Complete positivity and its robustness in the presence of initial correlations

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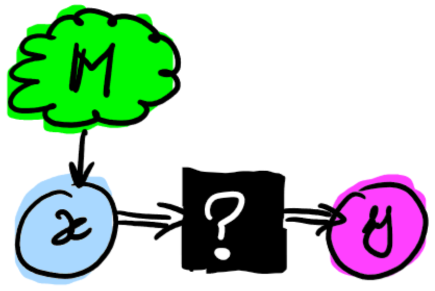
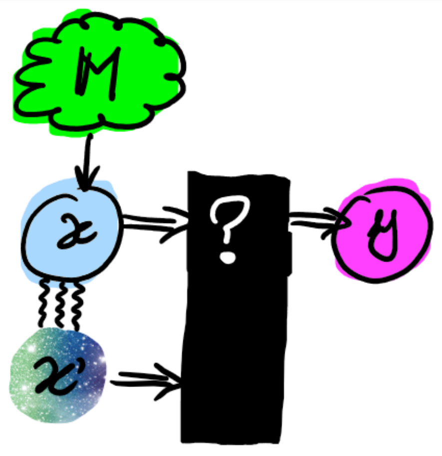
Iubilaei (50th) Symposium on Mathematical Physics
Toruń, Poland, 24 June 2018

The magic...



- M: message, X: input, Y: output
- we *would expect* that $I(X;M) \geq I(Y;M)$, i.e., "no free lunches in communication theory"
- what if we *observe* instead that $I(X;M) < I(Y;M)$?

...and the trick



the missing information was there all the time!
we couldn't see it, but we *knew*...

A lesson

When system and environment are initially correlated, we should not be surprised if:

1. the reduced dynamics of the system violates the data-processing inequality, or the second law, or behaves weird otherwise
2. the reduced dynamics of the system is not CP, or otherwise undefined

Question to be addressed in this talk

How to characterize those initial conditions (possibly including correlations) for which the reduced dynamics of the system are always well defined?

Formalization

- **datum**: initial set of possible system-ancilla (viz., environment) states $\mathsf{S}_{QE} = \{\rho_{QE} : \rho_{QE} \in \mathsf{S}_{QE}\}$
- **system's state set**: $\mathsf{S}_Q = \mathrm{Tr}_E[\mathsf{S}_{QE}]$

The Problem

To find conditions on S_{QE} guaranteeing that,
for any joint isometric evolution $V : QE \to Q'E'$,
there exists a corresponding CPTP map $\mathcal{V} : Q \to Q'$ such that

$$\mathcal{V}(\mathrm{Tr}_E[\rho_{QE}]) = \mathrm{Tr}_{E'}[V\rho_{QE}V^\dagger] \;,$$

for all $\rho_{QE} \in \mathsf{S}_{QE}$.

Remark. When the above property holds, we say that the set S_{QE} is **CPTP-reducible**.

The conventional starting point

Existence of an "assignment map"

One requires that

$$\rho_{QE} \neq \rho'_{QE} \implies \mathrm{Tr}_E[\rho_{QE}] \neq \mathrm{Tr}_E[\rho'_{QE}] \,,$$

that is, one requires the existence of a lifting (or *assignment map*) $\Phi : \mathsf{S}_Q \to \mathsf{S}_{QE}$ satisfying the consistency relation $(\mathrm{Tr}_E \circ \Phi)[\rho_Q] = \rho_Q$, for all $\rho_Q \in \mathsf{S}_Q$.

Remark. Essentially, the above means that $\mathrm{Tr}_E : \mathsf{S}_{QE} \to \mathsf{S}_Q$ is one-to-one.

Example. Simple initial conditions like $\rho_{QE} = \bar{\rho}_Q \otimes \omega_E$, for fixed $\bar{\rho}_Q$ and varying ω_E, cannot be treated in this approach.

An alternative idea

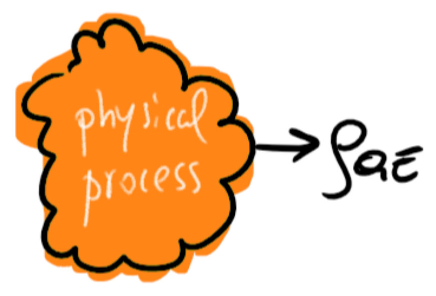
Existence of a "preparation"

We require that the set S_{QE} be originated by a filtering/preparation procedure. Mathematically speaking, we require the existence of an input system X and of a CP (maybe not TP) map $\mathcal{S} : X \to QE$ such that, for any $\rho_{QE} \in \mathsf{S}_{QE}$, there exists at least one density operator ρ_X such that

$$\rho_{QE} = \frac{\mathcal{S}(\rho_X)}{\mathrm{Tr}[\mathcal{S}(\rho_X)]}$$

Remark. All S_{QE} which are polytopes, are preparable

The meaning of preparability



- there exists a physical process that may or may not emit a compound system-environment state
- if it emits one, we know that it did and that the emitted state belongs to S_{QE}, *but we do not know which one*
- for example, imagine of "freezing" a strongly coupled open system dynamics at some arbitrary time, and add some filtering operation

