Incompatible incompatibilities, and how to make them compatible again

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Japanese-French Quantum Information 2023 workshop
Tokyo, 13 September 2023

references

- F.B., E. Chitambar, W. Zhou:
  A complete resource theory of quantum (POVMs) incompatibility as quantum programmability.
  Physical Review Letters 124, 120401 (2020)

- F.B., K. Kobayashi, S. Minagawa, P. Perinotti, A. Tosini:
  Unifying different notions of quantum (instruments) incompatibility into a strict hierarchy of resource theories of communication.
  Quantum 7, 1035 (2023)

POVMs and instruments

in this talk: all sets ($\mathbb{X}, \mathbb{Y}$ etc.) are finite, all spaces ($\mathscr{H}_A, \mathscr{H}_B$ etc.) are finite-dimensional

POVM: family **P** of positive semidefinite operators on $\mathscr{H}$ labeled by set $\mathbb{X}$, i.e., $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$, with $P_x \geqslant 0$ and $\sum_x P_x = \mathbb{1}$

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\varrho\, P_x]$

instrument: family $\{\mathcal{I}_x : A \to B\}_{x\in\mathbb{X}}$ of completely positive (CP) linear maps from $\mathscr{B}(\mathscr{H}_A)$ to $\mathscr{B}(\mathscr{H}_B)$, such that $\sum_x \mathcal{I}_x$ is trace-preserving (TP)

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\mathcal{I}_x(\varrho)]$, and corresponding post-measurement state is $\frac{1}{p(x)}\mathcal{I}_x(\varrho)$

incompatibility

In quantum theory, some measurements necessarily exclude others.

If all measurements were compatible, we would not have QKD, violation of Bell's inequalities, quantum speedups, etc.

Various formalizations:

- preparation uncertainty relations (e.g., Robertson)
- measurement uncertainty relations (e.g., Ozawa)
- incompatibility

compatible POVMs 1/2

Definition

given a family $\{\mathbf{P}^{(i)}\}_{i\in\mathbb{I}} \equiv \{P_x^{(i)}\}_{x\in\mathbb{X},i\in\mathbb{I}}$ of POVMs, all defined on the same system A, we say that the family is **compatible**, whenever there exists

- a "mother" POVM $\mathbf{O} = \{O_w\}_{w\in\mathbb{W}}$ on system A
- a conditional probability distribution $\mu(x|w,i)$

such that

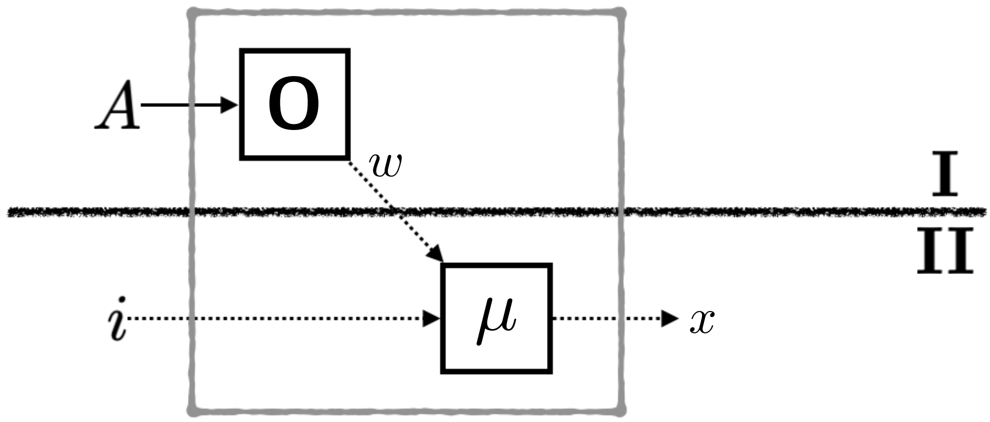
$$P_x^{(i)} = \sum_w \mu(x|w,i) O_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

But what does it mean, *operationally*, if I say that, e.g., a certain laboratory can only perform compatible measurement?

compatible POVMs 2/2

There's a bipartition hidden in the concept of (in)compatibility:



[F.B., E. Chitambar, W. Zhou; PRL 2020]

the problem

While there is consensus on a single notion of compatibility for POVMs, the situation is less clear for instruments...

classical compatibility 1/2

Definition (Heinosaari–Miyadera–Reitzner, 2014)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **classically compatible**, whenever there exist

- a mother instrument $\{\mathcal{H}_w : A \to B\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$

