INFORMATION-THEORETIC CHARACTERIZATION OF QUANTUM MEASUREMENTS

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MEASUREMENTS

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EXTRACT INFORMATION

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HOW TO QUANTIFY IT?

INFORMATION GAIN

THERMODYNAMIC COSTS (MEASUREMENT IMPLEMENTATION)

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INFORMATION-THEORETIC COSTS (MEASUREMENT SIMULATION)

THE STARTING POINT: POVMs

$$\rho \longrightarrow \boxed{\Lambda} \Longrightarrow X$$

$$\Lambda = \{\Lambda^x ; x \in X\}, \quad \Lambda^x \geq 0, \quad \sum_x \Lambda^x = I$$

$$X \sim p_X(x) = \Pr\{X = x\} = \mathrm{Tr}\{\rho \Lambda^x\}$$

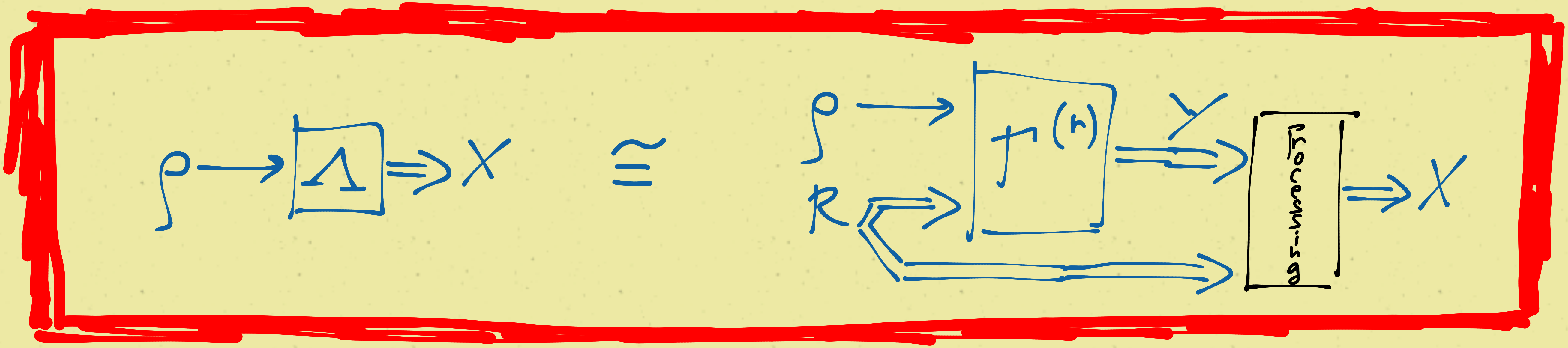
THE SET OF POVMs IS CONVEX!

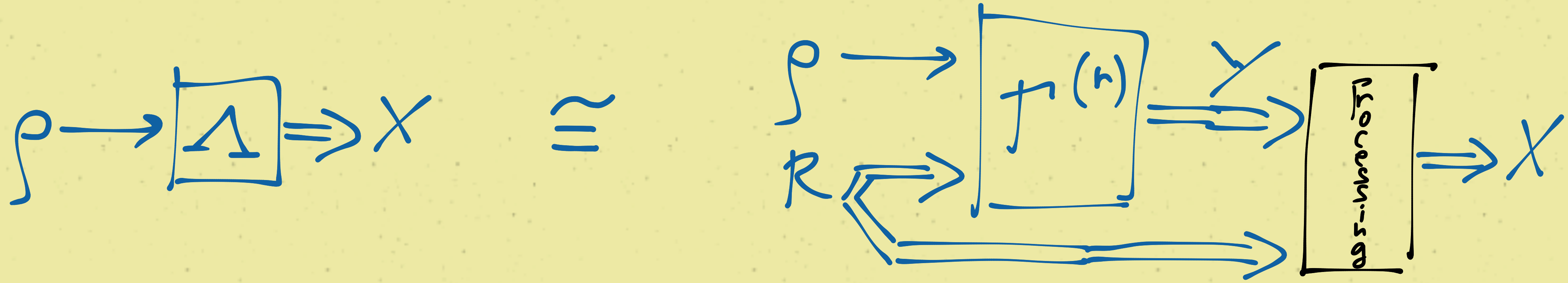
in general: $$\Lambda^{x} = \sum_{y\in Y}\sum_{r\in R} \overbrace{f(x|yr)}^{\text{possible classical post-processing}}\, p(r)\,\Gamma^{y|r}$$

two outcome (y,r) POVM

where $\Gamma^{y|r}\geq 0$, $\sum_{y\in Y}\Gamma^{y|r}=I$, $\forall r$

i.e.: $\Gamma^{(r)}=\{\Gamma^{y|r}\}_{y\in Y}$ is a POVM $\forall r$





WE SEE THAT THE OUTPUT RANDOM VARIABLE X CONTAINS (IN GENERAL) EXTRA-RANDOMNESS (REPRESENTED BY THE RANDOM VARIABLE R) WHICH IS COMPLETELY INDEPENDENT OF THE INPUT STATE.

$$\boxed{H(X) \overset{?}{=} \underbrace{I}_{\text{information}} + \underbrace{R}_{\text{extra-randomness}}}$$

BRIEF INCOMPLETE HISTORY OF INFORMATION GAIN IN Q.M.

INPUT STATE: ρ_Q POVM: $\{\Lambda^x\}_{x\in X}$

- GRÖNEWOLD (1971). PVM, LÜDERS

  $\rho \longrightarrow \rho_x = \Pi^x \rho \, \Pi^x \quad p_X(x)$

  $I := S(\rho) - \sum_x p_X(x) S(\rho_x)$

- LINDBLAD (1972). $I \geq 0$ for all Lüders measur.

