measurement sharpness and incompatibility: problems and solutions

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references

- F.B., E. Chitambar, W. Zhou:
  A complete resource theory of quantum (POVMs) incompatibility as quantum programmability.
  Physical Review Letters 124, 120401 (2020)
- F.B., K. Kobayashi, S. Minagawa, P. Perinotti, A. Tosini:
  Unifying different notions of quantum (instruments) incompatibility into a strict hierarchy of resource theories of communication.
  Quantum 7, 1035 (2023)
- F.B., K. Kobayashi, S. Minagawa:
  A complete and operational resource theory of measurement sharpness.
  Arxiv:2303.07737 (submitted)

POVMs, instruments, and channels

in this talk: all sets ($\mathbb{X}, \mathbb{Y}$ etc.) are finite, all spaces ($\mathscr{H}_A, \mathscr{H}_B$ etc.) are finite-dimensional

POVM: family $\mathbf{P}$ of positive semidefinite operators on $\mathscr{H}$ labeled by set $\mathbb{X}$, i.e., $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$, with $P_x \geqslant 0$ and $\sum_x P_x = \mathbb{1}$

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\varrho\ P_x]$

instrument: family $\{\mathcal{I}_x : A \to B\}_{x\in\mathbb{X}}$ of completely positive (CP) linear maps from $\mathscr{B}(\mathscr{H}_A)$ to $\mathscr{B}(\mathscr{H}_B)$, such that $\sum_x \mathcal{I}_x$ is trace-preserving (TP)

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\mathcal{I}_x(\varrho)]$, and corresponding post-measurement state is $\frac{1}{p(x)}\mathcal{I}_x(\varrho)$

channel: any CPTP linear map $\mathcal{E} : A \to B$; its trace-dual $\mathcal{E}^\dagger$ (CP and unit-preserving) is defined by $\mathrm{Tr}[\mathcal{E}(\varrho)\ X] = \mathrm{Tr}\big[\varrho\ \mathcal{E}^\dagger(X)\big]$, $\forall\varrho, \forall X$

the problem with measurement sharpness

sharp POVMs: conventional definition

definition (folklore): a POVM $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$ is called sharp whenever all its elements are projectors, i.e., $P_x P_{x'} = \delta_{x,x'} P_x$ for all $x, x' \in \mathbb{X}$

intuition: sharp POVMs are "sharp" because

- orthogonal projectors are "pointed"
- they can be measured in a repeatable, i.e., "clear-cut" way

sharpness as a resource: Paul Busch already in 2005 envisioned a "resource theory of sharpness" proposing a class of sharpness measures; most recent work is by Liu and Luo (2022), and by Mitra (2022)

transforming POVMs

POVMs can be transformed using:

- a quantum preprocessing, i.e., a channel $\mathcal{E}$ such that

$$\{P_x\}_{x\in\mathbb{X}} \mapsto \{\mathcal{E}^\dagger(P_x)\}_{x\in\mathbb{X}}$$

- a classical postprocessing, i.e., a conditional distribution $\mu(y|x)$ such that

$$\{P_x\}_{x\in\mathbb{X}} \mapsto \{\sum_x \mu(y|x) P_x\}_{y\in\mathbb{Y}}$$

- a convex mixture with another fixed POVM $\mathbf{T} = \{T_x\}_{x\in\mathbb{X}}$, i.e.

$$\{P_x\}_{x\in\mathbb{X}} \mapsto \{\lambda P_x + (1-\lambda) T_x\}_{x\in\mathbb{X}} \quad \lambda \in [0,1]$$

- a composition of the above

the problem with sharpness processing

Which processings are sharpness-non-increasing?

- quantum preprocessings: can turn non-sharp into sharp $\rightsquigarrow$ **ILLEGAL**
- classical postprocessings: can turn non-sharp into sharp $\rightsquigarrow$ **ILLEGAL**
- convex mixtures: legal if **T** is "maximally dull", but we need to characterize maximally dull POVMs first

the problem with measurements incompatibility

incompatibility

In quantum theory, some measurements necessarily exclude others.

If all measurements were compatible, we would not have QKD, violation of Bell's inequalities, quantum speedups, etc.