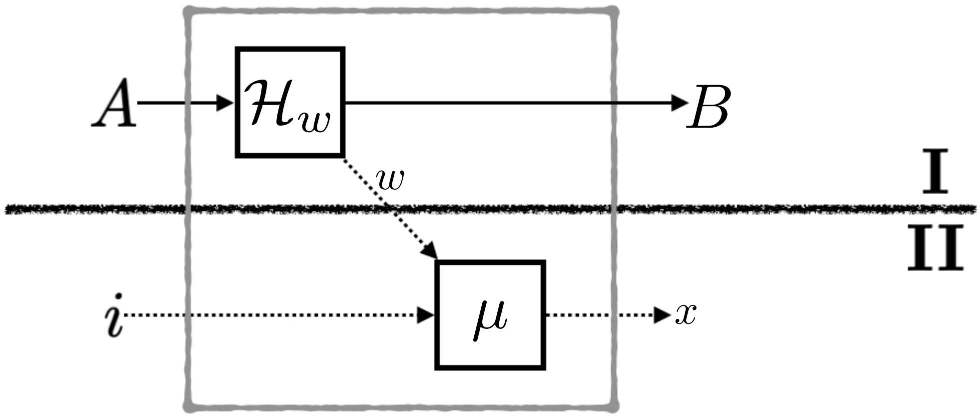
such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)\mathcal{H}_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

we call this "classical" because it involves only classical post-processings, but it is also called "traditional" [Mitra and Farkas; PRA 2022].

classical compatibility 2/2



crucially:

- no shared entanglement and communication is classical
- communication goes only from $\mathbf{I}$ to $\mathbf{II}$, i.e., the above is necessarily $\mathbf{II}\to\mathbf{I}$ non-signaling, see [Ji and Chitambar; PRA 2021]

marginalizing the mother

- without loss of generality (classical labels can be copied), compatible POVMs may be assumed to be recovered by marginalization, i.e.,

$$P_x^{(i)} = \sum_{x_j : j \neq i} O_{x_1, x_2, \ldots, x_n}$$

- the notion of "parallel compatibility" for instruments lifts the above insight to the quantum outputs

parallel compatibility 1/2

Definition (Heinosaari–Miyadera–Ziman, 2015)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **parallelly compatible**, whenever there exist

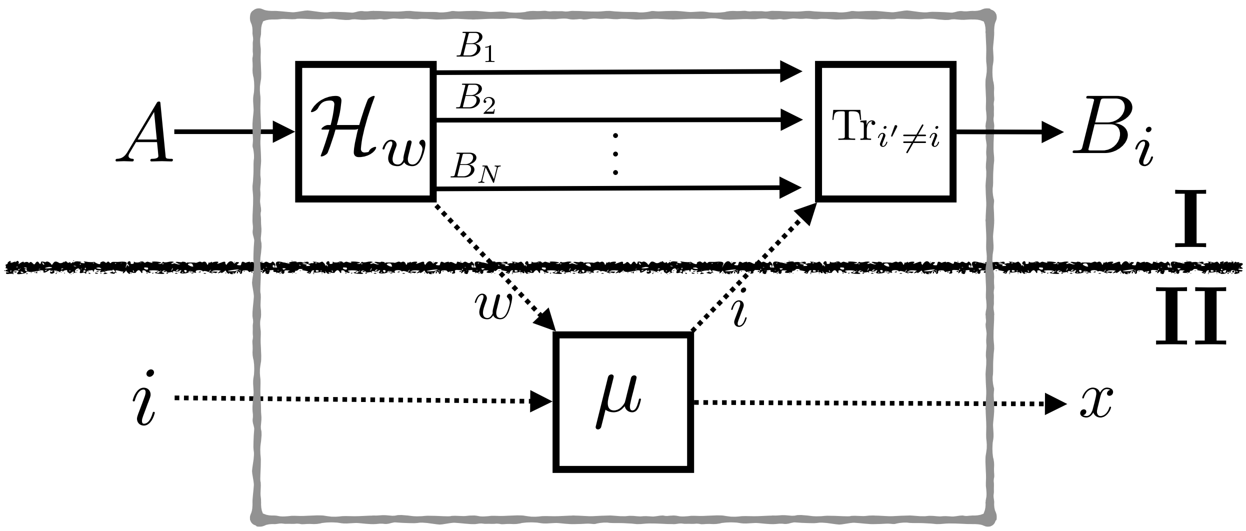
- a mother instrument $\{\mathcal{H}_w : A \to \otimes_{i\in\mathbb{I}} B_i\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$