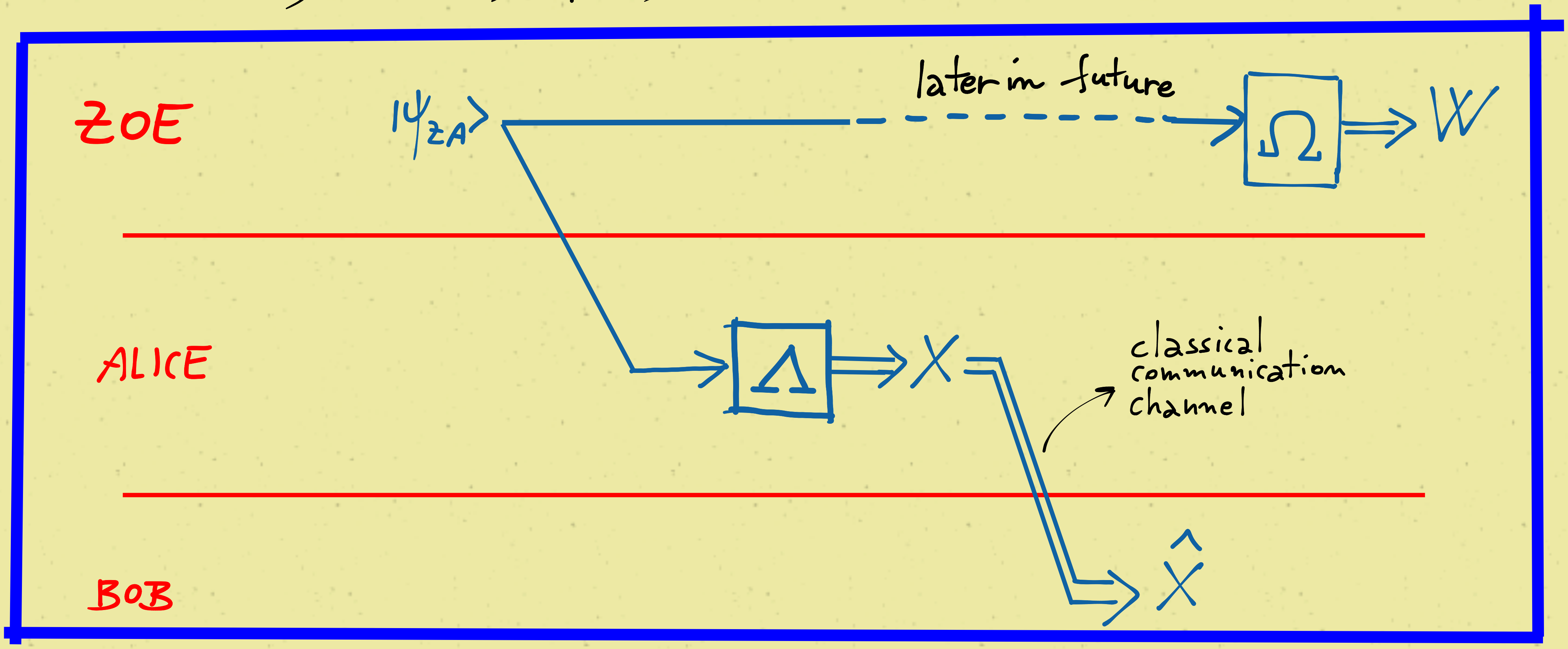
- OZAWA (1986). in general, $I \lesseqgtr 0$.

  necessary and sufficient conditions for $I \geq 0$.

- THE REST IS VERY RECENT STORY

MEASUREMENT SIMULATION

(Winter, 2004 ; Wilde, Hayden, FB, Hsieh, 2012)



DATA: joint state $|\psi_{ZA}\rangle$, POVM $\{\Lambda^x\}_{x\in X}$

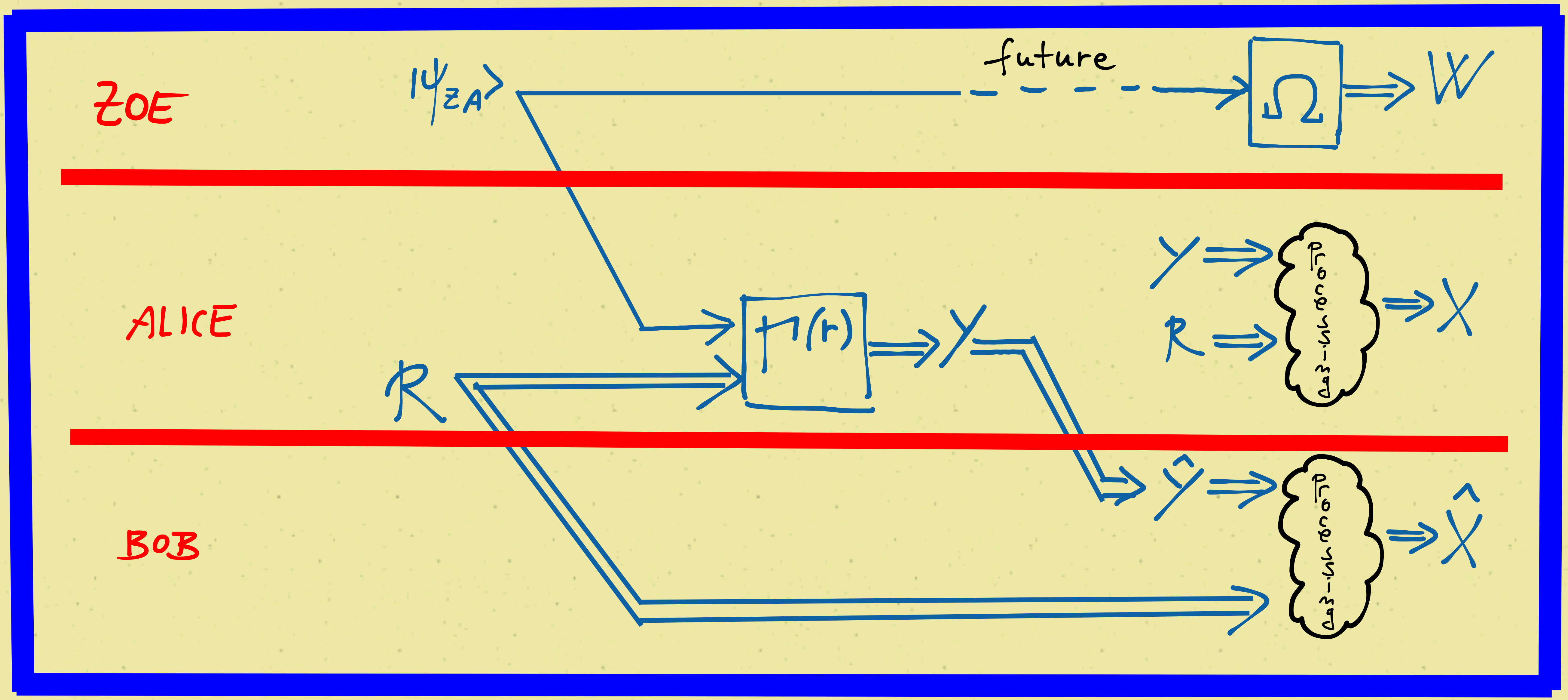
GOALS: • $\forall \{\Omega^w\}_{w\in W}$, $\Pr\{X=x, W=w\} = \Pr\{\hat{X}=x, W=w\}$ $\forall x, \forall w$

