notes)
100 + 50	»»» B = $300 and C = $600. Too much money already!		
100 + 20	»»» B = $240, C = $480 and D = $210 Too much money!		
...	Continuing the reasoning gives ...		
50 + 50	200	400	170

The table also allows us to work out the notes each person has to make these totals:

A: 2 x \$50

B: \$100 + 2 x \$50

C: 4 x \$100

D: \$100 + 3 x \$20 + \$10

Challenge 3 introduces the equations:

$A < 100$

$B = 2A$

$C = A + {}^{(B - A)}/_{2}$

$D = C + 20$

$D = B - 15$

Solving this set of simultaneous equations and using the other information in the challenge gives:

A: \$50 + \$20

B: \$100 + 2 x \$20

C: \$100 + \$5

D: \$100 + \$20 + \$5

One extension to the task is for the students to make up a similar problem of their own using the money in the task. This is certainly a non-trivial exercise! You might like to collect these student-made tasks and build a class set over time.

Painted Cubes

This task is highly visual and generates a range of algebraic relationships. The cubes strongly support the investigation and can be extended by including a large MAB cube from the classroom resources.

Students will quickly see that the number of cubes with three painted faces is constant and represents the eight corner cubes. They may also notice that the other values 'grow' at different rates. In fact, one is linear, one is quadratic and one is cubic. There is an algebraic connection between the size of the cube and the unit cubes with a particular number of faces painted. It is:

- 3 Painted Faces: 8
- 2 Painted Faces: $(n - 2) \times 12$
- 1 Painted Face: $(n - 2)^2 \times 6$
- 0 Painted Faces: $(n - 2)^3$

which, of course, gives the task a firm link with the task **Cube Numbers**.

The emphasis in the task is on *seeing why* these must be the relationships. Where is the $(n - 2)$ on the cube in each case? The ability to be visually algebraic in this way can lead to the solution of a problem more quickly than trying to manipulate the numerical data recorded in a table.

It is *not necessary* to have studied square and cubic functions in order to be able to use the geometric properties of squares and cubes to count the units and identify and generalise a pattern.

Find more information about this task in the Task Cameo Library at:

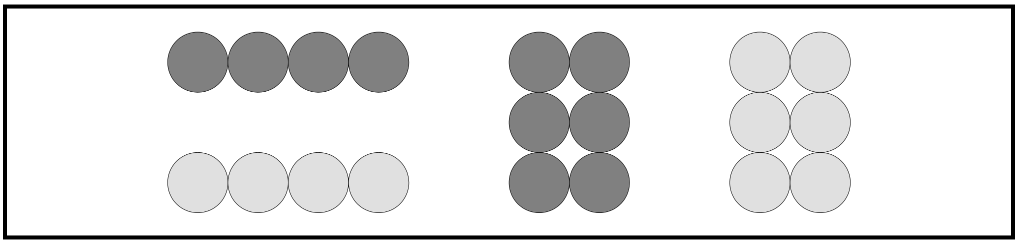
- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

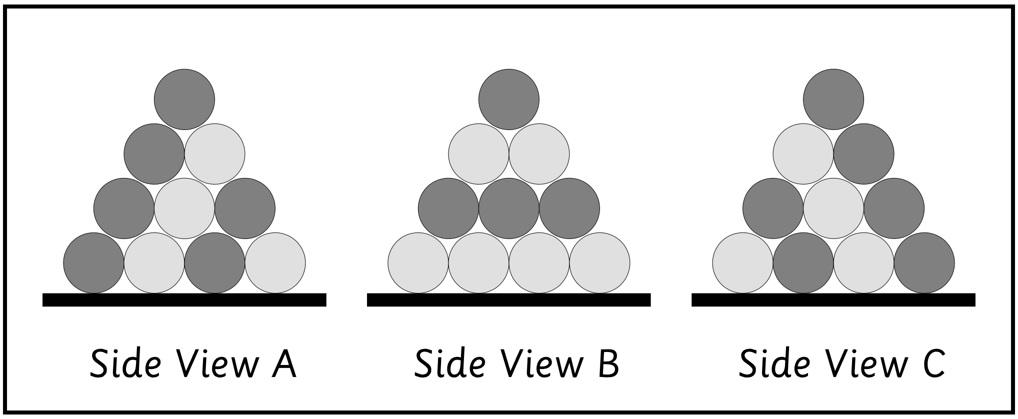
This information includes a PowerPoint report about the problem from two Year 8 students and a trainee teacher's work on developing the rule for the difference between two cubes from the problem.

Pyramid Puzzle

This apparently innocent 3D challenge has a considerable algebraic sting in its tail which is explored in Lesson 138, *Pyramid Puzzle & Other Algebra Excursions*. This lesson is suggested as the last stage in the Year 10 program to draw together the threads of the algebra course and open doors to further directions, for example, the strategy of 'proof by induction'. It is important that students have a chance to explore this task before that lesson, especially if you don't have a class set of the puzzle to use in the lesson.

The solution of the puzzle can be derived from the views below given that the pieces used are:





Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Red To Blue

Students may have experienced this task as a lesson if the school uses the MWA Space & Logic kit in Years 7 & 8. If so, it is an old friend, and there is value in meeting it in this form, whereby the young mathematicians are invited to tackle the challenge for themselves. They are supported by an Investigation Guide available in Appendix 1.

Answers to Investigation Guide

1. The table of moves using the (N - 1) rule for starting numbers to 10 is:

Number	1	2	3	4	5	6	7	8	9	10
Moves	No	2	No	4	No	6	No	8	No	10

2. If the starting number is odd it is impossible to solve the problem. If it is even, the number of moves is the same as the starting number. So, M = N.
3. Why the (N - 1) problem doesn't work for odd starting numbers is not easy to explain. Students would need to demonstrate their understanding that:
   - Each counter has to be involved in an odd number of turns in order to change colour.
   - After an odd number of moves either you return to a previous state, or there are an odd number of counters unchanged.
   - Each one of these odd number of counters still needs to be involved in an odd number of moves to change colour, but that means that after the next odd number of moves there will still be either a return to a previous state, or an odd number of counters unchanged.

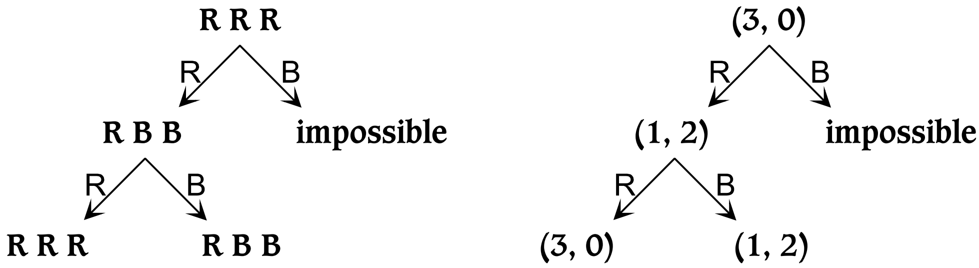
   Perhaps a logic tree diagram and the strategy of testing every possible combination is best to show why the odd number problems don't work. For example, if the start number is '3', the starting position is

   R R R

   which can also be written as

   (3, 0)

   If all except one tile is turned over, then we can focus on the disc that is not turned over. This colour is either R or B, so we can systematically follow the result of each possibility:



   which are both previous states in the problem.

   So the only possible states of the problem are:
   - (3, 0), (2, 1), (1, 2), and (0, 3)

   If we start with (3, 0) the only other state we can reach is (1, 2) and in both of these Red is ODD.

   A similar diagram for the '5' Red problem shows the total possible states are:
   - (5, 0) (4, 1), (3, 2), (2, 3), (1, 4) and (0, 5)

   The only states which can be reached starting with (5, 0) are:
   - (5, 0), (3, 2) and (1, 4)

and once again we notice that Red is always ODD.

In general, starting with (N, 0) where N is ODD, must Red always remain ODD?

The 'proof' revolves around the fact that if Red starts as ODD, any operation of leaving one counter unchanged will always change the Reds by an EVEN number hence leaving the number of Reds as ODD.
Starting with (N, 0), on the first move Red must go down to 1 (which is an odd number), and the number of Blues must be even (N - 1).
So (N, 0) becomes (1, N -1), and Red is still odd.
On the next move there are only two choices, keeping a Red or keeping a Blue. Keeping a Red merely changes it back to the original setting.
If we 'keep one Blue', consider the number of Reds we finish with after this move:

- The existing Red changes to Blue, so this reduces the Reds by 1.
- The existing number of Blues was EVEN, but since one is kept, an ODD number of Blues are changed back to Red so the Reds increase by an ODD number. So the total change in Reds is a decrease by 1 and an increase by an ODD number, which is a total change by an EVEN number, hence keeping the Reds as ODD.

At any stage in any odd numbered problem (N, 0) where N is ODD, if the state is (P, N - P) and P is odd, then (N - P) is necessarily even.

If you 'keep one Red' then:

- The P Reds reduces to 1, and
- The even (N - P) Blues all turn to Red increasing the Reds by (N - P). So the total number of Reds is now 1 + N - P

which must be ODD.
If you 'keep one Blue', then:

- The P Reds all change to Blue and there are 0 of these left.
- Of the even (N - P) Blues, (N - P) – 1 are changed to Red. So the total number of Reds is now 0 + (N - P) -1

which is ODD (since N - P was even).

Hence at no stage in the problem can the number of Reds be anything other than ODD.

4. The tree diagram will tend to help students keep track of the counters that don't turn, which is the critical part of the investigation. However it only becomes necessary (for some) to begin using this tool when there are 5 counters, because, for (N - 2):
   2 counters means turn 0, so the problem can't start.
   3 counters means turn 1 each time, so 3 moves will do it.
   4 counters means turn 2 each time, so 2 moves will do it.
   5 counters means turn 3 each time and it takes three moves.
   First don't change RR, then don't change RB, then don't change BB.
   The tree diagram not only illustrates this but shows why it is easy to 'lose the plot' if using a more guess and check approach.
5. These are open investigations, so responses will vary.

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Squares Around Squares

The focus of this task is the validation of alternative ways of visualising the generalisation. This is consistent with the mathematician's question:

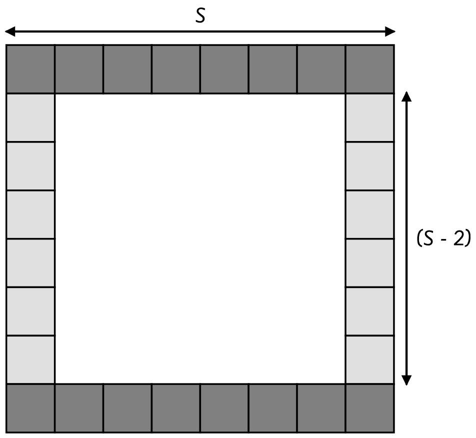
Can I check this another way?

If we don't encourage and validate alternative approaches to the same problem, then our students may become convinced that for any problem there is only *the way* to do it - and perhaps too frequently this comes to mean the teacher's way.

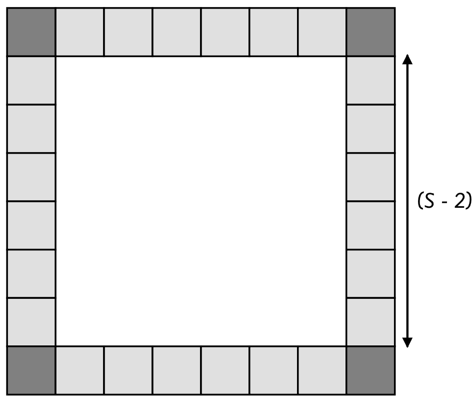
There is an Investigation Guide in Appendix 1 which extends the task into its iceberg.

Answers to Investigation Guide

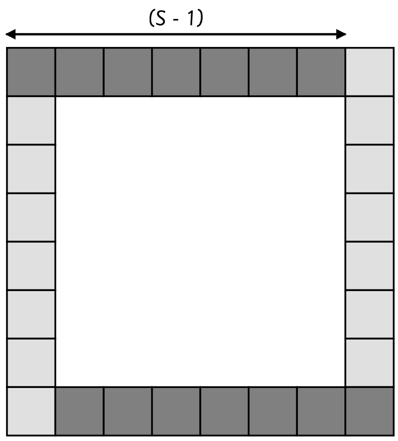
1. Student diagrams would be something like these:



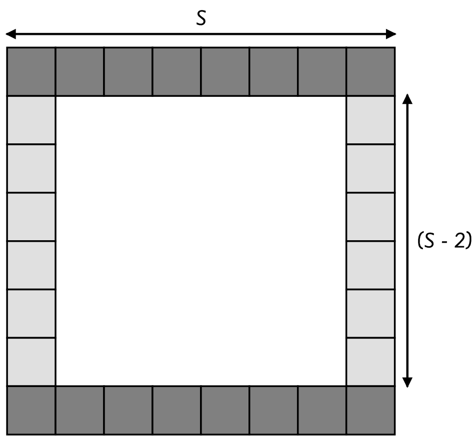
Angeline ... B = 2S + 2(S - 2)



Con ... B = 4(S - 2) + 4

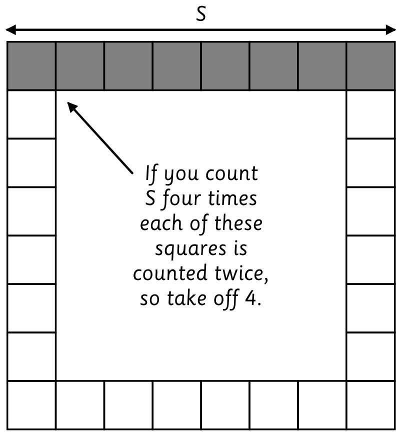


Ambrose ... B = 4(S - 1)



Tahlia ... B = 2[S + (S - 2)]

2. The visualisation in this question represents the difference between two squares. Like the formulas above $S^2 - (S - 2)^2$ can be simplified, and student explanations would, ideally, not only have the algebra to show that in all cases B = 4S - 4, but would also find a visualisation for this:



Simplified ... B = 4S - 4

3. Since the central colour in odd squares is only on the border every second round of squares, the size of the odd square matters:
   - First check whether the size number is of the form (4p + 1) or the form (4p + 3) where p is a whole number (0, 1, 2, 3...)
   - If (4p + 1), the sequence to be added is:
     $T_A = 1 + 16 + 32 + 40 + ...[4(S - 4) - 4] + (4S - 4)$
     $T_A = 1 + 16[1 + 2 + 3 + 4 + ... + {}^{(S - 1)}/_4]$
     The section in the brackets is the sum of natural numbers, and applying that formula appropriately leads to
     $T_A = 1 + {}^{(S - 1)(S + 3)}/_2$
   - If (4p + 3), use the result for the equivalent (4p + 1).
4. To find T_B, subtract T_A from S^2. So, for the form (4p + 1):
   $T_B = (S - 1)^2 \div 2$
5. The table which starts a similar chain of thinking for even squares is:

Size	2	4	6	8	10	12	14	16
Colour A	4	4	24	24	60	...	...	...