Various formalizations:

- preparation uncertainty relations (e.g., Robertson)
- measurement uncertainty relations (e.g., Ozawa)
- incompatibility

compatible POVMs 1/2

Definition

given a family $\{\mathbf{P}^{(i)}\}_{i\in\mathbb{I}} \equiv \{P_x^{(i)}\}_{x\in\mathbb{X},i\in\mathbb{I}}$ of POVMs, all defined on the same system A, we say that the family is **compatible**, whenever there exists

- a "mother" POVM $\mathbf{O} = \{O_w\}_{w\in\mathbb{W}}$ on system A
- a conditional probability distribution $\mu(x|w,i)$

such that

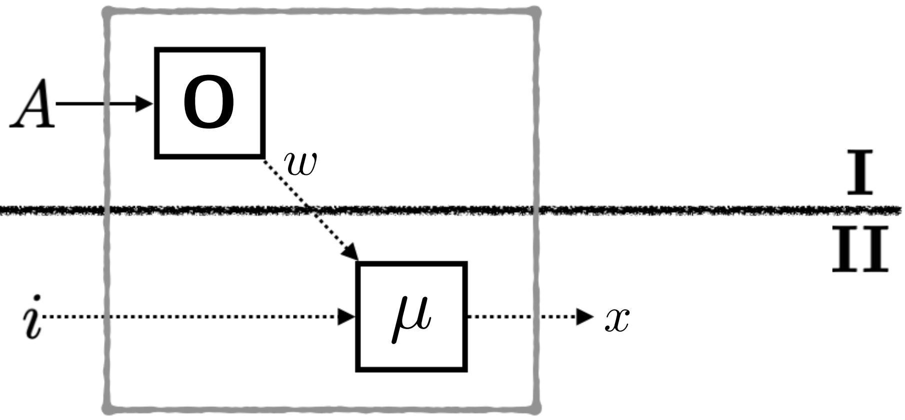
$$P_x^{(i)} = \sum_w \mu(x|w,i) O_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

But what does it mean, *operationally*, if I say that, e.g., a certain laboratory can only perform compatible measurement?

compatible POVMs 2/2

There's a bipartition hidden in the concept of (in)compatibility:



[F.B., E. Chitambar, W. Zhou; PRL 2020]

While there is consensus on a single notion of compatibility for POVMs, the situation is less clear for instruments...

classical compatibility 1/2

Definition (Heinosaari–Miyadera–Reitzner, 2014)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **classically compatible**, whenever there exist

- a mother instrument $\{\mathcal{H}_w : A \to B\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$

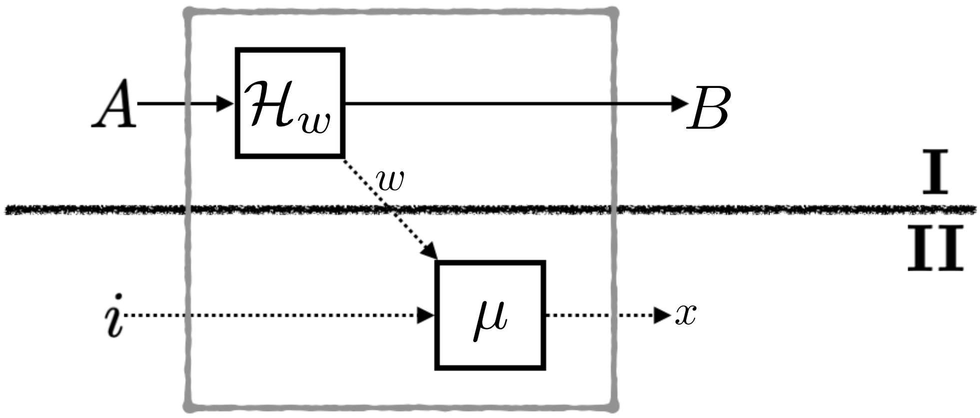
such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)\mathcal{H}_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

we call this "classical" because it involves only classical post-processings; it is also called "traditional" [Mitra and Farkas; PRA 2022].

classical compatibility 2/2



crucially:

- no shared entanglement and communication is classical
- communication goes only from $\mathbf{I}$ to $\mathbf{II}$, i.e., the above is necessarily $\mathbf{II}\to\mathbf{I}$ non-signaling, see [Ji and Chitambar; PRA 2021]

marginalizing the mother

- without loss of generality (classical labels can be copied), compatible POVMs may be assumed to be recovered by marginalization, i.e.,

$$P_x^{(i)} = \sum_{x_j : j \neq i} O_{x_1, x_2, \ldots, x_n}$$

- i.e., *everything* that needs to be output is output *simultaneously* by the mother

- the notion of "parallel compatibility" for instruments extends the above insight from the classical outcomes to the quantum outputs

parallel compatibility 1/2

Definition (Heinosaari–Miyadera–Ziman, 2015)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x \in \mathbb{X}, i \in \mathbb{I}}$, we say that the family is **parallelly compatible**, whenever there exist

