The Petz Map in Maths, Information Theory, and Physics: an Overview

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Conventions

- finite dimensions (operators $\equiv$ matrices)
- states, viz., density matrices: $\rho \geq 0$, $\mathrm{Tr}\,\rho = 1$ $(\rho, \sigma, \dots)$
- pure states, e.g, $|\psi_Q\rangle$ and $|\psi\rangle\langle\psi|_Q = \psi_Q$
- we denote supp$\rho :=$ (ker$\rho)^\perp$
- completely positive trace-preserving linear maps, viz. quantum channels, are denoted by $\mathcal{E}, \mathcal{F}, \dots$
- trace-dual map $\mathcal{E}^\dagger$ is defined by $\mathrm{Tr}[\mathcal{E}(X)\ Y] = \mathrm{Tr}[X\ \mathcal{E}^\dagger(Y)]$ for all X, Y
- fidelity (sometimes, *squared* fidelity) $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$

Umegaki's relative entropy

Definition (Relative entropy)

For $A, B \geq 0,\ A \neq 0$,

$$D(A\|B) := \begin{cases} \operatorname{Tr}[A(\log A - \log B)] \ , & \text{if } \operatorname{supp} A \subseteq \operatorname{supp} B \ , \\ +\infty \ , & \text{otherwise} \end{cases}$$

Useful properties:

- Klein's inequality: $\operatorname{Tr} A \geq \operatorname{Tr} B \implies D(A\|B) \geq 0$
- $B \leq B' \implies D(A\|B) \geq D(A\|B')$
- $-D(\rho\|I) = S(\rho) \qquad (= -\operatorname{Tr} \rho \log \rho)$
- **monotonicity**: $D(\rho\|\sigma) \geq D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma))$ for all channels $\mathcal{E}$ and all states ρ, σ

Origin of the transpose map

Question. For which triples $(\rho, \sigma, \mathcal{E})$, $D(\rho\|\sigma) = D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma))$?

Petz (1986,1988)

If and only if $\tilde{\mathcal{E}}_\sigma(\bullet) := \sqrt{\sigma}\mathcal{E}^\dagger \left[\frac{1}{\sqrt{\mathcal{E}(\sigma)}} \bullet \frac{1}{\sqrt{\mathcal{E}(\sigma)}} \right] \sqrt{\sigma}$ satisfies

$$\tilde{\mathcal{E}}_\sigma \circ \mathcal{E}(\rho) = \rho \ .$$

(The other equality $\tilde{\mathcal{E}}_\sigma \circ \mathcal{E}(\sigma) = \sigma$ is satisfied *by construction*.)

Remark. The map $\tilde{\mathcal{E}}_\sigma$ is already CPTP on $\operatorname{supp}[\mathcal{E}(\sigma)]$, but it can always be extended to a linear map CPTP *everywhere*.

Remark. Notice that $\tilde{\mathcal{E}}_\sigma$ in general is *not* the linear inverse of $\mathcal{E}$!

Re-discovered in quantum error correction

- state ρ_Q purified by $|\psi_{RQ}\rangle$, that is, $\mathrm{Tr}_R[\psi_{RQ}] = \rho_Q$
- given a state ρ_Q and a channel $\mathcal{E} : Q \to Q$, the entanglement fidelity is $F_e(\rho, \mathcal{E}) := \langle\psi_{RQ}|(\mathrm{id}_R \otimes \mathcal{E}_Q)(\psi_{RQ})|\psi_{RQ}\rangle$

Barnum and Knill (2002)

Given a state ρ_Q and a channel $\mathcal{E} : Q \to Q'$,

$$\left[\max_{\mathcal{R}:Q'\to Q} F_e(\rho, \mathcal{R} \circ \mathcal{E})\right]^2 \le F_e(\rho, \tilde{\mathcal{E}}_\rho \circ \mathcal{E}) \ .$$

Petz's transpose map can be used as decoder to achieve the quantum capacity (Beigi–Datta–Leditzky 2017). Belavkin's "pretty good measurement" (1975) can also be rederived as a special case.