   - In this case one set of numbers is of the form (4p + 2) and the other (4p + 4) and for (4p + 2):
     $T_A = 4 + 20 + 36 + 52 + ... [4(S - 4) - 4] + (4S - 4)$
     $T_A = 4[1 + 5 + 9 + 13 + ... + (S - 1)]$
     $T_A = 4[{}^{S(S + 2)}/_8]$
     $T_A = {}^{S(S + 2)}/_2$

   Some students may have derived this by working with the table values anyway, but that is the point. We all see things in different ways and can all learn from each other.

Find more information about this task in the Task Cameo Library at:
- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Staircase
The crux of this task is how to sum the natural numbers which, in its turn is the crux of so many other challenges in this kit. The **Double Staircase** notes above and the Maths300 lesson *Staircases* provide all the necessary background

information. Emphasise the section on the card which encourages students to think about the sum of the numbers in more than one way, and encourage *What happens if...?* questions like:

What happens if the staircase goes up 2 blocks at a time?

All the formulas related to Arithmetic Sequences can be developed by following this thinking through.

Find more information about this task in the Task Cameo Library at:

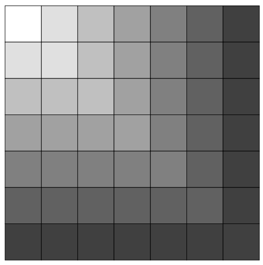
- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Tetrahedron Triangles

Again we have a task with a very visual/concrete basis leading to mathematics which has integrity in terms of the expectation of traditional school curriculum at this level. The Investigation Guide (Appendix 1) picks up where the task leaves off by stating the result of the challenges on the card and introducing the next question. A mathematician is never finished with a problem. They are only finished with it 'for now'.

The Guide leads further into traditional curriculum by including exercises in substitution into a quadratic, solution of quadratic equations, graphing a quadratic and using a spreadsheet as a tool.

Answers to Investigation Guide



1. The triangles grow by adding on a new row with an odd number of unit triangles. So the total of unit triangles in a larger equilateral triangle of size N is the sum of the first N odd numbers. Adding consecutive odd numbers from 1 always results in a square:
2. 324, 576, 1600, 2500, 10000, 4 million
3. 7, 11, 15, 28
   1440 - mindless application of the formula produces 18.97, which students need to realise makes no sense in the context. An appropriate answer would be Size 18 with 144 triangles left over, or perhaps 4 more triangles needed to make Size 19.
   1500 - Size 19 with 56 triangles left.
4. Instructions might be something like:
   - Find the nearest multiple of 4 less than or equal to your number. (Use your knowledge of divisibility tests - see Number & Computation kit).
   - Divide this by 4. If you get a perfect square, the square root is the size of the tetrahedron.
   - If you don't get a perfect square repeat these steps until you do.
5. Students are being asked to graph the parabola $T = 4S^2$. However, in the context the acceptable parts of this graph are those within the domain of whole numbers. By 'dotting in' the curve students can give visual explanation to the 'any number of triangles' situation explored in question 4.
6. The spreadsheet exercise is an application of two formulas. $T = 4S^2$ and the sum of the square numbers which result. It is an approach which suits the mathematical purposes of many real world situations. That is, the approach gives an answer to any sequence of sizes asked of it, but it doesn't provided one 'rule' which covers every case.

Symbolically the same problem could be tackled using:
$N = 4 \times 1^2 + 4 \times 2^2 + 4 \times 3^2 + \ldots + 4 \times S^2$
$N = 4(1^2 + 2^2 + 3^2 + \ldots + S^2)$
but how do we sum the square numbers? This is taken up in the lesson *Pyramid Puzzle & Other Algebra Excursions*, which is suggested as a plenary set of lessons for the algebra course in Years 9 & 10.

Tower Of Hanoi

As indicated on the card, this puzzle has been part of human culture for a very long time. Its Euro-centric history records it as being invented by the French mathematician Edouard Lucas and being first marketed as a toy in 1883. However, Lucas was no doubt influenced by an older Hindu legend usually know as the Tower of Brahma. There are now many software applets available on the Web which simulate the puzzle, and thus provide the opportunity to turn this task into a whole class lesson. One of the nicest is at: http://www.mazeworks.com/hanoi

Teachers who prefer real hands-on as opposed to simulated hands-on can easily make multiple versions of the puzzle with different size washers on nails banged into wood off-cuts. In co-operation with the craft faculty, students may be able to make the school's class set.

The problem is an example of recursive mathematics, which is a study in its own right. Recursive formulas lend themselves to programming which is, no doubt, one reason why there are several interactive forms available.

The Investigation Guide (Appendix 1) leads the task into the iceberg which develops from the question *What happens if we change the number of discs?*. It also introduces the strategy of breaking the problem into parts.

Answers to Investigation Guide

1 - 5. The table suggests there are two parts to the problem, but really there are three. They are:

- moves to reveal the bottom disc
- one move to shift the bottom disc
- a repeat of moves to shift the pile back on top of the bottom disc

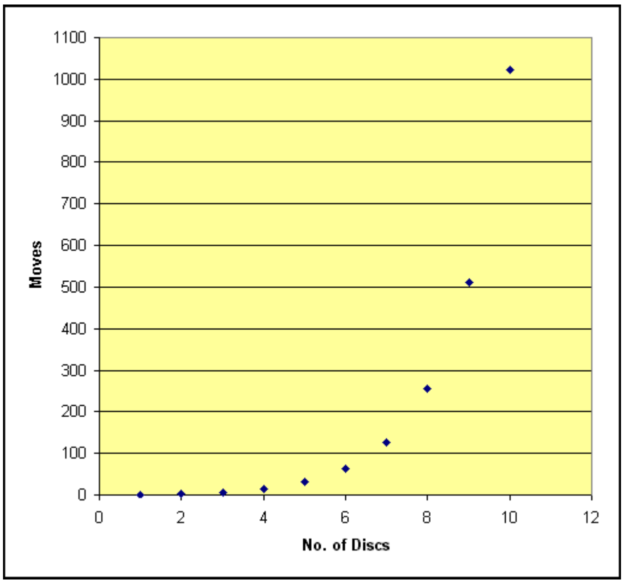
This is the explanation behind the 'plus 1' in Fatima's description. Perhaps the table is more instructive if presented as:

Number in Pile (N)	1	2	3	4	5	6	7	8
Moves to reveal bottom disc	0	1	3	7	15	31	63	127
Move for bottom disc	1	1	1	1	1	1	1	1
Moves after shifting bottom disc	0	1	3	7	15	31	63	127
Total moves (T)	1	3	7	15	31	63	127	255

The terms in the second and fourth rows are given by $2^{N-1} - 1$, so:

$T = [2^{N-1} - 1] + [2^{N-1} - 1] + 1$
$= 2^{N-1}(1 + 1) - 1 - 1 + 1$
$= 2^{N-1} \times 2 - 1$
$= 2^N - 1$

A graph of this function is shown.



Numbers of the form $2^N - 1$ represent the largest digit that can be written in binary number columns, in the same sense as numbers of the form $10^N - 1$ give the largest number in decimal number columns. This is the background to the 'If you have time' challenge on the sheet.

6.

Discs	Moves	Time
10	1023	17.05 hours
12	4095	68.25 hours
13	8191	136.5 hours
20	1,048,575	728.2 days
30	1,073,741,823	2042.9 years
50	1.126×10^{15}	approximately 2.14 billion years

7. If the monks used the computer simulation it would still take approximately: 4×10^{19} years to shift 100 discs. How old is the earth supposed to be?

Find more information about this task in the Task Cameo Library at:

- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Triangles & Colours

Here is another case of algebra appearing in unexpected places. The problem appears to be one of combinatorics, which it is, but asking the question *What happens if we change the number of colours?* opens an extensive investigation, the results of which can be generalised. The task only asks the students to predict the number of triangles for five colours (which is 35), but opens the door to increasing the number of colours to gather more data and look for a pattern.

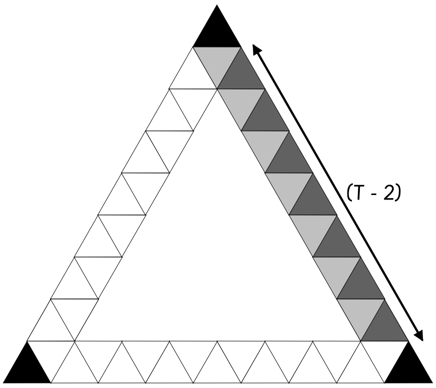
The whole class investigation in the companion Maths300 lesson uses many more sticks in more colours to model how a mathematician would work towards being able to predict the number of triangles for any number of colours. The rule is a relatively complex cubic and, paradoxically, is probably better understood in its non-simplified form.

Triangles Around Triangles

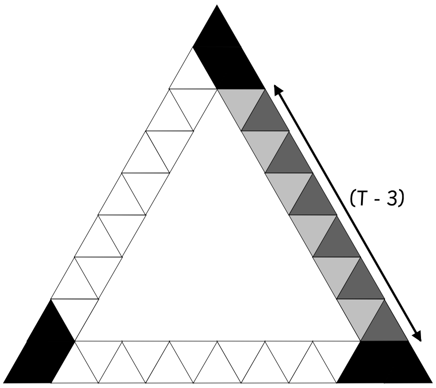
This task is a partner to **Squares Around Squares** and also emphasises validation of alternative ways of visualising the generalisation. There is an Investigation Guide in Appendix 1 which extends the task into its iceberg.

Answers to Investigation Guide

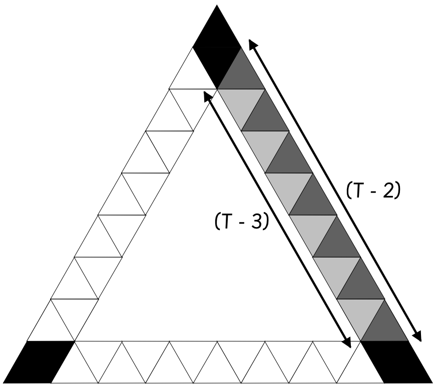
1. Student diagrams would be something like these:



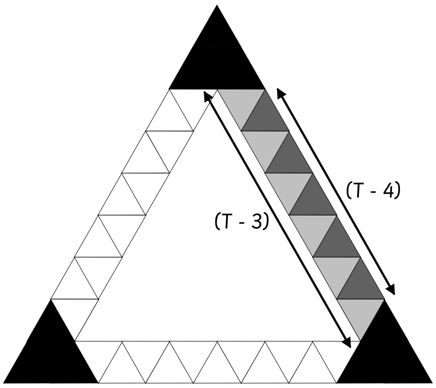
Conrad ... B = 3[2(T - 2)] + 3



Ina ... B = 3[2(T - 3)] + 3 x 3

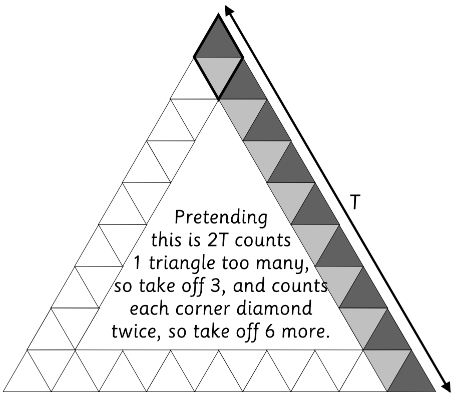


Ulla ... B = 3(T - 2) + 3(T - 3) + 3 x 2



Dominic ... B = 3(T - 4) + 3(T - 3) + 3 x 4

2. The visualisation in this question represents the difference between two squares. Like the formulas above $T^2 - (T - 3)^2$ can be simplified, and student explanations would, ideally, not only have the algebra to show that in all cases B = 6T - 9, but would also find a visualisation for this:



Simplified ... B = 6T - 9 = 3(2T - 3), T > 1

3. In G_1 triangles, the border colour and the central colour match on every second round of triangles, so the size of the G_1 triangle matters:
   - ♦ First check whether the size number is of the form (6p + 1) or the form (6p + 4) where p is a whole number (0, 1, 2, 3...)