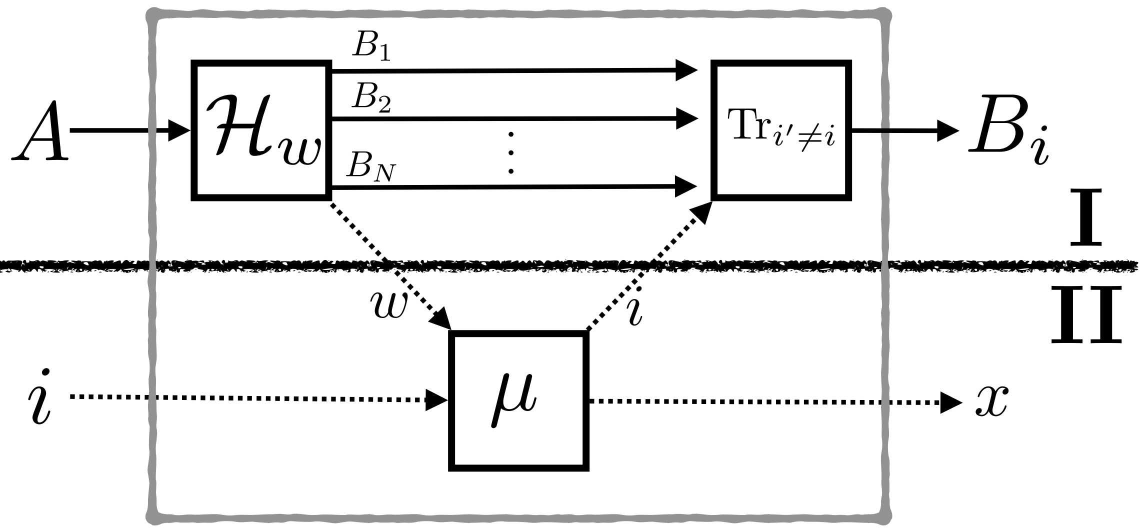
- a mother instrument $\{\mathcal{H}_w : A \to \otimes_{i \in \mathbb{I}} B_i\}_{w \in \mathbb{W}}$
- a conditional probability distribution $\mu(x|w, i)$

such that

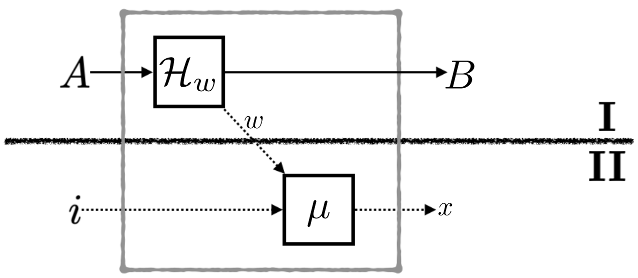
$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w, i) [\mathrm{Tr}_{B_{i' : i' \neq i}} \circ \mathcal{H}_w] ,$$

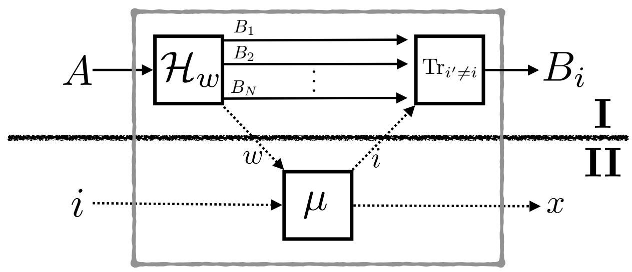
for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

parallel compatibility 2/2



parallel compatibility VS classical compatibility





- parallel compatibility is able to go beyond no-signaling, hence, parallel compatibility $\not\Rightarrow$ classical compatibility
- parallel compatibility has nothing to do with the "no information without disturbance" principle, because non-disturbing instruments are never parallelly compatible
- hence classical compatibility $\not\Rightarrow$ parallel compatibility

two simple examples

two instruments, classically compatible but parallelly incompatible

$$\left\{\frac{1}{2}\text{id}, \frac{1}{2}\text{id}\right\} \text{ and } \left\{\frac{2}{3}\text{id}, \frac{1}{3}\text{id}\right\}$$

two instruments, parallelly compatible but classically incompatible

$$\{\langle 0| \bullet |0\rangle |0\rangle\langle 0| \ , \ \langle 1| \bullet |1\rangle |0\rangle\langle 0|\} \text{ and } \{\langle 0| \bullet |0\rangle |1\rangle\langle 1| \ , \ \langle 1| \bullet |1\rangle |1\rangle\langle 1|\}$$

this is not OK...