such that

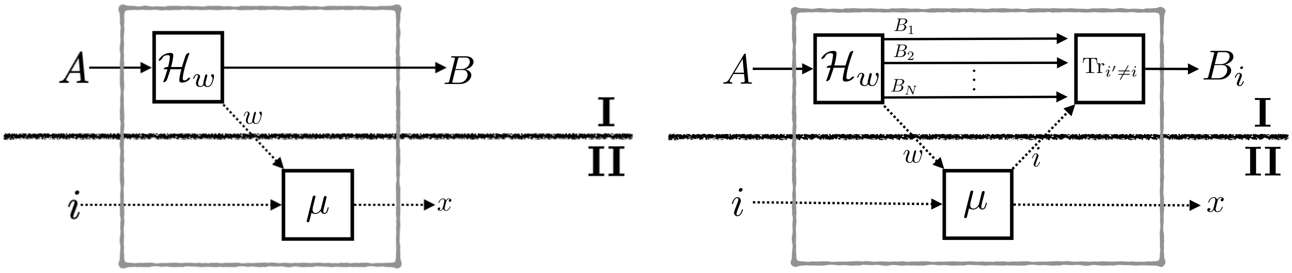
$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathrm{Tr}_{B_{i':i'\neq i}} \circ \mathcal{H}_w] \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

parallel compatibility 2/2



parallel compatibility VS classical compatibility



- parallel compatibility is able to go beyond no-signaling, hence, parallel compatibility $\not\Rightarrow$ classical compatibility
- parallel compatibility has nothing to do with the "no information without disturbance" principle, because non-disturbing instruments are never parallelly compatible
- hence classical compatibility $\not\Rightarrow$ parallel compatibility

bridging the two camps: q-compatibility

Definition

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **q-compatible**, whenever there exist

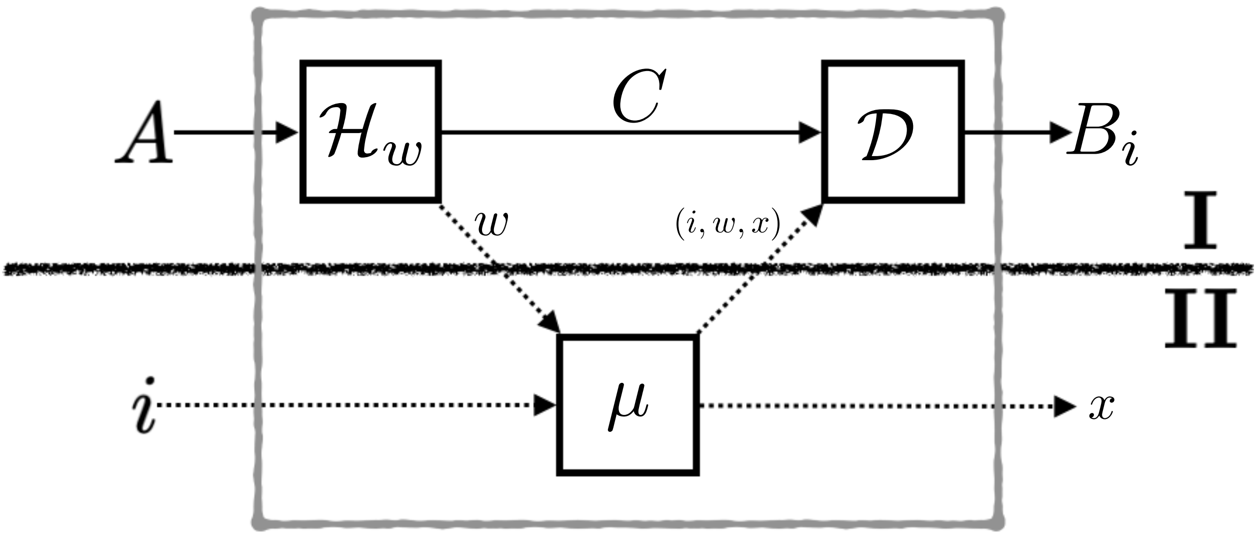
- a mother instrument $\{\mathcal{H}_w : A \to C\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$
- a family of postprocessing channels $\{\mathcal{D}^{(x,w,i)} : C \to B_i\}_{x\in\mathbb{X}, w\in\mathbb{W}, i\in\mathbb{I}}$

such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathcal{D}^{(x,w,i)} \circ \mathcal{H}_w] \ ,$$