- If (6p + 1), the sequence to be added is:
  $N_A = 1 + 33 + 69 + 105 + ... + 3(2T - 3)$
  $N_A = 1 + 3[11 + 23 + 35 ... + (2T - 3)]$
  The sequence in the brackets is an arithmetic progression (or Staircase) with a common difference of 12. We know this can be summed by adding the first and the last terms, multiplying by the number of terms and halving the result.
  We know the first and last terms, but how many terms are there?
  The count is given by the 'p' number, corresponding to T. Think of
putting the sequence:
  0, 1, 2, 3, 4, ...
  in one to one correspondence with the terms in N_A:
  1, 33, 69, 105, ...
  The 1, which is not included in what we are trying to count, matches with zero. The other terms match the 'p' number from which they were derived. So the number of terms in series = $^{(T - 1)}/_6$ and with appropriate algebraic manipulation:
  $N_A = 1 + {}^{(T - 1)(T + 4)}/_2$

  There is no doubt that this is sophisticated reasoning. In fact it is reasoning which parallels that of Andrew Wiles when he solved Fermat's Last Theorem. His proof relied on a way of counting particular sets, in the same way as the derivation relies on finding a way of counting the terms in the series for N_A.
- If (6p + 4), use the result for the equivalent (6p + 1).

4 To find N_B, subtract N_A from T^2. So, for the form (6p + 1):
  $T_B = {}^{(T - 1)(T - 2)}/_2$

5. Similar, albeit quite disciplined, reasoning will find ways to count the Colour A and Colour B triangles in G_2 and G_3 triangles.

Find more information about this task in the Task Cameo Library at:
- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Two Squares

This task is a partner to **Difference Between Two Squares**. It approaches the task from a measurement/arithmetic point of view and focuses on the evaluation of differences between square numbers. The other task attempts to develop a visual/transformational image of the concept involved. Through these two experiences students come to understand the reasoning which developed the tool:
- $a^2 - b^2 = (a - b)(a + b)$

which they will so often need to use. We trust this task is one more contribution to the hypothesis that algebra makes sense.

Find more information about this task in the Task Cameo Library at:
- http://www.mathematicscentre.com/taskcentre/iceberg.htm#cameos

Lesson Comments

- These comments introduce you to each Maths300 lesson. The complete plan is easily accessed through the lesson library available to members at:
  http://www.maths300.com
  where they are listed alphabetically by lesson name.

Algebra Charts

Learning to work like a mathematician involves structuring curriculum around modelling how a mathematician works, an invitation to personally put that model into practice and practising the skills of a mathematician. This lesson strongly supports the latter of these components.

Changing a series of closed text book questions into a puzzle chart format seems to be sufficient 'shift' to increase student involvement and enjoyment. Then adding a software environment with a timing feature seems to increase motivation even more. The software deserves repeated visits because it is accessible at multiple levels. Students can enter at a point where they feel successful, accept a personal challenge and leave feeling more confident to tackle a harder challenge the next time. Finally challenging students to design one of these types of charts adds yet another dimension and a potential assessment item.

This lesson is modelled heavily on the structure of *Number Charts* (Lesson 84) and, in fact, *Algebra Charts* evolved because of the reported success of *Number Charts*. It is strongly suggested that Number Charts be revisited as a prelude to this algebra version.

Crazy Animals

At first glance *Crazy Animals* is a Chance and Data lesson. Indeed it is likely the students have met it in this context in the past. This time however we meet the old friend at a fork in its development which looks at whole and part animals which might be formed. The lesson plan is recorded in the Whole & Part Appendix in the lesson. Animals are made at random and then classifying them into 3-part, 2-part, 1-part or 0-part giraffes creates a fascinating extended investigation involving probability, problem solving and finally algebra. The generalisation involved leads to the development of the expansion of $(N + 1)^3$.

Eric The Sheep

Eric is likely to be another old friend because it too, like *Crazy Animals*, is so rich in content that it can be used at many ages to achieve many different learning outcomes. At this level too, as you will see from some of the teachers' comments in the lesson plan, *Eric The Sheep* can mean different things to students of differing abilities. It may be that you have unconfident students at this level who need every bit of the concrete experience involved to be willing to make and test their hypotheses. Even your more able students are likely to first hypothesise a doubling relationship. Collecting more data shows this cannot be the case. On the other hand, it may be that with a different group a brief introductory activity puts you on your way to discussing discontinuous functions, domain and range. Or perhaps you challenge some students to find a general formula to cover all cases where there are **N** sheep in front of Eric and he sneaks past **P** of them each time the **S** shearers at the other end take their collection of sheep.

There is something in this problem for teachers at every level. If students claim *We've done this one*, you can ask them to recount what they remember, and invite them to continue learning with: *Great, let's see what we can learn from it this time.*

Find My Pattern

Based on a task invented by Year 4 students, this lesson presents challenges for much older students. It is easy enough to make up the two step patterns involved in the lesson (eg: patterns like + 6 followed by -7), but it is not as straightforward to analyse the resulting sequences to determine how the pattern was constructed. Doing so is more like the work of a mathematician, ie: hunting through data looking for the rules of an underlying pattern, rather than 'sitting at a desk' inventing patterns. The lesson involves application of numerical and graphical skills and can be introduced kinaesthetically. It has a particular supportive place in this kit because there are several challenging tasks (eg: **How Many Triangles?**) which produce patterns that are better understood by looking at them as the combination of two patterns, in a similar way to the patterns which are the focus of this lesson.

Jumping Kangaroos

Many teachers use *Jumping Kangaroos* as a way of introducing quadratic relationships. The algebra behind the number of moves is of the form:

$$y = x(x + 2)$$

but the approach to it can be through physical involvement, concrete small group work which develops a table of values and pattern recognition. Logic and facility with numbers is all that is necessary to reach the generalisation. One strategy which helps find a solution is to look one move ahead to avoid a situation where two kangaroos of the same type are together.

Once the generalisation is obtained (and this has often been achieved by older primary students) it can become a jumping off point for deeper investigation of the class of function involved, ie: quadratics. Perhaps the focus would be on the graphical form of the table of values, since students sometimes expect that graphing ordered pairs will result in a straight line.

Or you might ask students about the number of moves for various kangaroos each side, to involve them in quadratic substitution. Also, asking backwards questions, such as *There were 99 moves, how many kangaroos each side?*, you are introducing the solution of quadratic equations.

An additional merit of the lesson is that, within it, algebra makes sense. There is no need to stop with the symbolic generalisation. Indeed, to do so misses an opportunity. There *is* a way to give meaning to the formula above in terms of the movements of the kangaroos. Equally, in the context, there is a different meaning which can be found for the equivalent expression: $y = x^2 + 2x$

and indeed any other generalisation students develop to explain the puzzle. The lesson notes refer to all these aspects and include an investigation sheet to support the lesson.

Staircases

This lesson is largely self-directed using an investigation sheet that leads into the iceberg of the initial task. You will need linking cubes such as Multi-Link or Unifix, or wooden cubes, or square tiles to build staircases. The visual pattern of the steps in the staircase is a hint that the iceberg is an associated number/algebra pattern. The discovery of that pattern opens the door to further algebra and to a visual representation of the pattern in graphical form.

There are many such concrete, visual situations which link to algebra in this way. Students may have already experienced some which have a linear function basis. In this kit the focus on visualising algebra continues with an emphasis on quadratic relationships. *Staircases* is basic to much of the work in this unit because it establishes an expression for the sum of the natural numbers which is needed in so many of the other challenges in the kit.

Painted Cubes

Take a cube made of unit cubes and paint the outside surface of it. How many unit cubes have 0, 1, 2, 3, faces painted? This is an extraordinarily rich investigation that can succeed on many levels, especially if you provide cubes. The tactile, concrete counting challenge quickly exposes complex patterns which can be described algebraically and lead into cubic relationships. The lesson includes work on substitution and solution of these cubics and could easily be extended into a graphical dimension. The 3D context introduces a strong visual, as well as tactile, dimension to the learning. The small group collection and discussion of the data is also a strong feature. The lesson plan is supported by an investigation sheet.

Pyramid Puzzle

This lesson brings together all aspects of the work in this kit, and so has its place as a plenary sequence of lessons in the Year 10 Planner. Even though the lesson develops into a broad range of algebra which could be tackled symbolically, it is strongly suggested you begin the lesson with sufficient Pyramid Puzzles. Experience shows that even Year 12 students need to handle the material in order to get a grip on the number patterns involved. But the lesson goes far beyond that and at all stages it is supported by concrete material. For example, you will need plenty of linking cubes. It also offers strong links to the history of mathematics.

Triangles & Colours

With sufficient coloured pop-sticks (which are readily obtained from craft shops) the task is turned into a whole class investigation. You have five different coloured sticks. How many different coloured triangles can be made? The large scale colourful layout of triangles on each table makes this lesson very appealing. The group work, problem solving, algebraic patterns and extension possibilities make it mathematically 'rich'. One interesting aspect is the way the underlying algebraic rules are found and presented. The main rule students are likely to find for the number of triangles is:

$$n + n(n - 1) + {}^{[n(n - 1)(n - 2)]}/_{6}$$

In this form it is easily explainable, but if simplified to: $(n^3 + 3n^2 + 2n)/6$ the meaning is 'lost'. The simplest algebraic form is not necessarily the 'best'.

Part 3:
Value
Adding

The Poster Problem Clinic

Maths With Attitude kits offer several models for building a Working Mathematically curriculum around tasks. Each kit uses a different model, so across the range of 16 kits, teachers' professional learning continues and students experience variety. The Poster Problem Clinic is an additional model. It can be used to lead students into working with tasks, or it can be used in a briefer form as a opening component of each task session.

> *I was apprehensive about using tasks when it seemed such a different way of working. I felt my children had little or no experience of problem solving and I wanted to prepare them to think more deeply. The Clinic proved a perfect way in.*

Careful thought needs to be given to management in such lessons. One approach to getting the class started on the tasks and giving it a sense of direction and purpose is to start with a whole class problem. Usually this is displayed on a poster that all can see, perhaps in a Maths Corner. Another approach is to print a copy for each person. A Poster Problem Clinic fosters class discussion and thought about problem solving strategies.

Starting the lesson this way also means that just prior to liberating the students into the task session, they are all together to allow the teacher to make any short, general observations about classroom organisation, or to celebrate any problem solving ideas that have arisen.

One teacher describes the session like this:

> *I like starting with a class problem - for just a few minutes - it focuses the class attention, and often allows me to introduce a particular strategy that is new or needs emphasis.*

It only takes a short time to introduce a poster and get some initial ideas going. The class discussion develops a way of thinking. It allows class members to hear, and learn from their peers, about problem solving strategies that work for them.

> *If we don't collectively solve the problem in 5 minutes, I will leave the problem 'hanging' and it gives a purpose to the class review session at the end.*
> *Sometimes I require everyone to work out and write down their solution to the whole class problem. The staggered finishing time for this allows me to get organised and help students get started on tasks without being besieged.*
> *I try to never interrupt the task session, but all pupils know we have a five minute review session at the end to allow them to comment on such things as an activity they particularly liked. We often close then with an agreed answer to our whole class problem.*

A Clinic in Action

The aims of the regular clinic are:

- to provide children with the opportunity to learn a variety of strategies
- to familiarise children with a process for solving problems.

The following example illustrates a structure which many teachers have found successful when running a clinic.

Preparation

For each session teachers need:

- a Strategy Board as below
- a How To Solve A Problem chart as below
- to choose a suitable problem and prepare it as a poster
- to organise children into groups of two or three.

The Strategy Board can be prepared in advance as a reference for the children, or may be developed *with* the children as they explore problem solving and suggest their own versions of the strategies.

The problem can be chosen from

- a book
- the task collection
- prepared collections such as Professor Morris Puzzles which can be viewed at: http://www.mathematicscentre.com/taskcentre/resource.htm#profmorr

The example which follows is from the task collection. The teacher copied it onto a large sheet of paper and asked some children to illustrate it. *The teacher also changed the number of sheep to sixty* to make the poster a little different from the one in the task collection.

The Strategy Board and the How To Solve A Problem chart can be used in any maths activity and are frequently referred to in Maths300 lessons.

The Clinic

The poster used for this example session is:

> Eric the Sheep is lining up to be shorn before the hot summer ahead. There are sixty [60] sheep in front of him. Eric can't be bothered waiting in the queue properly, so he decides to sneak towards the front.
>
> Every time one [1] sheep is taken to be shorn, Eric <u>then</u> sneaks past two [2] sheep. How many sheep will be shorn <u>before</u> Eric?

This Poster Problem Clinic approach is also extensively explored in Maths300 Lesson 14, *The Farmer's Puzzle*.

Strategy Board

DO I KNOW A SIMILAR PROBLEM?	ACT IT OUT
GUESS, CHECK AND IMPROVE	DRAW A PICTURE OR GRAPH
TRY A SIMPLER PROBLEM	MAKE A MODEL
WRITE AN EQUATION	LOOK FOR A PATTERN
MAKE A LIST OR TABLE	TRY ALL POSSIBILITIES
WORK BACKWARDS	SEEK AN EXCEPTION
BREAK INTO SMALLER PARTS	...

How To Solve A Problem

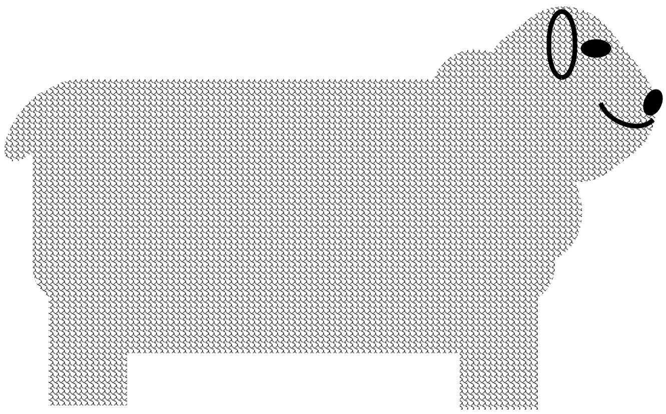
SEE & UNDERSTAND	Do I understand what the problem is asking? Discuss
PLANNING	Select a strategy from the board. Plan how you intend solving the problem.
DOING IT	Try out your idea.
CHECK IT	Did it work out? If so reflect on the activity. If not, go back to step one.