Equivalent representation

The existence of a preparation is equivalent to the following:

Steerability

We require that there exists a tripartite density operator ϖ_{RQE} such that, for any $\rho_{QE} \in \mathsf{S}_{QE}$, there exists an operator $\pi_R \geq 0$ such that

$$\rho_{QE} = \frac{\mathrm{Tr}_R[\varpi_{RQE}\ (\pi_R \otimes I_{QE})]}{\mathrm{Tr}[\varpi_{RQE}\ (\pi_R \otimes I_{QE})]}\ .$$

Example. For the set of states $\rho_{QE} = \bar{\rho}_Q \otimes \omega_E$ (where $\bar{\rho}_Q$ is fixed and ω_E varies), there exists no assignment map; nonetheless it can be steered from $\varpi_{RQE} = \Psi^+_{RE} \otimes \bar{\rho}_Q$.

Consequences of this formulation

Characterization

Let the set S_{QE} be **preparable/steerable**. The following are equivalent:

1. the set S_{QE} is **CPTP-reducible**
2. the set S_{QE} is **steerable from Markov state** ϖ_{RQE}, i.e., such that $I(R;E|Q)=0$

Remark. Thanks to recent results on approximate reversibility, all the above conditions are "robust" against small deviations.

Example: initial factorization condition

This is the traditional "textbook" situation:

- $\mathsf{S}_{QE} \triangleq \{\rho_Q \otimes \bar{\omega}_E : \text{for fixed } \bar{\omega}_E\}$
- $\varpi_{RQE} = \Psi^+_{RQ} \otimes \bar{\omega}_E$
- $I(R;E|Q)_\varpi = 0$

Remark. Pechukas (PRL, 1994) advocated for the need of going *beyond* the factorization assumption.

Example: zero-discord sets

This counterexample was found by Rodriguez-Rosario, Modi, Kuah, Shaji, and Sudarshan in 2008:

- $\mathsf{S}_{QE} \triangleq \left\{ \rho_{QE}^{\overrightarrow{p}} = \sum_{i=1}^{N} p_i |i\rangle\langle i|_Q \otimes \bar{\omega}_E^{(i)} : \text{ for varying } \overrightarrow{p} \right\}$
- in this case, S_{QE} is a polytope
- $\varpi_{RQE} = N^{-1} \sum_{i=1}^{N} |i\rangle\langle i|_R \otimes |i\rangle\langle i|_Q \otimes \bar{\omega}_E^{(i)}$
- $I(R;E|Q)_{\varpi} = 0$

Question. Are there other possibilities?

Example: discordant sets

No! Shabani and Lidar (2009) published a proof, according to which null discord would be, not only sufficient, but also necessary for CPTP-reducibility.

Yes! The above was disproved by the following counterexample (Brodutch, Datta, Modi, Rivas, Rodriguez-Rosario, 2013):

- $\mathsf{S}_{QE} \triangleq \left\{ \rho^p_{QE} = p\bar{\rho}^{(\alpha)}_{QE} + (1-p)\bar{\rho}^{(\beta)}_{QE} \right\}$, where $\bar{\rho}^{(\alpha)}_{QE} = \frac{1}{2}|0\rangle\langle 0|_Q \otimes \bar{\omega}^{(0)}_E + \frac{1}{2}|+\rangle\langle +|_Q \otimes \bar{\omega}^{(1)}_E$ and $\bar{\rho}^{(\beta)}_{QE} = |2\rangle\langle 2|_Q \otimes \bar{\omega}^{(2)}_E$
- this is also a polytope
- $\varpi_{RQE} = \frac{1}{2}|\alpha\rangle\langle\alpha|_R \otimes \bar{\rho}^{(\alpha)}_{QE} + \frac{1}{2}|\beta\rangle\langle\beta| \otimes \bar{\rho}^{(\beta)}_{QE}$
- $I(R;E|Q)_\varpi = 0$

A negative example

- $\mathsf{S}_{QE} \triangleq \{\bar{\rho}_Q \otimes \omega_E : \text{for fixed } \bar{\rho}_Q\}$
- an assignment map does not exists, because *all elements* of S_{QE} have the same reduced state on Q
- $\varpi_{RQE} = \Psi^+_{RE} \otimes \bar{\rho}_Q$
- $I(R;E|Q)_\varpi = 2\log d > 0$

The above example is, in a sense, trivial; and yet, it is outside the scope of the assignment map formalism.

Further consequences

- all counterexamples to the factorization condition involve separable states
- can we have CPTP-reducible sets containing entangled states?
- yes: starting from tripartite states with $I(R;E|Q)_\varpi = 0$, it is easy to construct a lot of counterexamples
- however, if we requires that S_Q contains **all possible density operators** on $\mathcal{H}_Q$, then **the factorization condition is the only one that works**

Conclusions

- existence of assignment maps replaced by **preparability**
- preparability is equivalent to **steerability**
- then, CPTP-reducibility is equivalent to the Markov condition $I(R;E|Q) = 0$
- easy to check, easy to use to construct a lot of counterexamples, and it **recovers the factorization condition** (if S_Q contains all possible pure states of Q)
- it is **robust** against small deviations

la fine