• $\Pr\{X=x, \hat{X}=x\}=1$, $\forall x\in X$ (feedback condition)

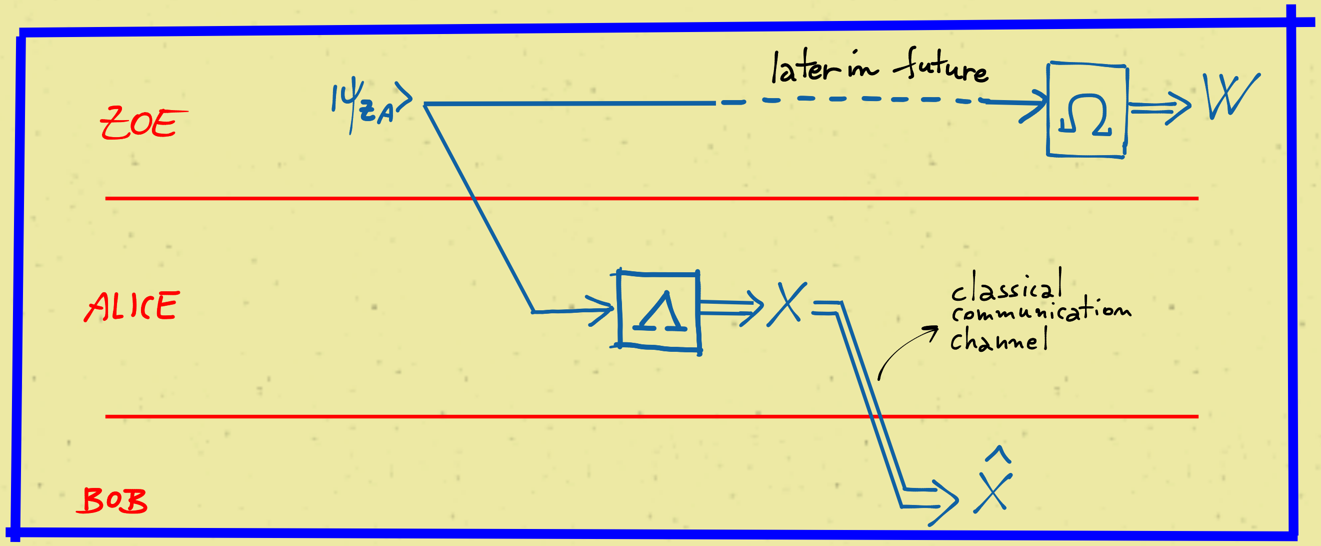
NAIF SOLUTION: zip $X \Longrightarrow H(X)$ bits need to be sent

EFFICIENT SIMULATION:

EXPLOITS CONVEXITY !!!

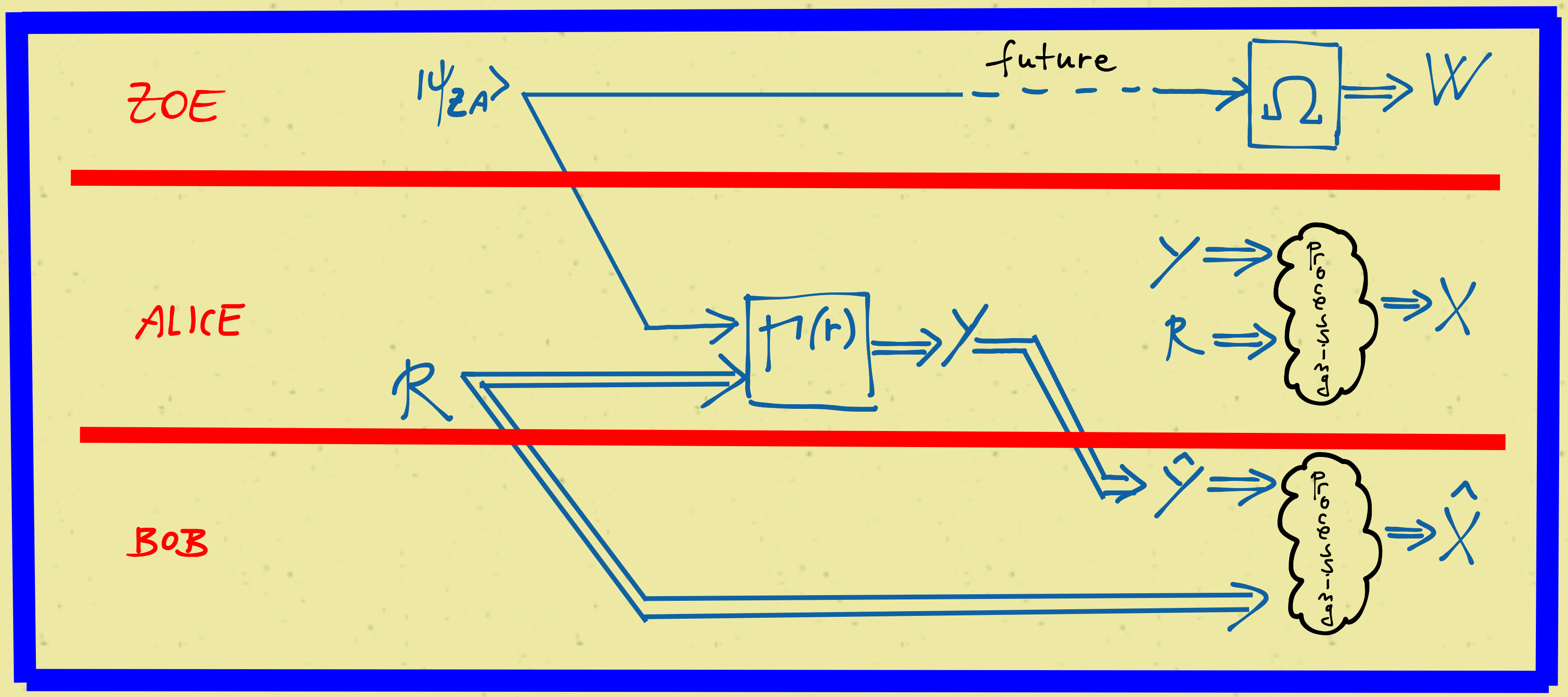


MAIN IDEA: EXTRA RANDOMNESS NEEDS NOT BE COMMUNICATED!