Step 1

- Tell the children that we are at Stage 1 of our four stage plan ... **See & Understand** ... Point to it! Read the problem with the class. Discuss the problem and clarify any misunderstandings.
- If children do not clearly understand what the problem is asking, they will not cope with the next stage. A good way of finding out if a child understands a problem is for her/him to retell it.
- Allow time for questions - approximately 3 to 5 minutes.

Step 2

- Tell the children that we are at Stage 2 of our four stage plan ... **Planning**. In their groups children select one or more strategies from the Strategy Board and discuss/organise how to go about solving the problem.
- Without guidance, children will often skip this step and go straight to Doing It. It is vital to emphasise that this stage is simply planning, not solving, the problem.
- After about 3 minutes, ask the children to share their plans.



Plan 1

Well we're drawing a picture and sort of making a model.

Can you give me more information please Brigid?

We're putting 60 crosses on our paper for sheep and the pen top will be Eric. Then Claire will circle one from that end, and I will pass two crosses with my pen top.

Plan 2

Our strategy is Guess and Check.

That's good Nick, but how are you going to check your guess?

Oh, we're making a model.

Go on ...

> *John's getting MAB smalls to be sheep and I'm getting a domino to be Eric and the chalk box to be the shed for shearing.*

Plan 3

> *We are doing it for 3 sheep then 4 sheep then 5 sheep and so on. Later we will look at 60.*

Great so you are going to try a simpler problem, make a table and look for a pattern.

This sharing of strategies is invaluable as it provides children who would normally feel lost in this type of activity with an opportunity to listen to their peers and make sense out of strategy selection. Note that such children are not given the answer. Rather they are assisted with understanding the power of selecting and applying strategies.

Step 3

- Tell the children that we are at Stage 3 of our four stage plan ... **Doing It**. Children collect what they need and carry out their plan.

Step 4

- Tell the children that we are at Stage 4 of our four stage plan ... **Check It**. Come together as a class for groups to share their findings. Again emphasis is on strategies.

> *We used the drawing strategy, but we changed while we were doing it because we saw a pattern.*

So Jake, you used the Look For A Pattern strategy. What was it?

> *We found that when Eric passed 10 sheep, 5 had been shorn, so 20 sheep meant 10 had been shorn ... and that means when Eric passes 40 sheep, 20 were shorn and that makes the 60 altogether.*

Great Jake. How would you work out the answer for 59 sheep or 62 sheep?

Sharing time is also a good opportunity to add in a strategy which no one may have used. For example:

> *Maybe we could've used the Number Sentence strategy, ie: 1 sheep goes to be shorn and Eric passes two sheep. That's 3 sheep, so perhaps, 60 divided into groups of 3, or 60 ÷ 3 gives the answer.*

Round off the lesson by referring to the Working Mathematically chart. There will be many opportunities to compliment the students on working like a mathematician.

Curriculum Planning Stories

Our attitude is:

> *teachers improve their teaching by re-enacting stories from the classrooms of their colleagues*

In more than a decade of using tasks and many years of using the detailed whole class lessons of Maths300, teachers have developed several models for integrating tasks and whole class lessons. Some of those stories are retold here. Others can be found at:

- http://www.mathematicscentre.com/taskcentre/plans.htm

Story 1: Threading

> *Educational research caused me a dilemma. It tells us that students construct their own learning and that this process takes time. My understanding of the history of mathematics told me that certain concepts, such as place value and fractions, took thousands of years for mathematicians to understand. The dilemma was being faced with a textbook that expected students to 'get it' in a concentrated one, two or three week block of work and then usually not revisit the topic again until the next academic year.*

A Working Mathematically curriculum reflects the need to provide time to learn in a supportive, non-threatening environment and...

> *When I was involved in a Calculating Changes PD program I realised that:*
>
> - *choosing rich and revisitable activities, which are familiar in structure but fresh in challenge each time they are used, and*
> - *threading them through the curriculum over weeks for a small amount of time in each of several lessons per week*
>
> *resulted in deeper learning, especially when partnered with purposeful discussion and recording.*

Calculating Changes:

- http://www.mathematicscentre.com/calchange

Story 2: Your turn

Some teachers are making extensive use of a partnership between the whole class lessons of Maths300 and small group work with the tasks. Setting aside a lesson for using the tasks in the way they were originally designed now seems to have more meaning, as indicated by this teacher's story:

> *When I was thinking about helping students learn to work like a mathematician, my mind drifted to my daughter learning to drive. She*

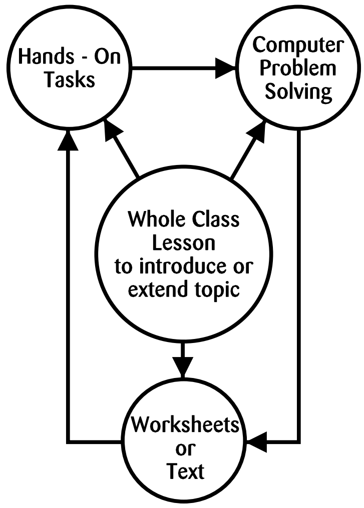
needed me to model how to do it and then she needed lots of opportunity to try it for herself.

That's when the idea clicked of using the Maths300 lessons as a model and the tasks as a chance for the students to have their turn to be a mathematician.

The Maths300 lessons illustrate how other teachers have modelled the process, so I felt I could do it too. Now the process is always on display on the wall or pasted inside the student's journal.

A session just using the tasks had seemed a bit like play time before this. Now I see it as an integral part of learning to work mathematically.

Story 3: Mixed Media



It was our staff discussion on Gardner's theory of Multiple Intelligences that led us into creating mixed media units. That and the access you have provided to tasks and Maths300 software.

We felt challenged to integrate these resources into our syllabus. There was really no excuse for a text book diet that favours the formal learners. We now often use four different modes of learning in the work station structure shown. It can be easily managed by one teacher, but it is better when we plan and execute it together.

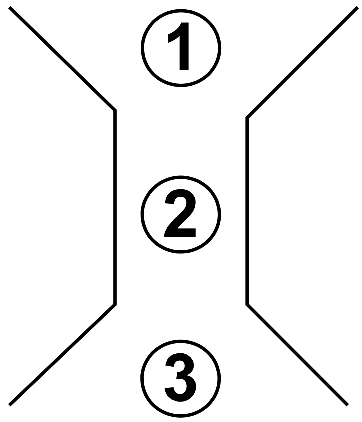
Story 4: Replacement Unit

We started meeting with the secondary school maths teachers to try to make transition between systems easier for the students. After considerable discussion we contracted a consultant who suggested that school might look too much the same across the transition when the students were hoping for something new. On the other hand our experience suggested that there needed to be some consistency in the way teachers worked.

We decided to 'bite the bullet' and try a hands-on problem solving unit in one strand. We selected two menus of twenty hands-on tasks, one for the primary and one for the secondary, that became the core of the unit. We deliberately overlapped some tasks that we knew were very rich and added some new ones for the high school.

Class lessons and investigation sheets were used to extend the tasks, within a three week model.

It is important to note that although these teachers structured a 3 week unit for the students, they strongly advised an additional *Week Zero* for staff preparation. The units came to be called Replacement Units.



Week Zero - Planning

Staff familiarise themselves with the material and jointly plan the unit. This is not a model that can be 'planned on the way to class'.

> *Getting together turned out to be great professional development for our group.*

Week 1 - Introduction

Students explore the 20 tasks listed on a printed menu:

- students explore the tip of the task, as on the card
- students move from task to task following teacher questioning that suggests there is more to the task than the tip
- in discussion with students, teachers gather informal assessment information that guides lesson planning for the following week.

> *We gave the kids an 'encouragement talk' first about joining us in an experiment in ways of learning maths and then gave out the tasks. The response was intelligent and there was quite a buzz in the room.*

Week 2 - Formalisation

> *It was good for both us and the students that the lessons in this week were a bit more traditional. However, they weren't text book based. We used whole class lessons based on the tasks they had been exploring and taught the Working Mathematically process, content and report writing.*
>
> *Assessment was via standard teacher-designed tests, quizzes and homework.*

Week 3 - Investigations

> *We were most delighted with Week 3. Each student chose one task from the menu and carried out an in-depth investigation into the iceberg guided by an investigation sheet. They had to publish a report of their investigation and we were quite surprised at the outcomes. It was clear that the first two weeks had lifted the image of mathematics from 'boring repetition' to a higher level of intellectual activity.*

Story 5: Curriculum shift

> *I think our school was like many others. The syllabus pattern was 10 units of three weeks each through the year. We had drifted into that through a text book driven curriculum and we knew the students weren't responding.*

Our consultant suggested that there was sameness about the intellectual demands of this approach which gave the impression that maths was the pursuit of skills. We agreed to select two deeper investigations to add to each unit. It took some time and considerable commitment, but we know that we have now made a curriculum shift. We are more satisfied and so are the students.

The principles guiding this shift were:

- Agree

  The 20 particular investigations for the year are agreed to by all teachers. If, for example, Cube Nets *is decided as one of these, then all the teachers are committed to present this within its unit.*

- Publish

  The investigations are written into the published syllabus. Students and parents are made aware of their existence and expect them to occur.

- Commit

  Once agreed, teachers are required to present the chosen investigations. They are not a negotiable 'extra'.

- Value

  The investigations each illustrate an explicit form of the Working Mathematically process. This is promoted to students, constantly referenced and valued.

- Assess

  The process provides students with scaffolding for their written reports and is also known by them as the criteria for assessment. (See next page.)

- Report

  The assessment component features within the school reporting structure.

A Final Comment

Including investigations has become policy.

Why? Because to not do so is to offer a diminished learning experience.

The investigative process ranks equally with skill development and needs to be planned for, delivered, assessed and reported.

Perhaps most of all we are grateful to our consultant because he was prepared to begin where we were. We never felt as if we had to throw out the baby and the bath water.

Assessment

Our attitude is:

stimulated students are creative and love to learn

Regardless of the way you use your **Maths With Attitude** resource, a variety of procedures can be employed to assess this learning.

Where these assessment procedures are applied to task sessions and involve written responses from students, teachers will need to be careful that the writing does not become too onerous. Students who get bogged down in doing the writing may lose interest in doing the tasks.

In addition to the ideas below, useful references are:

- http://www.mathematicscentre.com/taskcentre/assess.htm
- http://www.mathematicscentre.com/taskcentre/report.htm

The first offers several methods of assessment with examples and the second is a detailed lesson plan to support students to prepare a Maths Report.

Journal Writing

Journal writing is a way of determining whether the task or lesson has been understood by the student. The pupil can comment on such things as:

- What I learned in this task.
- What strategies I/we tried (refer to the Strategy Board).
- What went wrong.
- How I/we fixed it.
- Jottings - ie: any special thoughts or observations

Some teachers may prefer to have the page folded vertically, so that children's reflective thoughts can be recorded adjacent to critical working.

Assessment Form

An assessment form uses questions to help students reflect upon specific issues related to a specific task.

Anecdotal Records

Some teachers keep ongoing records about how students are tackling the tasks. These include jottings on whether students were showing initiative, whether they were working co-operatively, whether they could explain ideas clearly, whether they showed perseverance.

Checklists

A simple approach is to create a checklist based on the Working Mathematically process. Teachers might fill it in following questioning of individuals, or the students may fill it in and add comments appropriately.

Pupil Self-Reflection

Many theorists value and promote metacognition, the notion that learning is more permanent if pupils deliberately and consciously analyse their own learning. The deliberate teaching strategy of oral questioning and the way pupils record their work is an attempt to manifest this philosophy in action. The alternative is the tempting 'butterfly' approach which is to madly do as many activities as possible, mostly superficially, in the mistaken belief that quantity equates to quality.

> *I had to work quite hard to overcome previously entrenched habits of just getting the answer, any answer, and moving on to the next task.*

Thinking about *what* was learned *how* it was learned consolidates and adds to the learning.

When it follows an extensive whole class investigation, a reflection lesson such as this helps to shift entrenched approaches to mathematics learning. It is also an important component of the assessment process. On the one hand it gives you a lot of real data to assist your assessment. On the other it prepares the students for any formal assessment which you may choose to round off a unit.

Introduction

Ask students to recall what was done during the unit or lesson by asking a few individuals to say what *they* did, eg:

> *What did you do or learn that was new?*
> *What can you now do/understand that is new?*
> *What do you know now that you didn't know 1 (2, 3, ...) lesson ago?*

Continuing Discussion

Get a few ideas from the first students you ask, then:

- organise 5 -10 minute buzz groups of three or four students to chat together with one person to act as a recorder. These groups address the same questions as above.
- have a reporting session, with the recorder from each group telling the class about the group's ideas.

Student comments could be recorded on the board, perhaps in three groups.

Ideas & Facts	Maths Skills	Process (learning) Skills

If you need more questions to probe deeper and encourage more thought about process, try the following:

> *What new things did you do that were part of* how *you learned?*
> *Who uses this kind of knowledge and skill in their work?*

Student Recording

Hand out the REFLECTION sheet (next page) and ask students to write their own reflection about what they did, based on the ideas shared by the class. Collect these for interest and, possibly, assessment information.