Re-discovered in quantum statistical mechanics

Crooks (PRA, 2008) rediscovers Petz's transpose map on the basis of physical reasoning:

- starts from equilibrium, that is, $\mathcal{E}(\omega) = \omega$
- picks a Kraus representation $\mathcal{E}(\bullet) = \sum_k E_k(\bullet)E_k^\dagger$
- defines a stochastic "trajectory" over the Kraus representation index: $p(\alpha, \beta|\omega) = \mathrm{Tr}[E_\beta \left(E_\alpha \omega E_\alpha^\dagger\right) E_\beta^\dagger]$
- assumes that the "reverse" process, with Kraus operators $\{\tilde{E}_k\}_k$, at equilibrium satisfies microscopic reversibility: $\tilde{p}(\beta, \alpha|\omega) := \mathrm{Tr}[\tilde{E}_\alpha \left(\tilde{E}_\beta \omega \tilde{E}_\beta^\dagger\right) \tilde{E}_\alpha^\dagger] \stackrel{!}{=} p(\alpha, \beta|\omega)$
- the above is satisfied if $\tilde{E}_k = \omega^{1/2} E_k^\dagger \omega^{-1/2}$, for all indices k
- $\implies$ Crooks' reverse process coincides with $\tilde{\mathcal{E}}_\omega$

Extension beyond equality: approximate reversibility

Junge–Renner–Sutter–Wilde–Winter (2018)

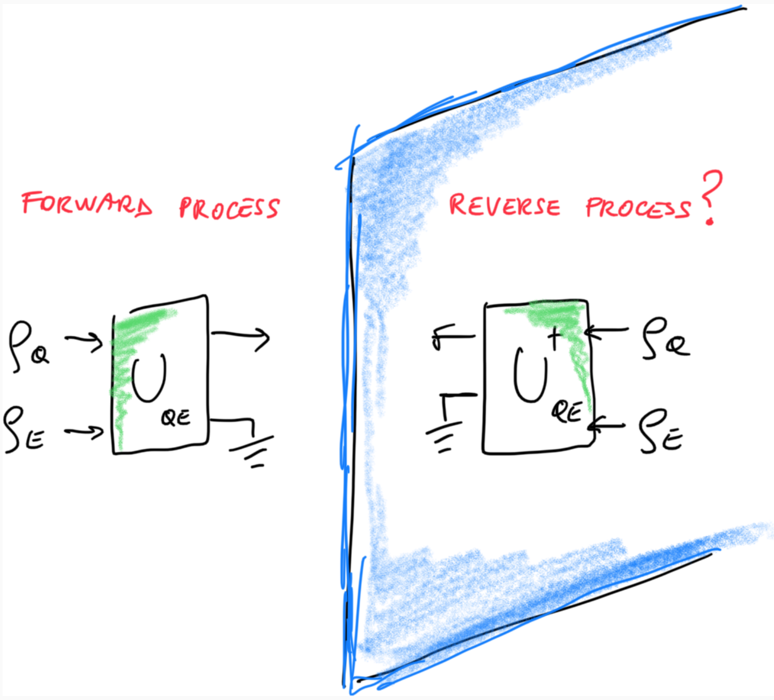
$$D(\rho\|\sigma) - D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \geq -\int_{-\infty}^{+\infty} \mathrm{d}t\; p(t) \log F(\rho, \tilde{\mathcal{E}}_\sigma^{t/2} \circ \mathcal{E}(\rho))$$
$$\geq -\log F(\rho, \mathcal{R} \circ \mathcal{E}(\rho)) \; ,$$

where

- $p(t) := \frac{\pi}{2(\cosh(\pi t)+1)}$ is a probability density
- $\tilde{\mathcal{E}}_\sigma^t(\bullet) := \sigma^{-it} \tilde{\mathcal{E}}_\sigma[\mathcal{E}(\sigma)^{it} \, (\bullet) \, \mathcal{E}(\sigma)^{-it}]\sigma^{it}$ are "rotated" Petz's maps
- $\mathcal{R} := \int_{-\infty}^{+\infty} \mathrm{d}t \; p(t) \tilde{\mathcal{E}}_\sigma^{t/2}$

What is the Petz transpose map and how to implement it?

The problems with "reversal"



- What is it? A (*the*?) time-reverse? Other symmetry reverse?
- How to achieve it?

The quest for a "physical implementation"

The underlying philosophy is that a channel represents a process that "actually happens".

Problem. Given a circuit implementation of a channel, *algorithmically* construct a circuit implementing its Petz transpose.