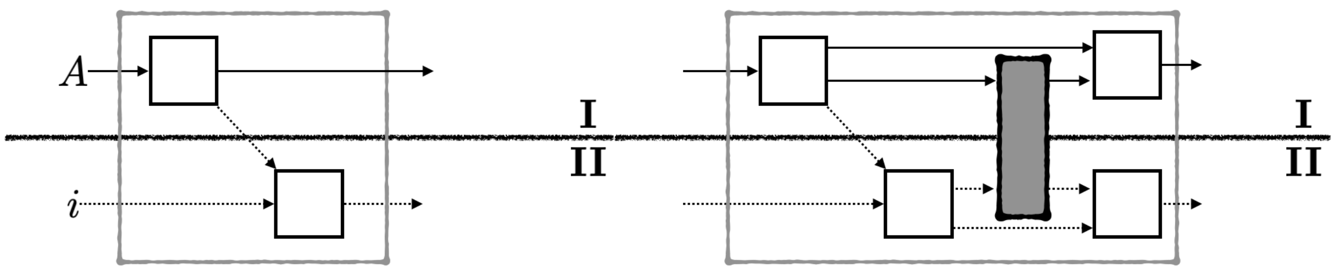
for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

q-compatibility as a circuit



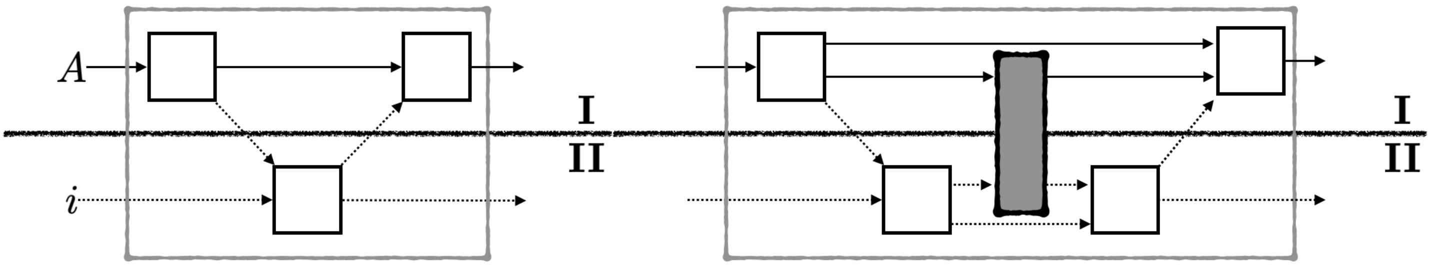
classical compatibility: $C \equiv B_i$ and $\mathcal{D}^{(x,w,i)} = \text{id}$
parallel compatibility: $C \equiv \bigotimes_i B_i$ and $\mathcal{D}^{(x,w,i)} = \mathrm{Tr}_{B_{i'}:i' \neq i}$

free operations for classical incompatibility



- all cassically compatible devices can be created for free
- if the initial device (the dark gray inner box) is classically compatible, the final device is also classically compatible

free operations for q-incompatibility



- all q-compatible devices can be created for free
- if the initial device (the dark gray inner box) is q-compatible, the final device is also q-compatible

going beyond q-compatibility

something is missing

two POVMs can only be simultaneously compatible

consider the two instruments

$$\mathcal{I}_1(\bullet) = pU_1 \bullet U_1^\dagger \qquad \mathcal{J}_1(\bullet) = |0\rangle\langle 0| \bullet |0\rangle\langle 0|$$
$$\mathcal{I}_2(\bullet) = (1-p)U_2 \bullet U_2^\dagger \qquad \mathcal{J}_2(\bullet) = |1\rangle\langle 1| \bullet |1\rangle\langle 1|$$

the corresponding POVMs, i.e., $\{p\mathbb{1}, (1-p)\mathbb{1}\}$ and $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ are compatible, but the two instruments are not (even q-compatible)

except that, in a sense, they actually are!
just not simultaneously so: do $\mathcal{I}$, keep the outcome, undo the unitary, and finally do $\mathcal{J}$ (in the opposite order (i.e., first $\mathcal{J}$, then $\mathcal{I}$) it would not work)

no-exclusivity

Definition

an instrument $\{\mathcal{I}_x : A \to B_1\}_{x\in\mathbb{X}}$ **does not exclude** another instrument $\{\mathcal{J}_y : A \to B_2\}_{y\in\mathbb{Y}}$, whenever there exist