REFLECTION

me looking at me learning

NAME: CLASS:

Working With Parents

Balancing Problem Solving with Basic Skill Practice

Many schools find that parents respond well to an evening where they have an opportunity to work with the tasks and perhaps work a task together as a 'whole class'. Resourced by the materials in this kit, teachers often feel quite confident to run these practical sessions. Comments from parents like:

> *I wish I had learnt maths like this.*

are very supportive. Letting students 'host' the evening is an additional benefit to the home/school relationship.

The 4½ Minute Talk

Charles Lovitt has considerable experience working with parents and has developed a crisp, parent-friendly talk which he shares below. Many others have used it verbatim with great success.

Why the Four and a Half Minute Talk?

When talking with parents about Problem Solving or the meaning of the term Working Mathematically, I have often found myself in the position, after having promoted inquiry based or investigative learning, of the parents saying:

> *Well - that's all very well - BUT ...*

at which stage they often express their concern for basic (meaning arithmetic) skill development.

The weakness of my previous attempts has been that I have been unable to reassure parents that problem solving does not mean sacrificing our belief in the virtues of such basic skill development.

One of the unfortunate perceptions about problem solving is that if a student is engaged in it, then somehow they are not doing, or it may be at the expense of, important skill based work.

This Four and a Half Minute Talk to parents is an attempt to express my belief that basic skill practice and problem solving development can be closely intertwined and not seen as in some way mutually exclusive.

(I'm still somewhat uncomfortable using the expression 'basic skills' in the above way as I am certain that some thinking, reasoning, strategy and communication skills are also 'basic'.)

Another aspect of the following 'talk' is that, as teachers put more emphasis on including investigative problem solving into their courses, a question arises about the source of suitable tasks.

This talk argues that we can learn to create them for ourselves by 'tweaking' the closed tasks that heavily populate out existing text exercises, and hence not be dependent on external suppliers. (Even better if students begin to create such opportunities for themselves.)

The Talk

In preparation, write the following graphic on the board:

CLOSED	OPEN	EXTENDED INVESTIGATION
		How many solutions exist? How do you know you have found them all?

I would like to show you what teachers are beginning to do to achieve some of the thinking and reasoning and communication skills we hope students will develop. I would like to show you three examples.

Example One: *6 + 5 = ?*

I write this question under the 'closed' label on the diagram:

CLOSED	OPEN	EXTENDED INVESTIGATION
$\begin{array}{r} 6 \\ +\,5 \\ \hline \end{array}$		How many solutions exist? How do you know you have found them all?

And I ask:

What is the answer to this question?

I then explain that:

We often ask students many closed questions such as 6 + 5 = ?

The only response the students can tell us is "The answer is 11." ... and as a reward for getting it correct we ask another twenty questions just like it.

What some teachers are doing is trying to *tweak* the question and ask it a different way, for example:

I have two counting numbers that add to 11. What might the numbers be?

[Counting numbers = positive whole numbers including zero]

I write this under the ‘open’ label on the diagram:

CLOSED	**OPEN**	**EXTENDED INVESTIGATION**
6 + 5 ___	? + ? 11	How many solutions exist? How do you know you have found them all?

What is the answer to the question now?

At this stage it becomes apparent there are several solutions:

The question is now a bit more open than it was before, allowing students to tell you things like 8 + 3, or 10 + 1, or 11 + 0 etc.

Let's see what happens if the teacher 'tweaks' it even further with the investigative challenge *or* extended investigation question:

How many solutions are there altogether?

and more importantly, and with greater emphasis on the second question:

How could you convince someone else that you have found them all?

Now the original question is definitely different - it still involves the skills of addition but now also involves thinking, reasoning and problem solving skills, strategy development and particularly communication skills.

Young students will soon tell you the answer is ‘six different ones’, but they must also confront the communication and reasoning challenge of convincing you that there are only six and no more.

Example Two: *Finding Averages*

Again, as I go through this example, I write it into the diagram on the board in the relevant sections.

The CLOSED question is: *11, 12, 13 - find the average*

Tweaking this makes it an OPEN question and it becomes:

I have three counting numbers whose average is 12. What might the numbers be?

Students will often say:

10, 12, 14 … or 9, 12, 15 … or even 12, 12, 12

After realising there are many answers, you can tweak it some more and turn it into an EXTENDED INVESTIGATION:

How many solutions exist? … AND …

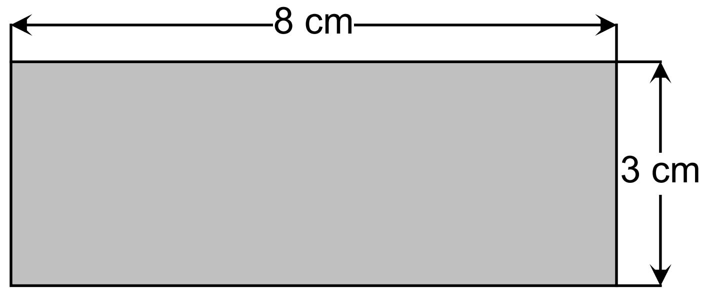
How do you know you have found them all?

Now the question is of a quite different nature. It still involves the arithmetic skill, but has something else as well - and that something else is the thinking, reasoning and communication skills necessary to find all of the combinations and convince someone else that you have done so.

By the time a student announces, with confidence, there are 127 different ways (which there are) that student will have engaged in all of these aspects, ie: the skill of calculating averages, (and some combination number theory) as well as significant strategy and reasoning experiences.

Example Three: *Finding the Area of a Rectangle*

A typical CLOSED question is:



Find the area. Find the perimeter.

The OPEN question is:

A rectangle has 24 squares inside:

What might its length and width be?

What might its perimeter be?

The EXTENDED INVESTIGATION version is:

Given they are whole number lengths, how many different rectangles are there? … AND …

How do you know you have found them all?

In summary, mathematics teachers are trying to convert *some* (not all) of the many closed questions that populate our courses and 'push' them towards the investigation direction. In doing so, we keep the skills we obviously value, but also activate the thinking, reasoning and justification skills we hope students will also develop.

This sequence of three examples hopefully shows two major features:

- That skills and problem solving can 'live alongside each other' and be developed concurrently.
- That the process of creating open-ended investigations can be done by anyone - just go to any source of closed questions and try 'tweaking' them as above. If it only worked for one question per page it would still provide a very large supply of investigations.

In terms of the effect of the talk on parents, I have usually found them to be reassured that we are not compromising important skill development (and nor do we want to). The only debate then becomes whether the additional skills of thinking, reasoning and communication are also desirable.

I've also been told that parents appreciate it because of the essential simplicity of the examples - no complicated theoretical jargon.

A Working Mathematically Curriculum

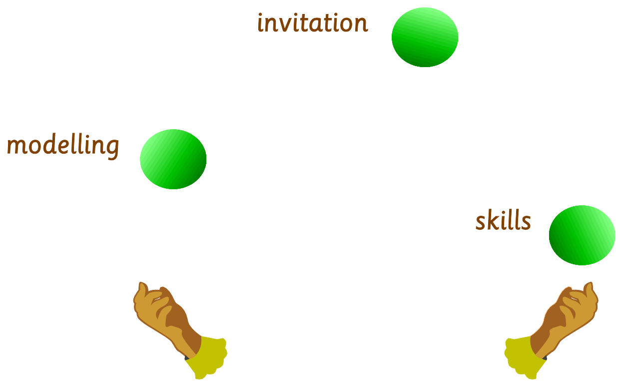
An Investigative Approach to Learning

The aim of a Working Mathematically curriculum is to help students learn to work like a mathematician. This process is detailed earlier (Page 8) in a one page document which becomes central to such a curriculum.

The change of emphasis brings a change of direction which *implies and requires* a balance between:

- the process of being a mathematician, and
- the development of skills needed to be a *successful* mathematician.

This journey is not two paths. It is one path made of two interwoven threads in the same way as DNA, the building block of life, is one compound made of two interwoven coils. To achieve a Working Mathematically curriculum teachers need to balance three components.



The task component of **Maths With Attitude** offers each pair of students an invitation to work like a mathematician.

The Maths300 component of **Maths With Attitude** assists teachers to model working like a mathematician.

Content skills are developed in context. They *are* important, but it is the application of skills within the process of Working Mathematically that has developed, and is developing, the human community's mathematical knowledge.

A focus for the Working Mathematically teacher is to help students develop mathematical skills in the context of problem posing and solving.

> *We are all 'born' with the same size mathematical toolbox, in the same way as I can own the same size toolbox as my motor mechanic. However, my motor mechanic has many more tools in her box than I and she has had more experience than I using them in context. Someone has helped her learn to use those tools while crawling under a car.*

Afzal Ahmed, Professor of Mathematics at Chichester, UK, once quipped:

> *If teachers of mathematics had to teach soccer, they would start off with a lesson on kicking the ball, follow it with lessons on trapping the ball and end with a lesson on heading the ball. At no time would they play a game of football.*

Such is not the case when teaching a Working Mathematically curriculum.

Elements of a Working Mathematically Curriculum

Working Mathematically is a K - 12 experience offering a balanced curriculum structured around the components below.

Hands-on Problem Solving Play

Mathematicians don't know the answer to a problem when they start it. If they did, it wouldn't be a problem. They have to play around with it. Each task invites students to play with mathematics 'like a mathematician'.

Skill Development

A mathematician needs skills to solve problems. Many teachers find it makes sense to students to place skill practice in the context of *Toolbox Lessons* which *help us better use the Working Mathematically Process* (Page 8).

Focus on Process

This is what mathematicians do; engage in the problem solving process.

Strategy Development

Mathematicians also make use of a strategy toolbox. These strategies are embedded in Maths300 lessons, but may also have a separate focus. Poster Problem Clinics are a useful way to approach this component.

Concept Development

A few major concepts in mathematics took centuries for the human race to develop and apply. Examples are place value, fractions and probability. In the past students have been expected to understand such concepts after having 'done' them for a two week slot. Typically they were not visited again until the next year. A Working Mathematically curriculum identifies these concepts and regularly 'threads' them through the curriculum.

Planning to Work Mathematically

The class, school or system that shifts towards a Working Mathematically curriculum will no longer use a curriculum document that looks like a list of content skills. The document would be clear in:

- choosing genuine problems to initiate investigation
- choosing a range of best practice teaching strategies to interest a wider range of students
- practising skills for the purpose of problem solving

Some teachers have found the planning template on the next page assists them to keep this framework at the forefront of their planning. It can be used to plan single lessons, or units built of several lessons. There are examples from schools in the Curriculum & Planning section of Maths300 and a Word document version of the template.

Unit Planning Page

Class

Topic

Pedagogy	Problem Solving	Skills
	In this topic how will I engage my students in the Working Mathematically process?	
How do I create an environment where students know what they are doing and why they have accepted the challenge?		Does the challenge identify skills to practise? Are there other skills to practise in preparation for future problem solving?

Notes

As a general guide:

- Find a problem(s) to solve related to the topic.
- Choose the best teaching craft likely to engage the learners.
- Where possible link skill practice to the problem solving process.

More on Professional Development

For many teachers there will be new ideas within **Maths With Attitude**, such as unit structures, views of how students learn, teaching strategies, classroom organisation, assessment techniques and use of concrete materials. It is anticipated (and expected) that as teachers explore the material in their classrooms they will meet, experiment with and reflect upon these ideas with a view to long term implications for the school program and for their own personal teaching.

Being explored 'on-the-job' so to speak, in the teacher's own classroom, makes the professional development more meaningful and practical for the teacher. This is also a practical and economic alternative for a local authority.

Strategic Use by Systems

From Years 3 - 10, **Maths With Attitude** is designed as a professional development vehicle by schools or clusters or systems because it carries a variety of sound educational messages. They might choose **Maths With Attitude** because:

- It can be used to highlight how investigative approaches to mathematics can be built into balanced unit plans without compromising skill development and without being relegated to the margins of a syllabus as something to be done only after 'the real' content has been covered.
- It can be used to focus on how a balance of concept, skill and application work can all be achieved within the one manageable unit structure.
- It can be used to show how a variety of assessment practices can be used concurrently to build a picture of student progress.
- It can be used to focus on transition between primary and secondary school by moving towards harmony and consistency of approach.
- It can be used to raise and continue debate about the pedagogy (art of teaching) that supports deeper mathematical learning for a wider range of students.

Teachers in Years K - 2 are similarly encouraged in professional growth through **Working Mathematically with Infants**, which derives from Calculating Changes, a network of teachers enhancing children's number skills from Years K - 6.

In supporting its teachers by supplying these resources in conjunction with targeted professional development over time, a system can fuel and encourage classroom-based debate on improving outcomes. There is evidence that by exploring alternative teaching strategies and encouraging curriculum shift towards Working Mathematically, learners improve and teachers are more satisfied. For more detail visit Research & Stories at:

- http://www.mathematicscentre.com/taskcentre/do.htm

We would be happy to discuss professional development with system leaders.

Web Reference

The starting point for all aspects of learning to work like a mathematician, including Calculating Changes, and the teaching craft which encourages it is:

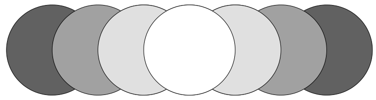
- http://www.mathematicscentre.com

Appendix 1: Investigation Guides

Can Stack - Investigation Guide

Do the task first. Use this Guide to find out more. Prepare a report.

Handy Harry's First Try



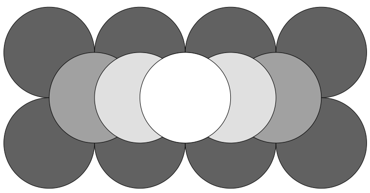
Looking from the top, the picture shows Handy Harry's first try at stacking. Investigate this stacking pattern and write a report on the following:

1. *Harry tells you the number of layers in the stack, eg: 4 or 5 or 6 or...*
   Explain how to work out the number of cans he needs.
2. *Harry tells you the number of cans he has.*
   Explain how to work out the number of layers he can build.
3. *Write a formula to connect the number of layers (L) with the number of cans (C).*
   Explain anything special about the numbers Harry can use in the formula.
4. *Draw a graph showing the cans for the first 10 layers.*
   You can use graph paper, a graphing calculator or a spreadsheet.
   Comment on what you see.