Results. Having a realization of the forward process does not mean that its reversal is also available: there is no simple "reversal button"! See e.g.: Quintino *et al.* (2019) and Gilyén *et al.* (2020).

Remark. Any channel's realization involves unobservable degrees of freedom (the inside of the black-box). Should the reverse depend on those?

However, some special cases are "easy". (But beware building your intuition based on them!)

Hamiltonian (i.e., unitary or one-to-one) dynamics

The following are equivalent:

- the channel $\mathcal{E}$ is unitary (that is, $\mathcal{E}(\bullet) = U \bullet U^\dagger$)
- the channel $\mathcal{E}$ is such that $\tilde{\mathcal{E}}_\sigma$ does not depend on the choice of σ
- the channel $\mathcal{E}$ is such that its linear inverse $\mathcal{E}^{-1}$ coincides with $\tilde{\mathcal{E}}_\sigma$ for some choice of σ

Moreover, for unitary channels all rotated Petz maps coincide.

Interpretation

The Petz transpose channel corresponds to "the movie shown backwards" (intuitive notion of "time-reversal") if and only if the channel is unitary.

Thermal operations (and a little more)

Alhambra–Wehner–Wilde–Woods (2018)

Consider a channel $\mathcal{E} : Q \to Q$ possessing a realization of the form

$$\mathcal{E}(\bullet_Q) = \mathrm{Tr}_E[U_{QE}\,(\bullet_Q \otimes \tau_E)\,U_{QE}^\dagger]\ ,$$

such that

$$U_{QE}\,(\omega_Q \otimes \tau_E)\,U_{QE}^\dagger = \omega_Q \otimes \tau'_E\ ,$$

for some steady state $\omega_Q > 0$. Then

$$\tilde{\mathcal{E}}_\omega(\bullet_Q) = \mathrm{Tr}_E[U_{QE}^\dagger\,(\bullet_Q \otimes \tau'_E)\,U_{QE}]\ .$$

Remark. A thermal operation has ω_Q and $\tau_E = \tau'_E$ as Gibbs states of the system's and bath's Hamiltonians, respectively.

In general, what is the relation between a channel and its Petz transpose?

Petz's transpose map as Bayesian retrodiction

The Bayes–Laplace Rule

Inverse Probability Formula

$$\underbrace{\mathcal{P}(H|D)}_{\text{inv. prob.}} \propto \underbrace{\mathcal{P}(D|H)}_{\text{likelihood/model}} \underbrace{\mathcal{P}(H)}_{\text{prior}}$$

where H is a hypothesis, D is the result of observation (i.e., data or evidence)

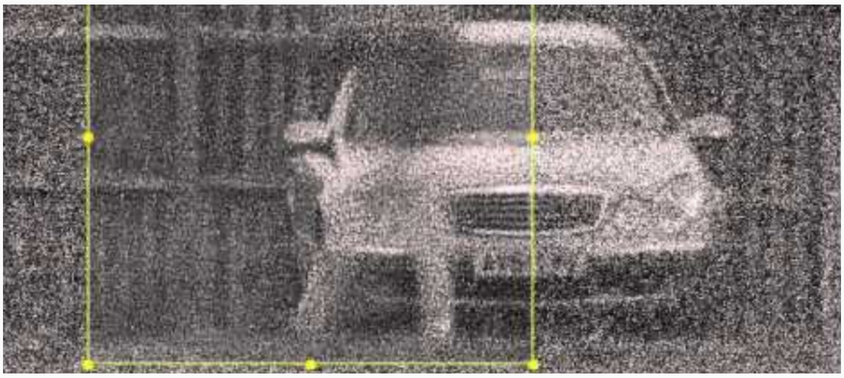
Meanings of the inverse probability

- it is the main *tool* of Bayesian statistics for problems like:
  - estimation (e.g.: how many red balls are in an urn?)
  - decision (e.g.: is ACME's stock a good investment? should I buy some? how much?)
  - inference and learning: predictive inference (e.g.: weather forecasts) and retrodictive inference (e.g.: what kind of stellar event possibly caused the Crab Nebula?)
- it measures the degree of belief that a rational agent should have in one hypothesis, among other mutually exclusive ones, given the data

Inference with noisy data or uncertain evidence

BUT! Bayes-Laplace Rule *does not* tell us how to update the prior in the face of uncertain data...