IDEAL SCENARIO

SIMULATED SCENARIO

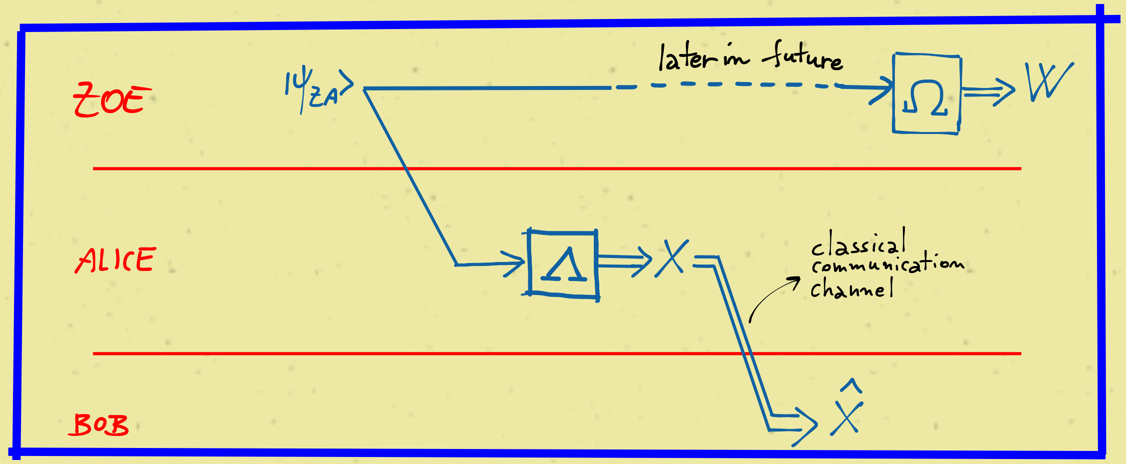


GOALS: • $\Pr\{W=w,\ \hat{X}=x\} = \Pr\{W=x,\ X=x\}$

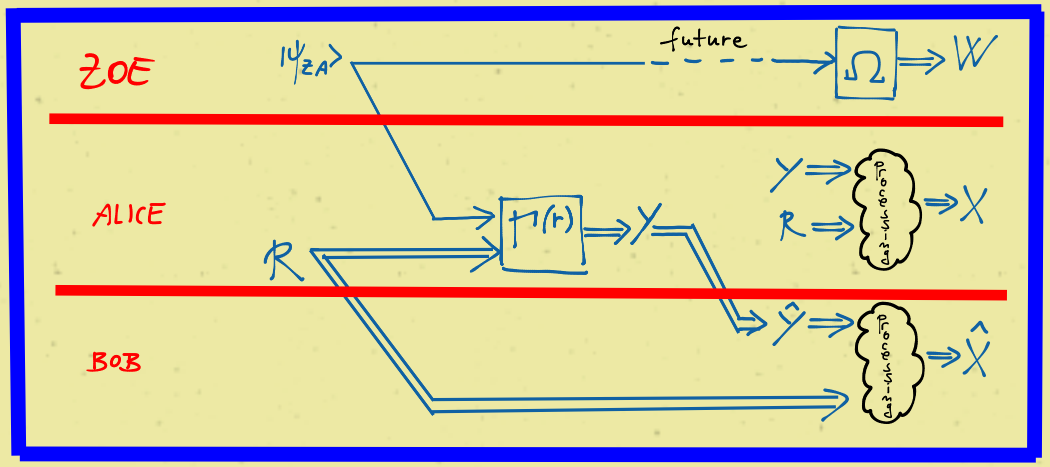
• $\Pr\{X=x,\ \hat{X}=x\}=1$ (feedback condition)

$\overset{\text{feedback}}{\Longrightarrow}$ deterministic post-proc's (permut.'s & grouping) only!

MEASUREMENT SIMULATION: RESOURCES COST



$\cong$

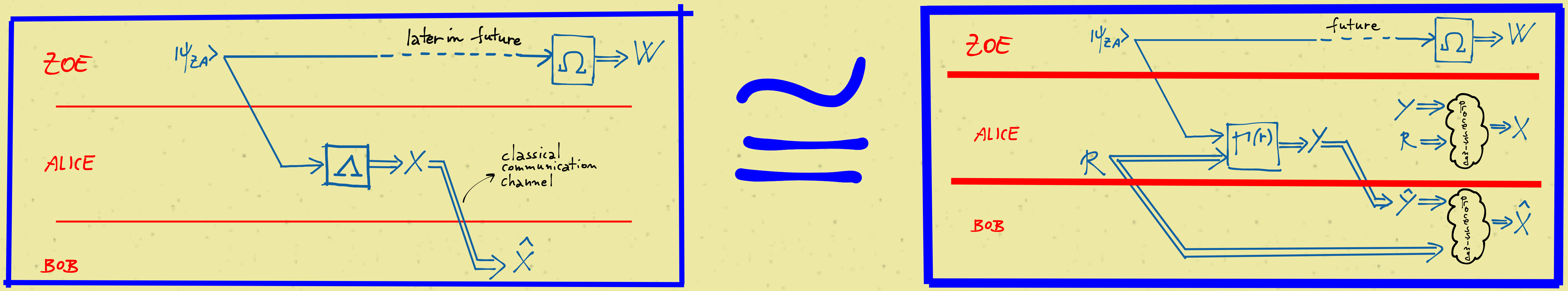


THE SIMULATION WITH FEEDBACK IS POSSIBLE IF AND ONLY IF:

1. $$H(Y) \geq I(Z:X) = S(\rho_Z) - \sum_x p_X(x) S(\rho_Z^x)$$

where $\rho_Z^x := [p_X(x)]^{-1} \operatorname{Tr}_A \{ (I_Z \otimes \Delta_A^x) |\psi_{ZA}\rangle\langle\psi_{ZA}| \}$