The Boss's Change

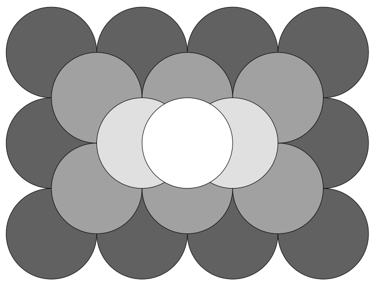
Harry's stacking method was unstable. The task card shows how the boss started off a new stack for him. But did the boss want him to keep stacking...

...like this...



Method 1

...or like this?



Method 2

"Build two rows on the bottom first Harry. They can be as long as you like. Then you can stack your way. That will make the stack more stable."

"Choose the size of the bottom layer first. The number of rows should be one less than the number of columns. Then stand every can above on four other cans. Except the top one. You just put that there to finish off."

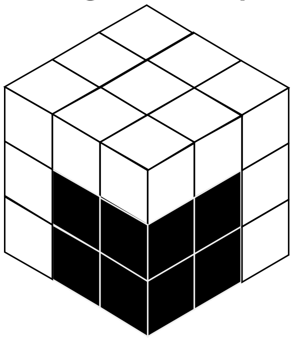
5. *For both of these methods report on challenges 1 to 4 above.*

If you have time, design a can stacking system of your own.
Report on challenges 1 to 4 above for your method.

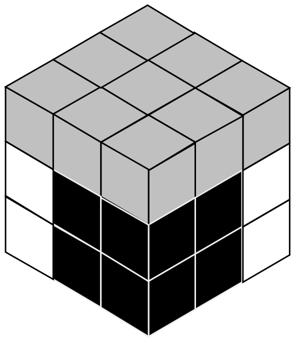
Cube Numbers - Investigation Guide

Do the task first. Use this Guide to find out more. Prepare a report.

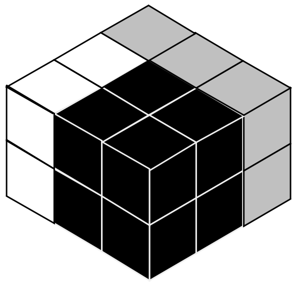
Taking Cubes Apart



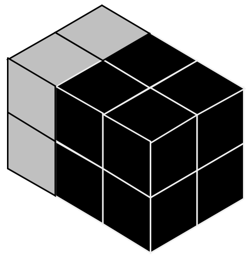
2^3 is inside $(2 + 1)^3$. Let's try to find it.



Slice off the top layer. This is 3^2.



Slice off a wall. This rectangle is $2(2 + 1)$.



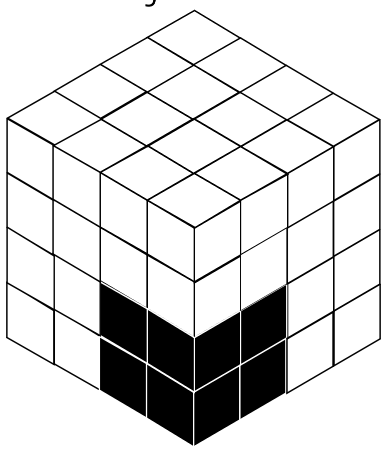
Slice off a square wall. This is 2^2.

You could slice in the same way for any size smaller cube, so, the smaller cube can be any size. Let's call it a^3. Then, working backwards from the smaller cube shows:
$a^3 + a^2 + a(a + 1) + (a + 1)^2 = (a + 1)^3$.

Taking Cubes Apart Another Way

In simplest form the expression above becomes: $a^3 + 3a^2 + 3a + 1 = (a + 1)^3$

1. *Use a size three cube made from the blocks to find:*
   - the three a^2 pieces
   - the three a pieces
   - the 1
2. *Use isometric paper to record how to take the cube apart to find the simplest form.*
3. *Work out the value of 1001^3 using both of the expressions above.*
   Comment on which one you would rather use for this purpose and why.
4. *Suppose the inner cube is 2 units smaller than the outer cube.*
   Imagine taking the cube apart to work out two expressions for $(a + 2)^3$.
   If you want to build the 4 x 4 x 4 cube you will need more cubes than are provided with the task.
5. *Suppose the inner cube is 3, or 4, or 5, ... units smaller than the outer cube.*
   Imagine taking these cubes apart until you can explain the simplest form of an expression for $(a + b)^3$.
   If you use linking cubes you can build sets of cubes, rectangle and square walls in different sizes to piece together.



If you have time, work out the simplest form of an expression for $(a - b)^3$.

Fold Up Houses - Investigation Guide

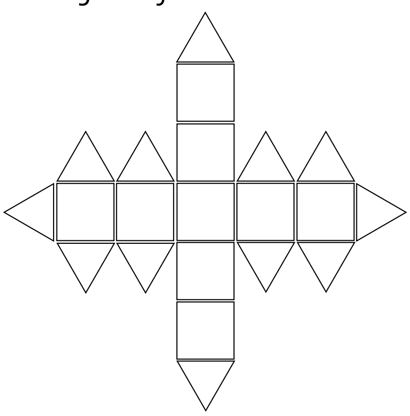
Do the task first. Use this Guide to find out more. Prepare a report.

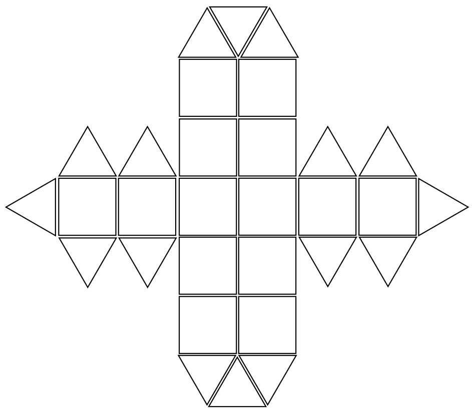
Helping The Workers

1. *Some workers learn best from words.*
   Write instructions for these workers to be able to work out the number of squares and triangles for any size fold-up house.
2. *Some workers learn best from symbols.*
   Write algebra rules for these workers to be able to work out the number of squares and triangles for any size fold-up house.
3. *Some workers learn best from numbers.*
   Make a table which makes it easy for these workers to look up the number of squares and triangles for fold-up houses up to Size 10.
4. *Some workers learn best from pictures.*
   Draw a graph which makes it easy for these workers to see the number of squares and triangles for fold-up houses up to Size 10.
5. *Use a table, a graph and algebra to find out about fold-up houses which have the same number of squares and triangles.*
6. *The company has 100 squares and 100 triangles in stock.*
   What is the largest fold-up house they could build from these?
   Are there any pieces left over? Explain.

Growing The Business

The company architect has invented a new product called Hulk Houses.
Hulk Houses are made on the ground like this and then folded up.
These are the layouts for the Size 1 and Size 2 Hulk Houses.





7. *Make the Size 2 Hulk House.*
8. *The architect asks you to prepare the Hulk House instructions for the workers.*
   Write your report for the Hulk House based on Challenges 1 - 6 above.
9. *Also in your report comment on the advantages and disadvantages of this design:*
   (a) In areas with heavy snow fall.
   (b) In areas with high average daily temperatures.

If you have time, design your own fold-up house and report on Challenges 1 - 6.

How Many Squares? - Investigation Guide

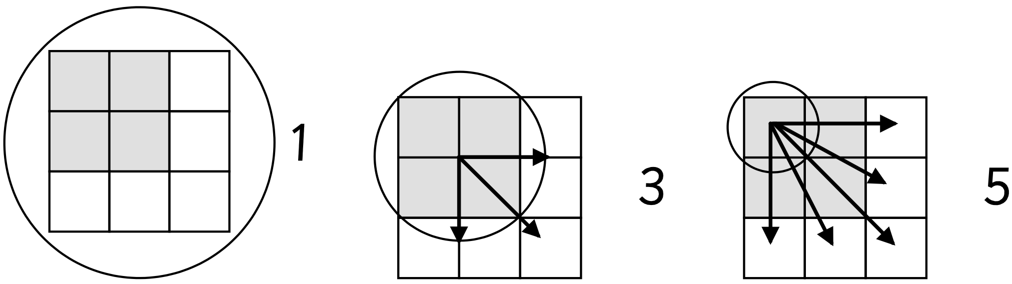
Do the task first. Use this Guide to find out more. Prepare a report.

Going Up A Square

1. Create a table which shows the number of 1x1, 2x2, 3x3, ... 8x8 squares for each size square up to S_8. Check your work against this table:

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
Total	1	5	14	30	55	91	140	204

2. Erin and Don noticed that each column in this table is made from the one before by adding on a square number, ie: $T_S = T_{S-1} + S^2$. But why?
   They started to explain it like this:
   - Each time you go up a size you add one column the same size as before, one row the same size as before and one extra corner square.
   - This makes 1 new square and new positions for sliding old squares. Example S_2 to S_3:



   Explore their ideas for other examples until you can explain what happens for any size. More odd numbers should appear. Explain how these odd numbers connect to S^2.
3. Use the formula $T_S = T_{S-1} + S^2$ to make a spreadsheet which works out the total of squares for sizes up to S_{20}.

A Different Formula

4. Your table in Q1 shows another way to work out the total for S_{20}. You could add up the first 20 square numbers. Use this method to check your spreadsheet answer for S_{20}.
5. The Ancient Greeks showed you could add up square numbers using this formula:

$$T_S = 1 + 4 + 9 + \ldots + S^2 = [S(S + 1)(2S + 1)] \div 6$$

   Check your spreadsheet answer for S_{20} using this formula.
   Use the formula to check each of the entries in the table in Q1.
6. Make a graph from the table in Q1. The points line on a shape called a cubic graph.

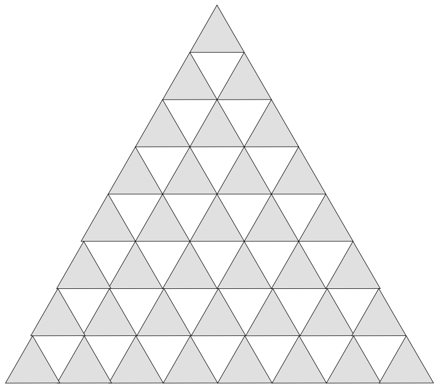
> If you have time, sketch an extension to your cubic graph to see how close it comes to the value you calculated for S_9.

How Many Triangles? - Investigation Guide

Do the task first. Use this Guide to find out more. Prepare a report.

Collecting Data

1. *Create a table which shows the number of Size 1, Size 2, Size 3 ... Size 8 triangles inside each large equilateral triangle up to E_8. It might make it easier for you to count if you look at triangles with point up first and then triangles with point down.*



Check your work against this table:

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
Total	1	5	13	27	48	78	118	170

2. *The Size 1 row in your table is easy to predict for any large triangle. For E_N, Size 1 (S_1) = N^2. But it's not so easy to find a rule to calculate Size 2 triangles for any large triangle. Try for a while to see if you can find one. It could help if you know how to use a spreadsheet to make a table, add a matching chart and add a trendline to the chart.*

Applying Strategies

3. *When one way doesn't work, mathematicians just try another way. In this case, since there are two types of triangle (point up/point down), perhaps the strategy of breaking a problem into smaller parts might help. For counting Size 2 triangles embedded in the largest equilateral triangles make a table which shows the point up count and the point down count for each large triangle. Eg:*

S_2 Triangles

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
Up		1	3	6				
Down		0	0	1				

4. *Use your table to predict the number of S_2 triangles for E_9 and E_{10}.*
5. *Explore S_1, S_3, S_4, S_5 triangles the same way.*
6. *Use what you learn to work out the total number of triangles for E_9 and E_{10}.*

If you have time, explore how you might find the total triangles for E_N.

Red To Blue - Investigation Guide

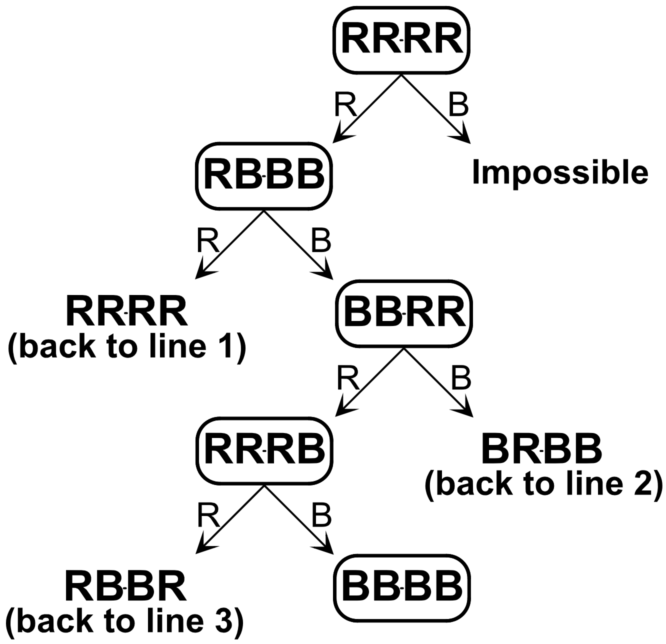
Reproducible Page

Do the task first. Use this Guide to find out more. Prepare a report.