- suppose that a noisy observation suggests a probability distribution $\mathcal{Q}(D)$ for the data (e.g., the license plate no.)



- how should we update our prior $\mathcal{P}(H)$ given *uncertain evidence* in the from $\mathcal{Q}(D)$?

Jeffrey's rule of probability kinematics

Vanilla Bayes:

$$\mathcal{P}(H|D) = \mathcal{P}(D|H)\mathcal{P}(H)/\mathcal{P}(D)$$

Extended Bayes:

$$\mathcal{P}(H|\mathcal{Q}(D)) = ?$$

Jeffrey's conditioning* (1965)

$$\mathcal{P}(H|\mathcal{Q}(D)) = \sum_D \underbrace{\mathcal{P}(H|D)}_{\text{inv. prob.}} \mathcal{Q}(D)$$

$$= \sum_D \frac{\mathcal{P}(D|H)\mathcal{P}(H)}{\sum_H \mathcal{P}(D|H)\mathcal{P}(H)} \mathcal{Q}(D)$$

* Jeffrey's rule was introduced *ad hoc*, but it can be proved from Bayes-Laplace Rule and Pearl's method of virtual evidence (1988)

Jeffrey's rule "promotes" Bayes' posterior distribution to a fully fledged channel.

Petz transpose in the classical case

- state $\rho \rightsquigarrow$ probability distribution $p(x)$
- channel $\mathcal{E} \rightsquigarrow$ discrete noisy channel $\varphi(y|x)$
- $\mathcal{E}(\rho) \rightsquigarrow [\varphi \circ p](y) = \sum_x \varphi(y|x)p(x)$
- Petz transpose $\tilde{\mathcal{E}}_\rho \rightsquigarrow \tilde{\varphi}_p(x|y) = \frac{1}{[\varphi \circ p](y)} \varphi(y|x)p(x)$
- hence, Petz's transpose map coincides with Jeffrey's rule!
- moreover, in the classical case there is only one Jeffrey–Petz reverse (i.e., all rotated maps coincide)

Approximate reversibility in the classical case

Li–Winter (2018)

In the classical case,

$$D(p\|q) - D(\varphi[p]\|\varphi[q]) \geq D(p\|[\tilde{\varphi}_q \circ \varphi]p) \; .$$

Remark. Notice that $D(p\|q) \geq -\log F(p, q)$, so the above is stronger than the best general quantum bounds we know.

Only for *(sub-)unital* CPTP maps $\mathcal{E}$, we have a similar bound (Buscemi–Das–Wilde 2016): $S(\mathcal{E}(\rho)) - S(\rho) \geq D(\rho\|(\mathcal{E}^\dagger \circ \mathcal{E})\rho)$.

Open question. What about a different relative entropy, like Belavkin–Staszewski's?

Case study: application in statistical mechanics

Satosi Watanabe

"The phenomenological one-wayness of temporal developments in physics is due to ***irretrodictability****, and not due to irreversibility."* S. Watanabe (1965)

Ed Jaynes

"To understand and like thermo we need to see it, not as an example of the n-body equations of motion, but as an example of the logic of ***scientific inference****."*

E.T. Jaynes (1984)

More concretely: to derive fluctuation relations with the reverse process as Bayesian retrodiction

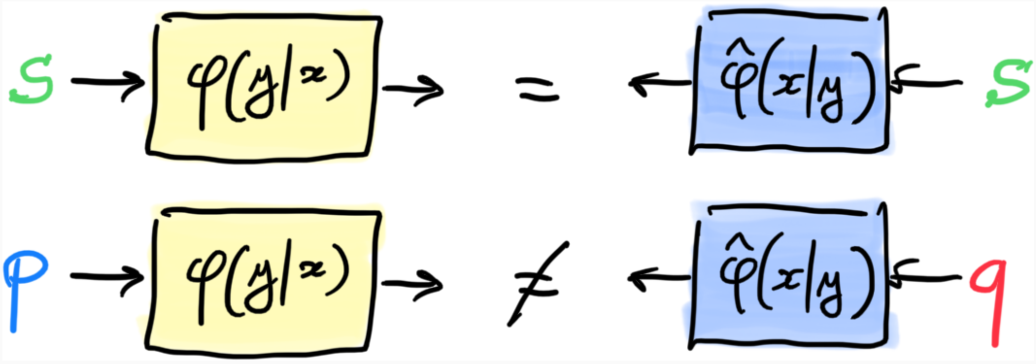
Construction of the reverse process as retrodiction