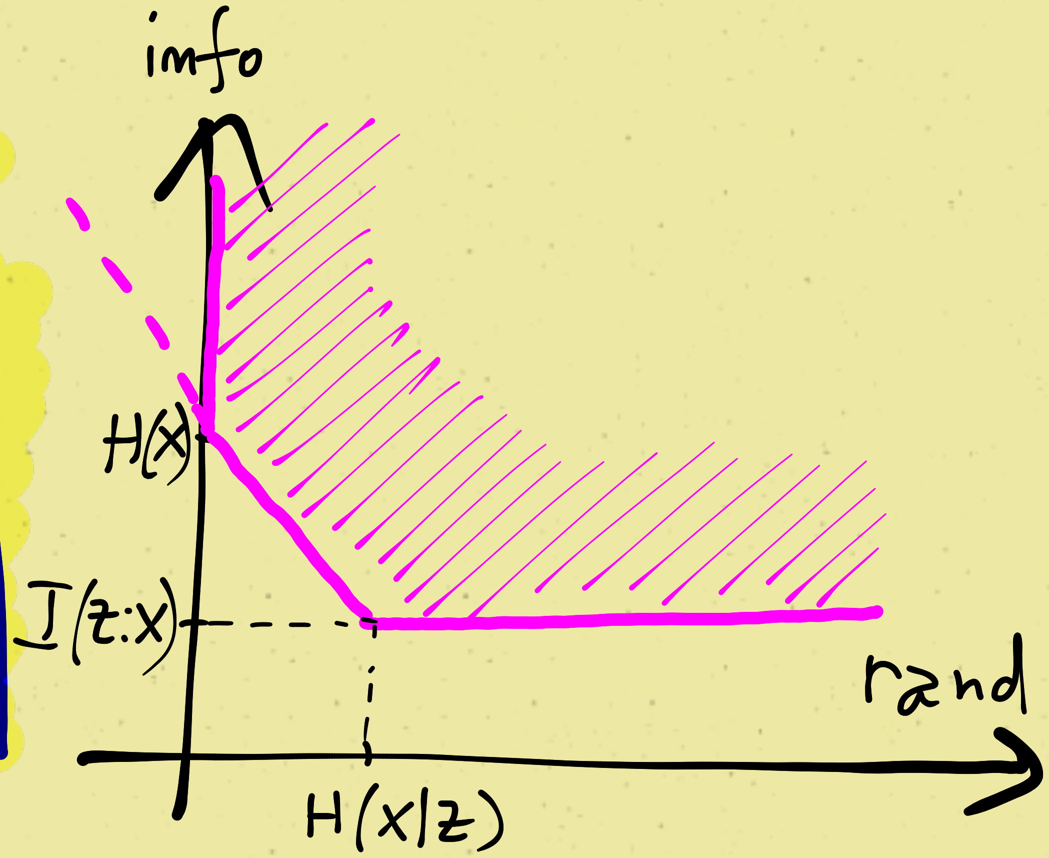
2. $$H(Y) + H(R) \geq H(X)$$



THE SITUATION IS THE FOLLOWING:

INFORMATION TO BE COMMUNICATED $\geq I(Z:X)$

INFO + RANDOMNESS $\geq H(X)$

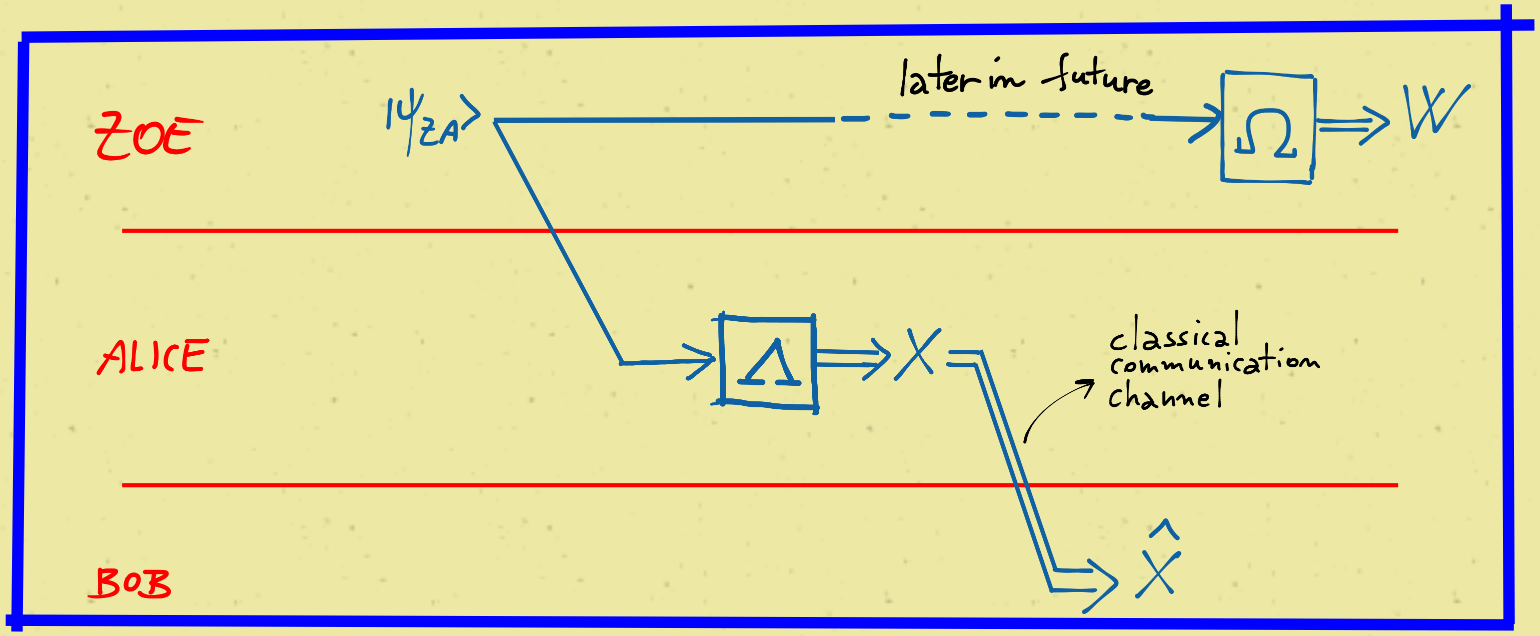


REMARK: REMEMBER HERE WE HAVE FEEDBACK: ALICE'S AND BOB'S POST-PROC.'S ARE LIMITED TO DETERMINISTIC ONES.