N - 1

1. *The card uses the rule of turning all except one. That is called the (N - 1) case. For the (N - 1) case make a table to show the minimum number of moves for all starting numbers of counters from 1 to 10.*
2. *Explain to someone else how to predict the minimum number of moves for any starting number of counters. Explain in words and as an algebra rule.*
3. *Try to explain why the (N - 1) case is impossible if the number of counters is odd.*

This tree diagram is a way of exploring the (N - 1) case starting with 4 counters. The letters on the arrows show the colours that DO NOT turn over.



N - 2

4. *Use tree diagrams to explore the (N - 2) case. At each step it will be RR, RB or BB that DOES NOT turn. Your aim is to explain to someone else how to predict the minimum number of moves for any number of counters.*

Other Rules

5. *Explore (N - 3), (N - 4), (N - 5) and any other rules you like. AIM: If someone else tells you any number of counters to start and any rule, you will be able to tell them the minimum number of moves.*

If you have time, try to explain the Super Person game below.

Super Person

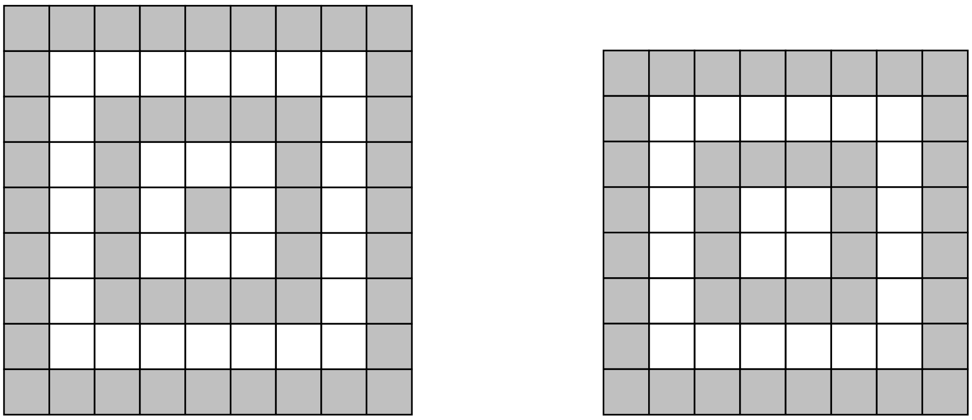
There is a party trick called Super Person that goes like this:

- *Super Person scatters a handful of coins on a table.*
- *A volunteer blindfolds Super Person.*
- *A volunteer is instructed to turn over any number of coins any number of times.*
- *The volunteer must call out Turn each time a coin is turned over.*
- *When finished, the volunteer is asked to cover only one coin with their hand and the blindfold is removed.*
- *Super Person has X-Ray vision and can immediately say whether the coin under the hand is heads or tails.*

What does Super Person have to know to be able to do this?

Squares Around Squares - Investigation Guide

Do the task first. Use this Guide to find out more. Prepare a report.



These pictures may help your thinking.
Use Square Dot paper if you need to make bigger pictures.

On The Border

1. The total number of squares (N) is easy to work out for any size square (S). $N = S^2$. But can you explain how these students pictured their rule to find the number of tiles in the square on the border (B)?
Draw a diagram to explain how each person might have been thinking.

Angeline	Con	Ambrose	Tahlia
$B = 2S + 2(S - 2)$	$B = 4(S - 2) + 4$	$B = 4(S - 1)$	$B = 2[S + (S - 2)]$

2. The border is what's left if you take the largest inside square (S - 2) away from the largest square (S). This makes the rule $B = S^2 - (S - 2)^2$. Write an explanation to convince someone else that these five rules all give the same answer for B.

Out To The Border

3. Odd squares have a single central square. This square has a colour - call it Colour A. Investigate the total number of Colour A squares for any size odd square. For example, you can see from the picture above that:

Size	1	3	5	7	9	11	13	15
Colour A	1	1	17	17	49	...	...	...

Challenge
If I tell you any size odd square, explain how to work out the number of Colour A squares.

4. The other squares are Colour B squares.
Challenge
If I tell you any size odd square, explain how to work out the number of Colour B squares.

5. Even squares have a central set of 4 squares. These squares have a colour - Colour A. Try the Colour A and Colour B challenges for even squares around squares.

Tetrahedron Triangles - Investigation Guide

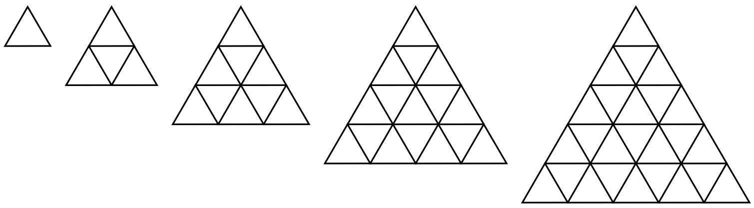
Do the task first. Use this Guide to find out more. Prepare a report.

Explaining The Formula

Zina explained the formula like this:
- *The size of the tetrahedron tells you the number of triangles along one edge.*
- *Square this number and you know the number of triangles in one face.*
- *Multiply that number by four because there are four faces.*
- *So, $T = S^2 \times 4$*

But why is the number of triangles in one face a square number?

1. *The size of each face grows row by row:*



Make a record of the number in each new row and explain how this makes the total of each face a square number.

Using The Formula

2. *Substituting*

Size	*Total Triangles*
9	
12	
20	
25	
50	
1000	

Copy & complete these tables

3. *Solving - Working Backwards*

Size	*Total Triangles*
	196
	484
	900
	3136
	1440
	1500

4. *Someone tells you they have any number of triangles and want to build a tetrahedron. Explain how to work out the size of the biggest tetrahedron they could make.*

Seeing The Formula

5. *Prepare a graph to help a person with a large collection of triangles work out the biggest tetrahedron they could build up to Size 20. Explain how to use your graph.*

Extending The Formula

6. *To build the Size 1 tetrahedron you need 4 triangles. To build Size 1 AND Size 2 you need 20 triangles - 4 for Size 1 and 16 for Size 2. Prepare a spreadsheet so that someone with a large collection of triangles can work out the longest sequence (Size 1 and Size 2 and Size 3 and...) of tetrahedrons they could make.*

Tower Of Hanoi - Investigation Guide

Do the task first. Use this Guide to find out more. Prepare a report.

Solving The Puzzle

The card tells you that the monks had spikes with 100 discs. Your puzzle has 5 discs. So, it must be okay to change the number of discs.

1. *Try the puzzle with starting piles of 1, 2, 3 and 4 discs.*
   Record the number of moves each time in a table like this:

Number in Pile (N)	1	2	3	4	5	6	7	8
Moves to shift bottom disc	1	2	4					
Moves after shifting bottom disc		1	3					
Total moves (T)	1	3	7					

Explaining The Puzzle

Fatima explained that you work out the total moves like this:

It all grows from the first disc. The next one is twice as many plus 1. The one after that is twice again plus 1. It keeps going on like that.

2. *Use your table and draw pictures to explain how she is thinking and why she is right.*
3. *Explain how you would work out the total number of moves if you knew the size of the starting pile.*
4. *Write a formula to connect the number of discs (N) with the total moves (T).*

Graphing The Formula

5. *Draw a graph (T, N) pairs up to N = 10. A spreadsheet might be the quickest way.*
   A graph which grows by multiplication in this way is said to grow exponentially.

Using The Formula

6. *The monks could move one disc per minute. Work out the time it would take to shift these piles of discs: 10 ... 12 ... 13 ... ... 20 ... ... 30 ... ... 50*
7. *Suppose the monks had a computer simulation of the puzzle which could make 1000 moves per second. Work out approximately how long it would take to move 100 discs.*

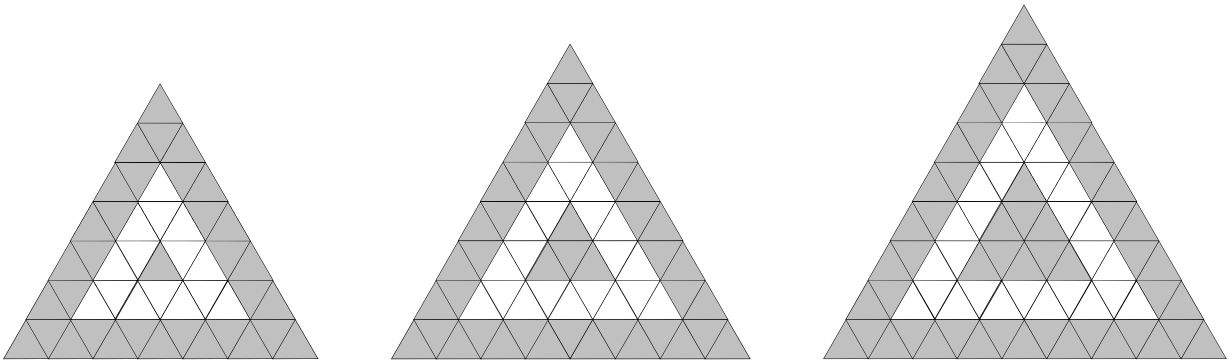
If you have time, try to explain how the table below tells you how to move for 4 discs.

	D	C	B	A
1				1
2			1	0
3			1	1
4		1	0	0
5		1	0	1
6		1	1	0
7		1	1	1
8	1	0	0	0

	D	C	B	A
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
12	1	1	0	0
13	1	1	0	1
14	1	1	1	0
15	1	1	1	1

Triangles Around Triangles - Investigation Guide

Do the task first. Use this Guide to find out more. Prepare a report.



These pictures may help your thinking.
Use Triangle Dot paper if you need to make bigger pictures.

On The Border

1. *The total number of tiles (N) is easy to work out for any size triangle (T).* $N = T^2$.
   But can you explain how these students pictured their rule to find the number of tiles in the triangle on the border (B)?
   Draw a diagram to explain how each person might have been thinking.

Conrad	Ina	Ulla	Dominic
$B = 3[2(T - 2)] + 3$	$B = 3[2(T - 3)] + 3 \times 3$	$B = 3(T - 2) + 3(T - 3) + 3 \times 2$	$B = 3(T - 4) + 3(T - 3) + 3 \times 4$

2. *The border is what's left if you take the largest inside triangle (T - 3) away from the largest triangle (T). This makes the rule* $B = T^2 - (T - 3)^2$. *Write an explanation to convince someone else that these five rules all give the same answer for B.*

Out To The Border

3. *Group 1 (G_1) triangles have a single centre triangle. This centre has a colour; Colour A. Investigate the total number of Colour A triangles for any G_1 triangle. For example, you can see from the picture above that:*

Size	1	4	7	10	13	16	19	22
Colour A	1	1	34	34	...	...	...	...

   Challenge
   If I tell you any size G_1 triangle, explain how to work out the number of Colour A triangles.
4. *The other triangles are Colour B triangles.*
   Challenge
   If I tell you any size G_1 triangle, explain how to work out the number of Colour B triangles.
5. *G_2 triangles have a Size 2 centre and G_3 triangles have a Size 3 centre. These centres have a colour - Colour A.*
   Try the Colour A and Colour B challenges for G_2 and G_3 triangles.

Appendix 2: Recording Sheets

Cube Numbers

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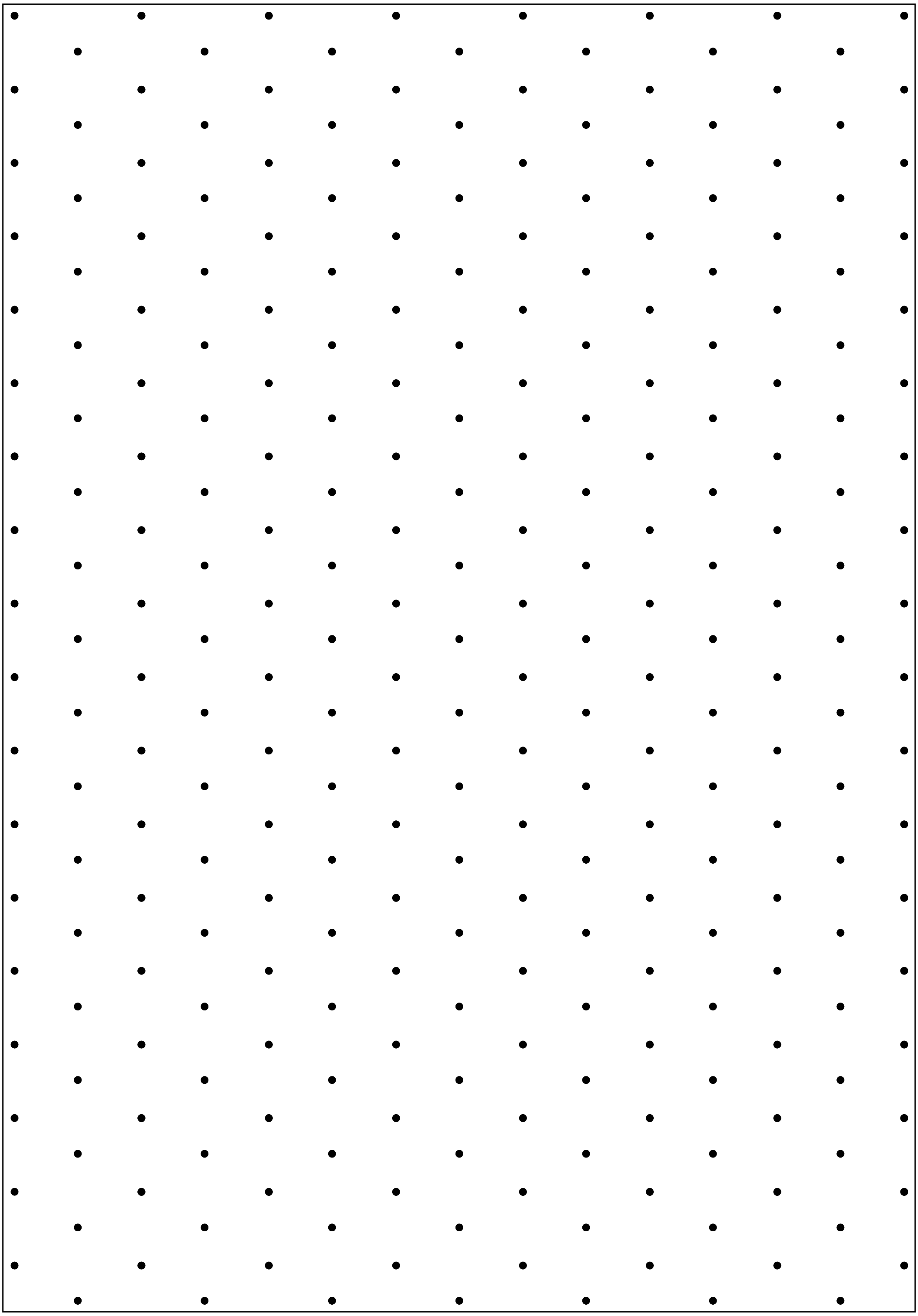

Names: .... Class: ....