- **physical setup**:
  - a stochastic transition rule: $\varphi(y|x)$
  - a steady (viz. invariant) state: $\sum_x \varphi(y|x)s(x) = s(y)$
- **Bayes–Jeffrey inversion at the steady state**:

$$s(y)\tilde{\varphi}(x|y) := s(x)\varphi(y|x) \iff \frac{\varphi(y|x)}{\tilde{\varphi}(x|y)} = \frac{s(y)}{s(x)}$$

- **two priors**:
  - predictor's prior: $p(x)$
  - retrodictor's prior $q(y)$
- **two processes**:
  - forward process (prediction): $\mathcal{P}_F(x,y) = \varphi(y|x)p(x)$
  - reverse process (retrodiction): $\mathcal{P}_R(x,y) = \tilde{\varphi}(x|y)q(y)$

A picture



- at the steady state: prediction = retrodiction
- otherwise: asymmetry (irreversibility, *irretrodictability*)

Quantifying irretrodictability

Idea: fluctuation relations as measures of divergence between prediction and retrodiction

- relative entropy:

$$D(\mathcal{P}_F \| \mathcal{P}_R) := \left\langle -\ln \frac{\mathcal{P}_R(x,y)}{\mathcal{P}_F(x,y)} \right\rangle_F =: \langle -\ln r(x,y) \rangle_F$$

⇝ more generally, one can use $D_f(\mathcal{P}_R \| \mathcal{P}_F) := \langle f(r(x,y)) \rangle_F$

- introduce probability density functions

⇝ $\Omega(x,y) := f(r(x,y))$ (total stochastic f-entropy production)

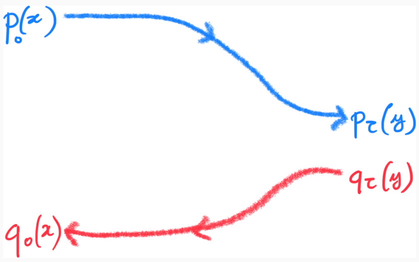
⇝ $\mu_F(\omega) := \sum_{x,y} \delta[\omega - \Omega(x,y)] \, \mathcal{P}_F(x,y)$

⇝ $\mu_R(\omega) := \sum_{x,y} \delta[\omega - \Omega(x,y)] \, \mathcal{P}_R(x,y)$

$$\implies \langle \omega \rangle_F = D_f(\mathcal{P}_R \| \mathcal{P}_F)$$

Examples of known results recovered by retrodiction

Example: driven closed system evolution



- driving protocol: $H(0) \to H(t) \to H(\tau)$
- $H(0) = (\epsilon_x)_x$, $H(\tau) = (\eta_y)_y$
- $\varphi(y|x) = \delta_{y,y(x)}$, i.e., one-to-one
- Hamiltonian $\implies$ $\tilde{\varphi}(x|y) \equiv \varphi(y|x)$
- $p_0(x) = e^{\beta(F-\epsilon_x)}$, $q_\tau(y) = e^{\beta(F'-\eta_y)}$

In this case,

$$\Omega(x,y) = \ln \frac{\mathcal{P}_F(x,y)}{\mathcal{P}_R(x,y)} = \ln \frac{\varphi(y|x)p(x)}{\tilde{\varphi}(x|y)q(y)} = \ln \frac{p(x)}{q(y)}$$
$$= \beta(F - \epsilon_x + F' + \eta_y) = \beta(W - \Delta F)$$

$$\implies \frac{\mu_F(W)}{\mu_R(W)} = e^{\beta(W-\Delta F)} \implies \langle W \rangle \geq \Delta F$$

Example: nonequilibrium steady states

- stochastic process $\varphi(y|x)$ with non-thermal steady state $s(x)$
- thermal equilibrium priors: $p(x) = q(x) \propto e^{-\beta\epsilon_x}$
- fluctuation variable:
  $\omega = \ln \frac{\mathcal{P}_F(x,y)}{\mathcal{P}_R(x,y)} = \ln \frac{p(x)}{q(y)} \frac{s(y)}{s(x)} = \beta(\epsilon_y - \epsilon_x) + (\ln s(y) - \ln s(x))$
- *nonequilibrium potential*: $V(x) := -\frac{1}{\beta} \ln s(x)$ (e.g., Manzano&al 2015)
- nonequilibrium potentials (usually introduced *ad hoc*) are understood here as remnants of Bayesian inversion
- $\implies \left\langle e^{\beta(\Delta E - \Delta V)} \right\rangle_F = 1 \implies D(p\|s) - D(\varphi[p]\|s) \geq 0$

But why known relations are compatible with Bayesian inversion?