THE FEEDBACK CONDITION

CONSIDER THE CASE $\Delta^{x} = p_X(x)\mathbb{1}$, $\forall x$

I.E. Δ EXTRACTS NO USEFUL INFO, LIKE THROWING A DICE

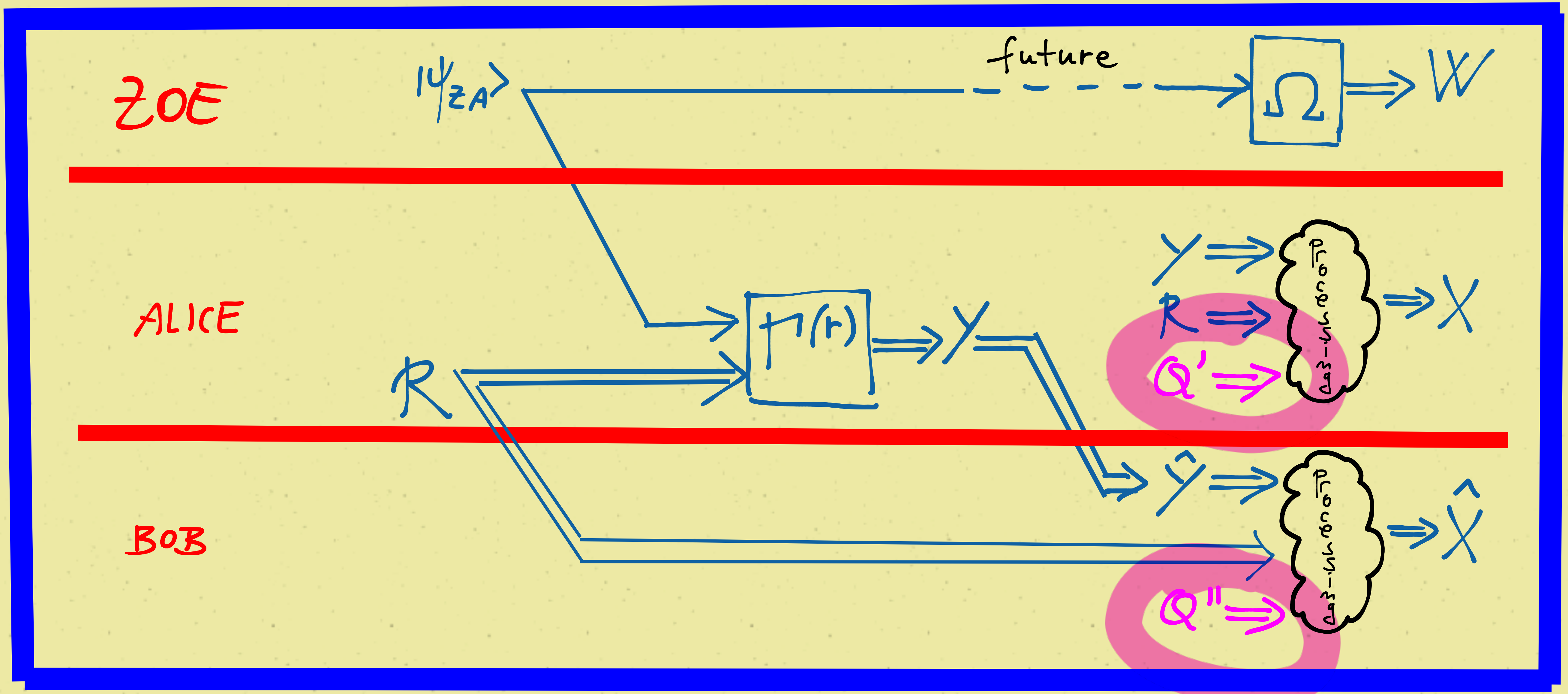


X is completely random; yet, we want $\hat{X} = X$.
(feedback)

- $I(X:Z) = 0 \implies$ NO COMMUNICATION NEEDED
- $\implies$ common randomness $\geq H(X)$

HOWEVER, DROPPING THE FEEDBACK CONDITION, THIS CASE CAN BE SIMULATED FOR FREE !! (Bob throws his dice locally...)

SIMULATION W/O FEEDBACK



ALICE AND BOB ARE ALLOWED TO USE PROBABILISTIC LOCAL POST-PROCESSINGS. ONLY CONDITION:

$$\Pr\{W=w, X=x\} = \Pr\{W=w, \hat{X}=x\}, \forall \Omega$$

W/O FEEDBACK: THE RESULT

DEFINITION: A REFINEMENT OF A POVM $\{\Lambda^x\}_{x\in X}$ IS A POVM $\{\Gamma^y\}_{y\in Y}$ SUCH THAT $\Lambda^x = \sum_y \mu(x|y)\,\Gamma^y$, FOR SOME COND. P.D. $\mu(x|y)$.

$$\boxed{\begin{array}{l} \text{INFO TO TRANSMIT} \geq I(Z:Y) \\ \text{INFO + COMMON RANDOMNESS} \geq I(ZX:Y) \end{array}}$$

(we can optimize over all possible refinements)

where quantities are computed w.r.t.

$$\omega_{ZYX} := \sum_x \sum_y \mu(x|y)\, p_Y(y)\, \rho_Z^y \otimes |y\rangle\langle y|_Y \otimes |x\rangle\langle x|_X$$

with $p_Y(y)\rho_Z^y = \mathrm{Tr}_A\{\psi_{ZA}\ \mathbb{1}_Z \otimes \Gamma_A^y\}$, and Γ refinement of Λ