How Many Squares?

Reproducible Page

Size of square (S)	1 x 1	2 x 2	3 x 3	4 x 4	5 x 5	6 x 6
No. tiles used						
No. squares formed (F)						

Names: .... Class: ....

How Many Squares?

Reproducible Page © Mathematics Task Centre

Size of square (S)	1 x 1	2 x 2	3 x 3	4 x 4	5 x 5	6 x 6
No. tiles used						
No. squares formed (F)						

Names: .... Class: ....

How Many Squares?

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Size of square (S)	1 x 1	2 x 2	3 x 3	4 x 4	5 x 5	6 x 6
No. tiles used						
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Names: .... Class: ....

How Many Squares?

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Size of square (S)	1 x 1	2 x 2	3 x 3	4 x 4	5 x 5	6 x 6
No. tiles used						
No. squares formed (F)						

Names: .... Class: ....

How Many Squares?

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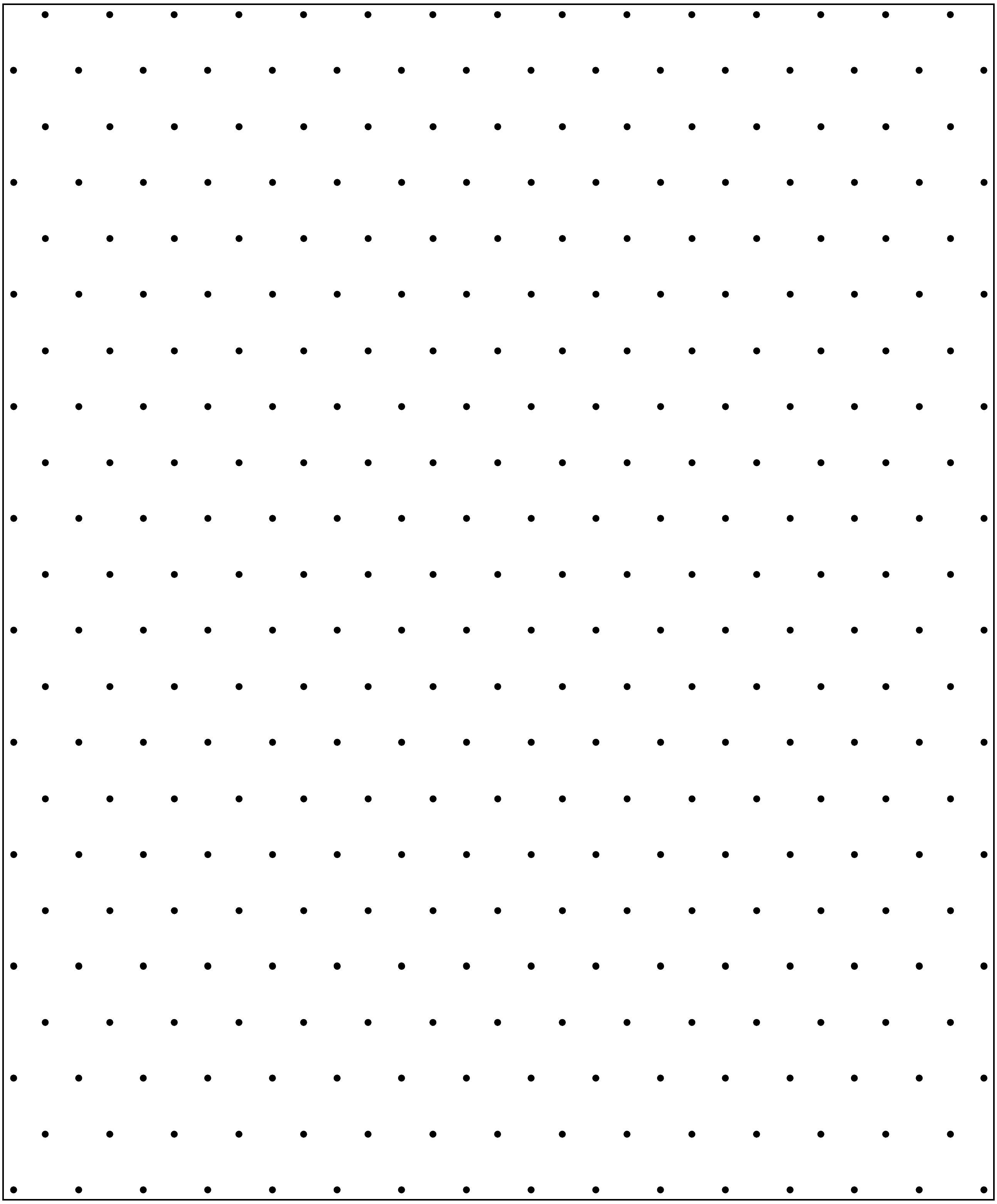
Size of square (S)	1 x 1	2 x 2	3 x 3	4 x 4	5 x 5	6 x 6
No. tiles used						
No. squares formed (F)						

Names: .... Class: ....

How Many Triangles?



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Squares Around Squares

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Tetrahedron Triangles

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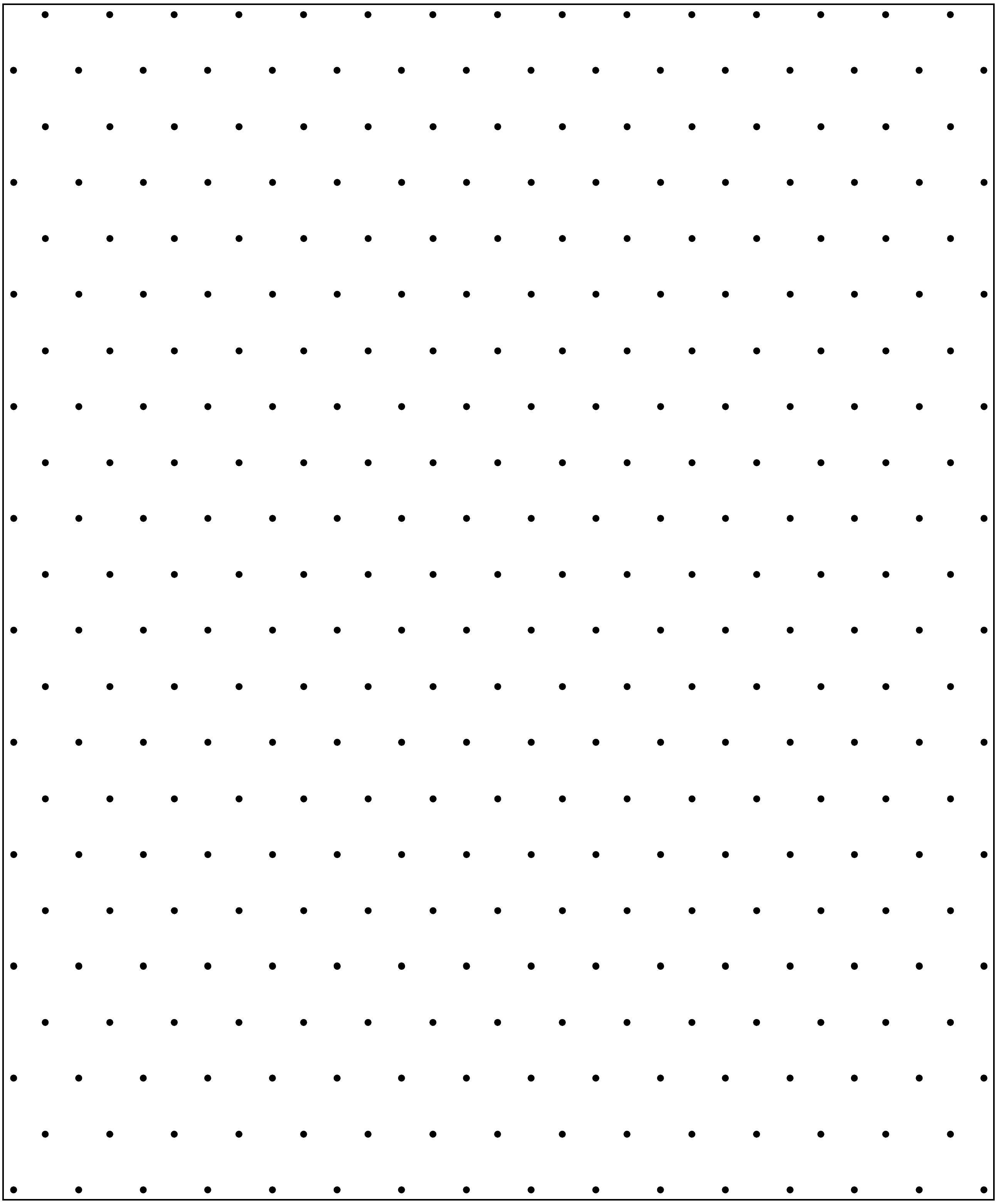

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Triangles Around Triangles

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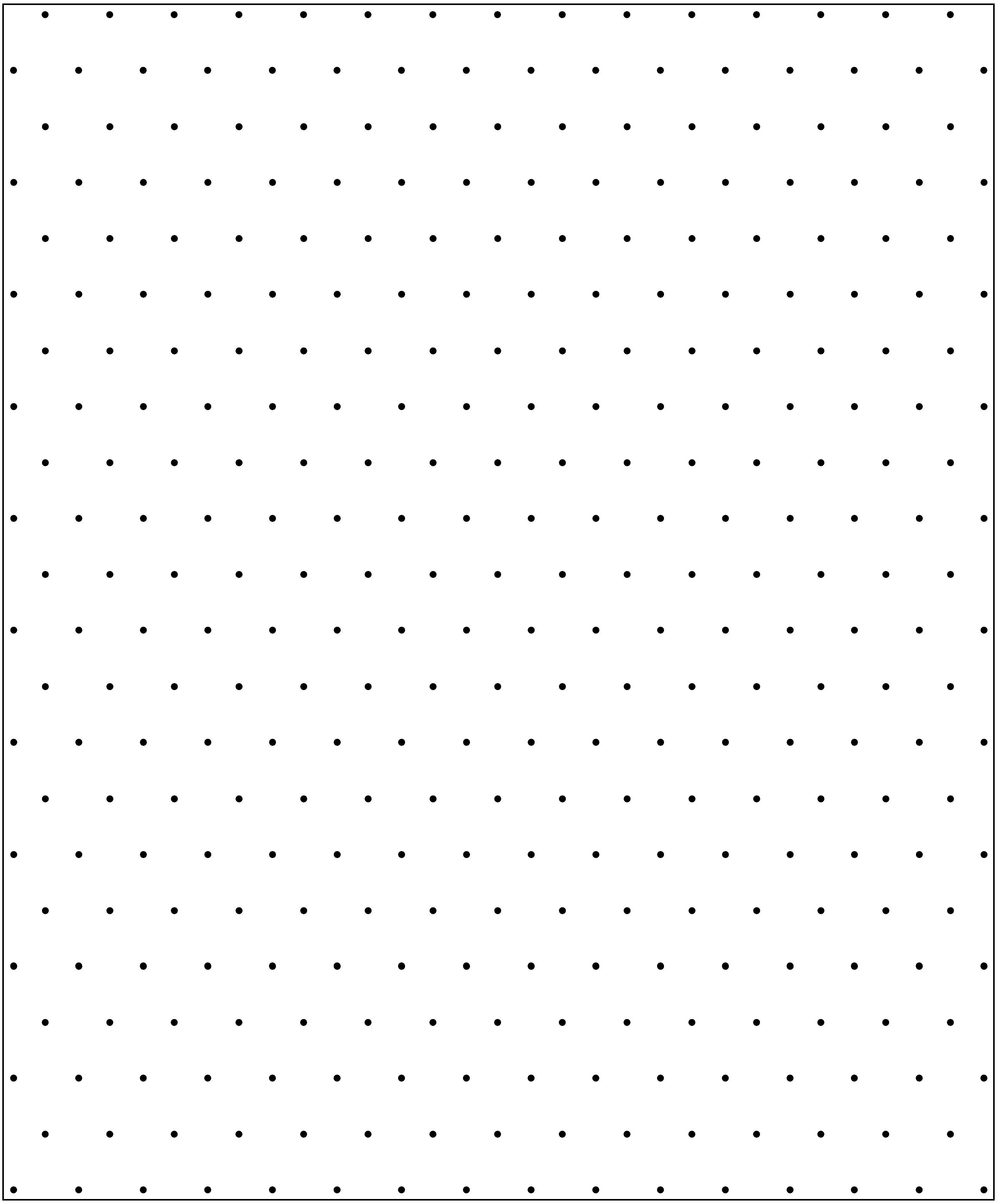

Names: .... Class: ....