Is that a necessity?

Sketch argument

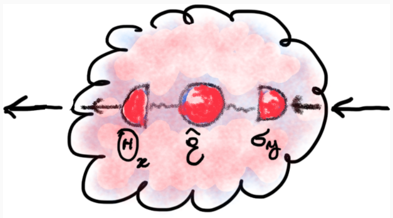
- $D(\mathcal{P}_F \| \mathcal{P}_R) = \left\langle \ln \frac{\mathcal{P}_F(x,y)}{\mathcal{P}_R(x,y)} \right\rangle_F$
- let us impose that the fluctuation variable is **local**:
  $\ln \frac{\mathcal{P}_F(x,y)}{\mathcal{P}_R(x,y)} = \Omega(x,y) \stackrel{!}{=} g_2(y) - g_1(x)$
  - $\implies \quad \frac{\mathcal{P}_F(y|x)}{\mathcal{P}_R(x|y)} = \frac{h_2(y)}{h_1(x)}$
  - $\implies \quad h_1(x)\mathcal{P}_F(y|x) = h_2(y)\mathcal{P}_R(x|y)$
  - sum over $x \implies \quad h_2(y) = \sum_x h_1(x)\mathcal{P}_F(y|x)$
- $\implies \quad \mathcal{P}_R(x|y) = \frac{1}{\sum_x h_1(x)\mathcal{P}_F(y|x)} h_1(x)\mathcal{P}_F(y|x)$

Hence, a Bayesian inverse-like form for the reverse process is **inevitable** if we want the fluctuating variable to have a local form!

Finally, what about the quantum case?

Quantum retrodiction and the Petz map

- assume $\varphi(y|x) = \mathrm{Tr}[\Pi_y \; \mathcal{E}(\rho_x)]$
- let $s(x)$ be invariant distribution
- according to the formalism of *quantum retrodiction*:
  - $\Sigma := \sum_x s(x)\rho_x$
  - $\tilde{\rho}_y := \frac{1}{s(y)}\sqrt{\mathcal{E}(\Sigma)}\Pi_y\sqrt{\mathcal{E}(\Sigma)}$
  - $\tilde{\Pi}_x := s(x)\frac{1}{\sqrt{\Sigma}}\rho_x\frac{1}{\sqrt{\Sigma}}$
  - $\tilde{\mathcal{E}}_\Sigma(\bullet) := \sqrt{\Sigma}\left\{\mathcal{E}^\dagger\left[\frac{1}{\sqrt{\mathcal{E}(\Sigma)}}(\bullet)\frac{1}{\sqrt{\mathcal{E}(\Sigma)}}\right]\right\}\sqrt{\Sigma}$



- Bayesian inversion works seamlessly
  $\tilde{\varphi}(x|y) = \mathrm{Tr}[\tilde{\Pi}_x \; \tilde{\mathcal{E}}_\Sigma(\tilde{\rho}_y)]$

Conclusion

Conceptual take-home messages

1. the reversal in general depends on a choice of prior (exceptions: Hamiltonian processes)
2. classically, Petz's map coincides with Jeffrey's retrodiction
3. in quantum theory, it agrees with previous proposals for "quantum retrodiction"
4. however, we still lack of a thorough mathematical theory of quantum inference (Petz map is not unique in general!)
5. reverse processes in statistical mechanics better seen as "reverse inferences" rather than "time-reverses"
6. the inferential approach circumvents the problem of "realization"

thank you

About these ideas

Two papers:

- with V. Scarani. *Fluctuation relations from Bayesian retrodiction*. Phys. Rev. E (2021). arXiv:2009.02849 [quant-ph]
- with C.C. Aw and V. Scarani. *Fluctuation Theorems with Retrodiction rather than Reverse Processes*. AVS Quantum Science (to appear). arXiv:2106.08589 [cond-mat.stat-mech]