WITH AND W/O FEEDBACK: COMPARISON

WITH FEEDBACK

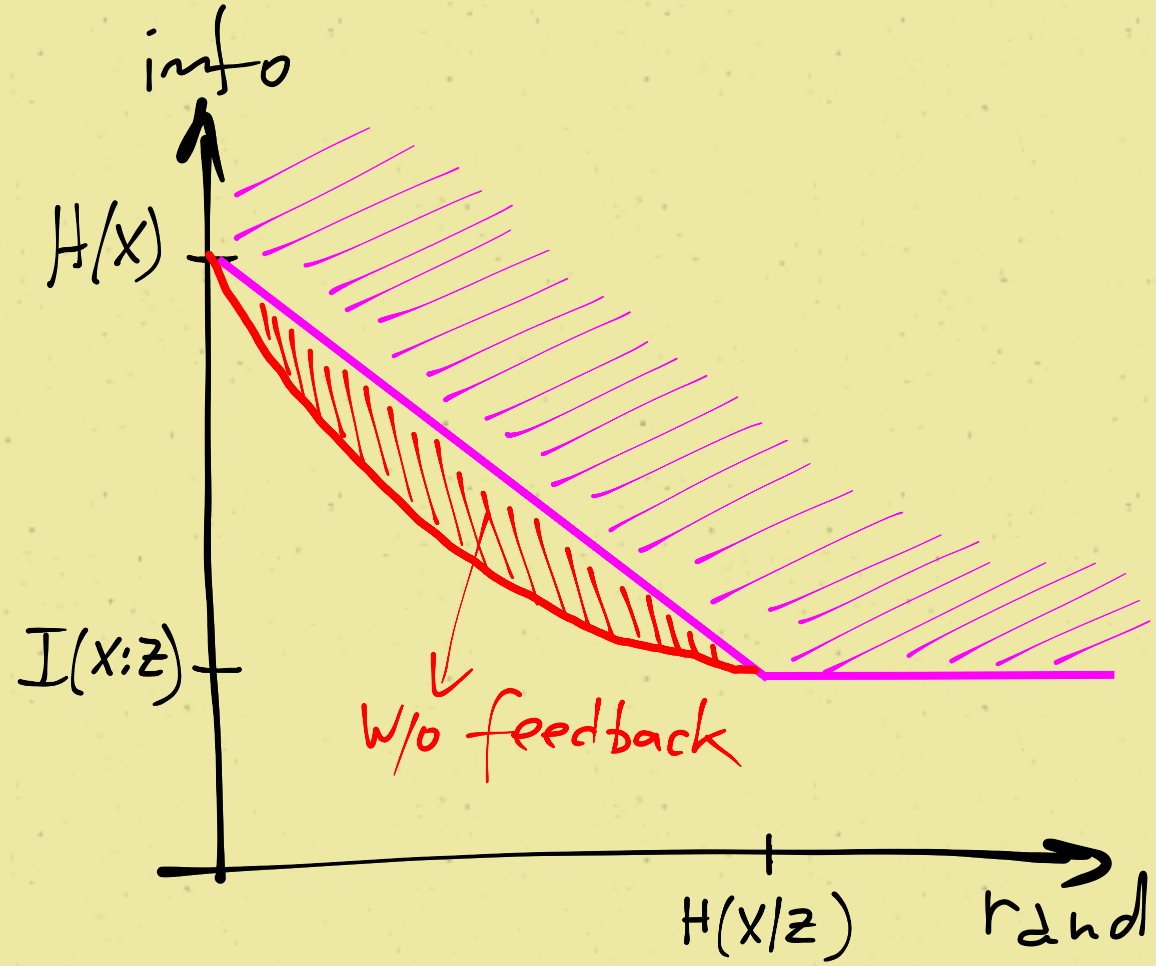
INFORMATION TO BE COMMUNICATED $\geq I(Z:X)$

INFO + RANDOMNESS $\geq H(X)$

W/O FEEDBACK

INFO TO TRANSMIT $\geq I(Z:Y)$

INFO + COMMON RANDOMNESS $\geq I(ZX:Y)$



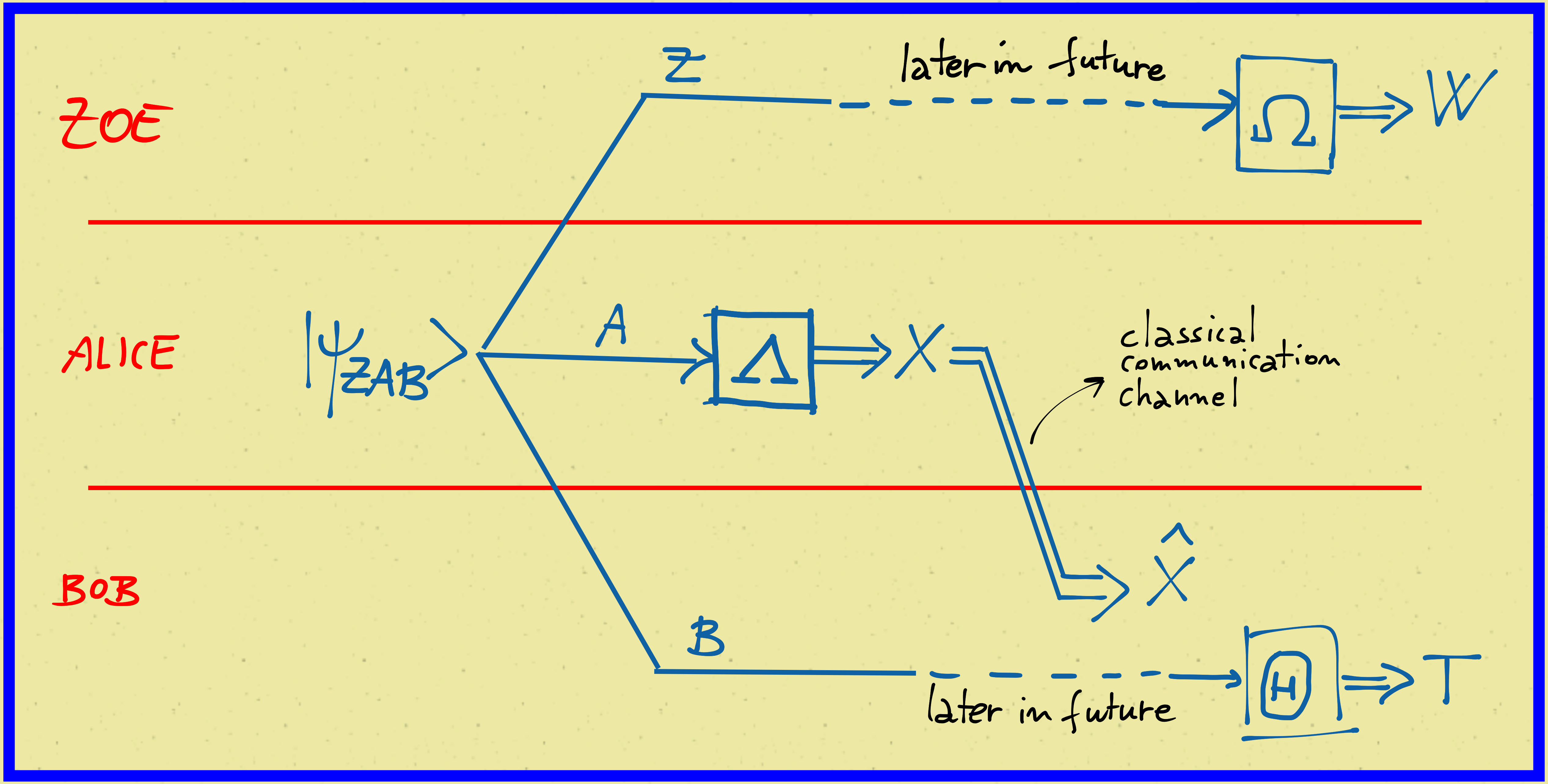
data processing $Y \to X$

- $I(Z:Y) \geq I(Z:X)$
- $I(ZX:Y) \leq H(X)$

possible because $I(X:Y|Z) \leq H(X|Z)$

cfr. Wyner's common info

THE CASE OF QUANTUM SIDE-INFO



- $\Pr\{X=x, W=w, T=t\} = \Pr\{\hat{X}=x, W=w, T=t\} \quad \forall \Omega, \forall H$
- $\Pr\{X=x, \hat{X}=x\}=1$ (feedback)