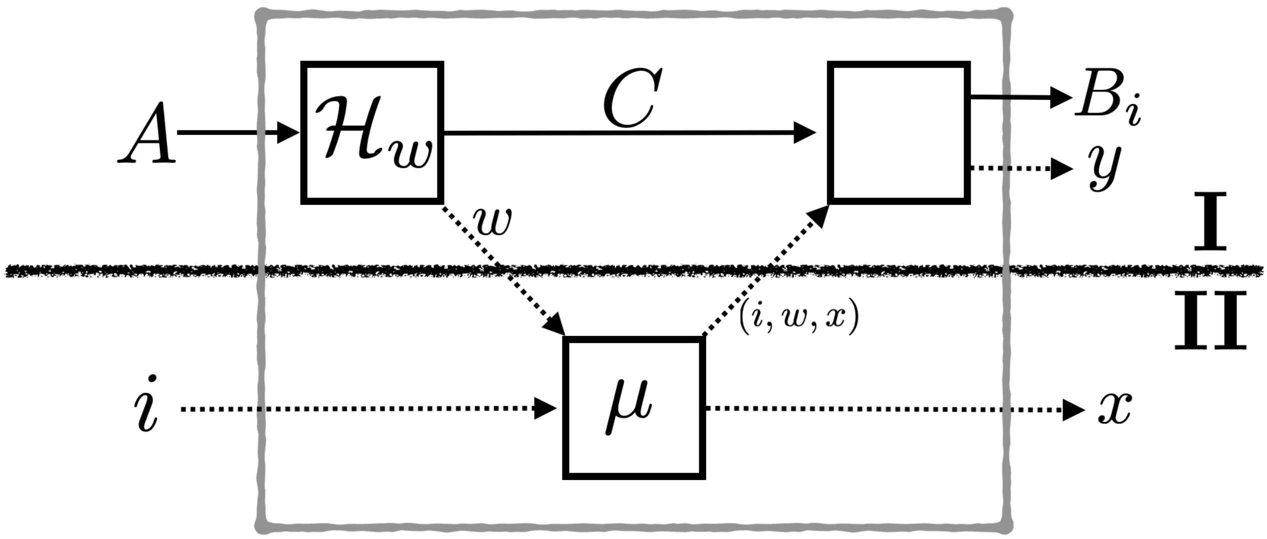
- a mother instrument $\{\mathcal{H}_w : A \to C\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w)$
- a family of postprocessing channels $\{\mathcal{D}^{(x,w)} : C \to B_1\}_{x\in\mathbb{X}, w\in\mathbb{W}}$
- a family of instruments $\{\mathcal{K}_y^{(w)} : C \to B_2\}_{w\in\mathbb{W}, y\in\mathbb{Y}}$

such that

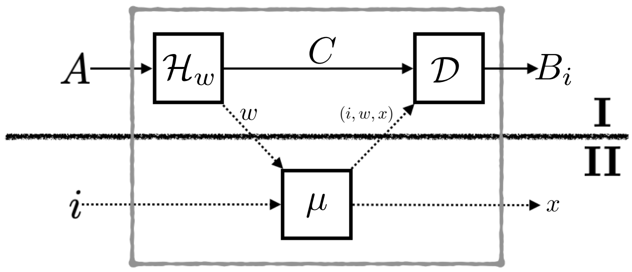
$$\mathcal{I}_x = \sum_w \mu(x|w)[\mathcal{D}^{(x,w)} \circ \mathcal{H}_w] , \qquad \mathcal{J}_y = \sum_w \mathcal{K}_y^{(w)} \circ \mathcal{H}_w ,$$

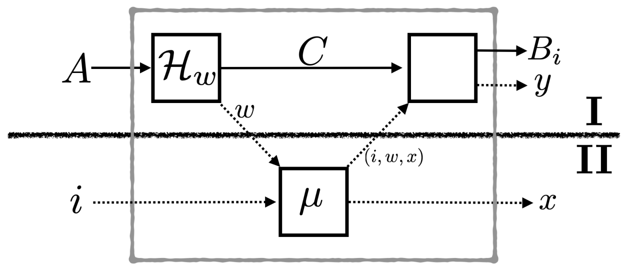
for all $x \in \mathbb{X}$ and all $y \in \mathbb{Y}$.

no-exclusivity as a circuit



no-exclusivity VS q-compatibility





in the definition of no-exclusivity, the post-processing box can be an instrument, but the non-excluded outcome is with agent $\mathbf{I}$ (agent $\mathbf{II}$ only has non-excluding outcomes)

intuition: it is possible to provide classical information sufficient to reconstruct the outcome of one instrument, without introducing a level of disturbance that would exclude the other

conclusion

take home messages

- no need to argue about the "correct" definition of compatibility for instruments: q-compatibility provides an overarching framework
- incompatibility is an essentially nonlocal concept, i.e., quantum information transmission, either in space (quantum channel) or in time (quantum memory)
- still a lot to do to understand no-exclusivity; are there connections with causal order?

thank you for your attention