WYNER'S COMMON INFORMATION (1975)

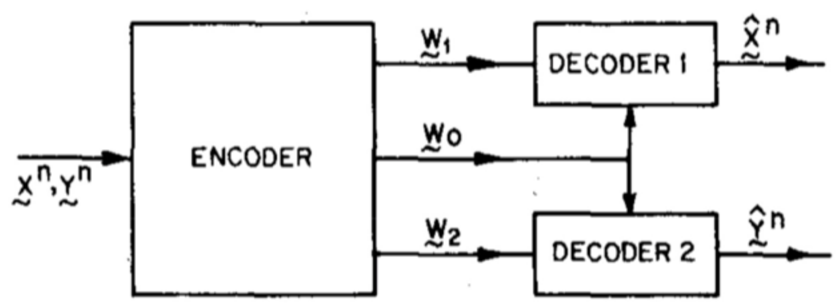


Fig. 1.

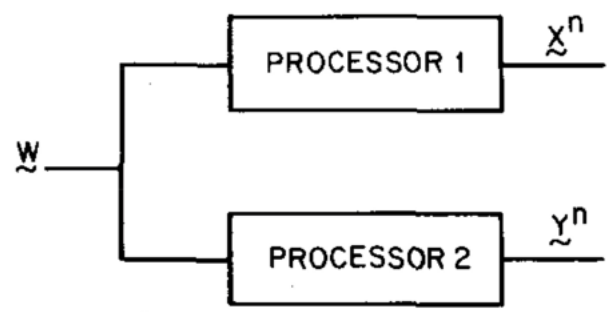


Fig. 3.

PROBLEM: HOW TO QUANTIFY THE INFORMATION THAT TWO RANDOM VARIABLES HAVE IN COMMON.

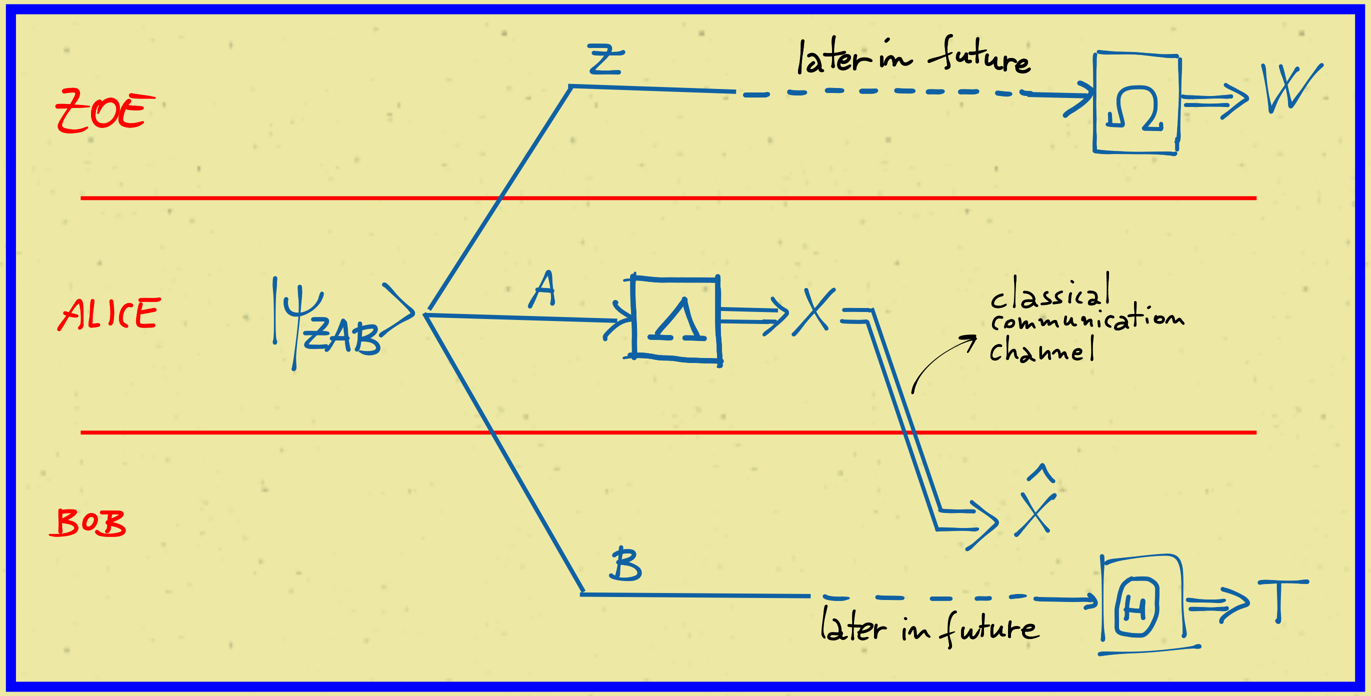
EXAMPLE: $X = (Q', W)$ $\quad Y = (Q'', W)$

SOLUTION: $C(X,Y) = \min_{W} I(W : XY)$, with $X \to W \to Y$ MARKOV

PROPERTIES:
- $C(X,Y) \leq \min\{H(X), H(Y)\}$
- $C(X,Y) \geq I(X:Y)$ $\left[\Leftarrow I(W:XY) \geq I(X:W) \geq I(X:Y) \right]$

data-processing

WITH QUANTUM SIDE-INFORMATION



INFORMATION TO BE SENT $\geq I(X:Z|B)$

INFORMATION + COMMON RANDOMNESS $\geq H(X|B)$

cfr. with previous cases

WITH FEEDBACK

INFORMATION TO BE COMMUNICATED $\geq I(Z:X)$

INFO + RANDOMNESS $\geq H(X)$

W/O FEEDBACK

INFO TO TRANSMIT $\geq I(Z:Y)$

INFO + COMMON RANDOMNESS $\geq I(ZX:Y)$