a solution for sharpness

new definition: sharp$^\sharp$ POVMs

Definition

A given POVM $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$ is sharp$^\sharp$ whenever the set

$$\operatorname{range}\mathbf{P} := \left\{\mathbf{p} \in \mathbb{R}_+^{|\mathbb{X}|} : \exists \varrho \text{ state}, p_x = \operatorname{Tr}[\varrho\ P_x]\,, \forall x\right\}$$

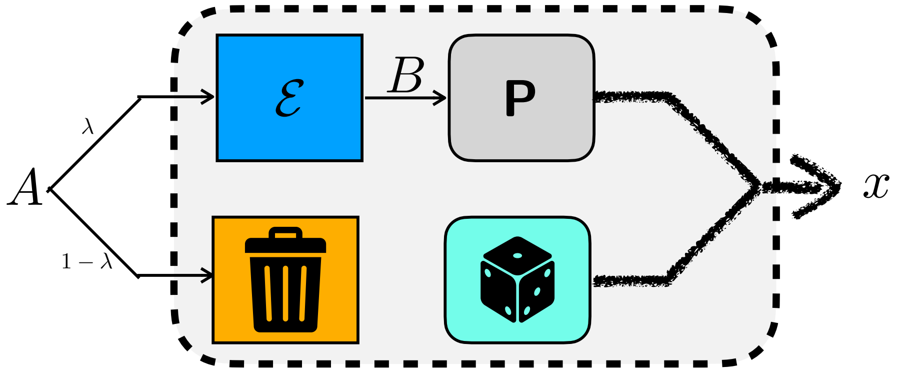
coincides with the entire probability simplex ("sharp"!) on $\mathbb{X}$.
It is dull$^\sharp$ whenever $\operatorname{range}\mathbf{P}$ is a singleton.

- sharp$^\sharp$ $\iff$ $\forall x, \exists|\psi_x\rangle : P_x|\psi_x\rangle = |\psi_x\rangle$ ($\implies$ $P_x|\psi_{x'}\rangle = 0$ for $x \neq x'$)
- sharp$^\sharp$ $\implies$ $\dim\mathscr{H} \geqslant |\mathbb{X}|$ $\implies$ nondegenerate observables are "canonical"
- excluding null POVM elements, sharp $\underset{\not\Leftarrow}{\Rightarrow}$ sharp$^\sharp$ $\underset{\Leftarrow}{\Rightarrow}$ repeatably measurable
- dull$^\sharp$ $\iff$ $P_x \propto \mathbb{1}$, for all $x \in \mathbb{X}$

fuzzifying operations as affine maps

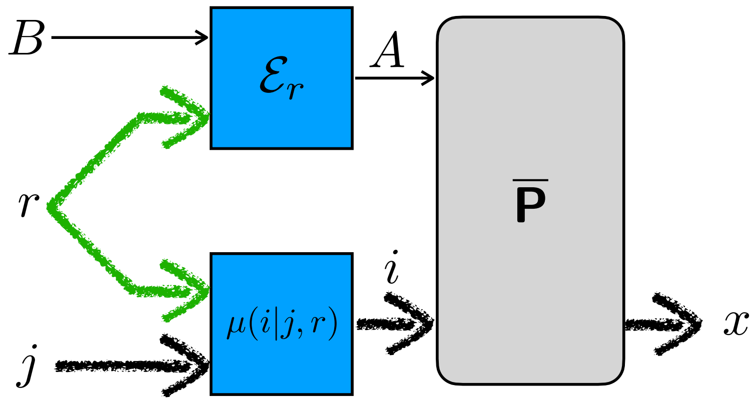
for sharpness$^\sharp$:

- quantum preprocessing: **LEGAL**
- convex mixture with any dull$^\sharp$ POVM: **LEGAL**



$$P_x \longmapsto \lambda\mathcal{E}^\dagger(P_x) + (1-\lambda)p(x)\mathbb{1}\ , \qquad \forall x \in \mathbb{X}$$

fuzzifying operations as linear maps (LOSR)



where

- $i, j \in \{0, 1, 2, \ldots, |\mathbb{X}|\}$ label all possible programs
- $\overline{\mathbf{P}} = (\mathbf{P}, \mathbf{T}^{(1)}, \mathbf{T}^{(2)} \ldots, \mathbf{T}^{|\mathbb{X}|})$ is a programmable POVM with $|\mathbb{X}| + 1$ program states, with $\mathbf{T}^{(i)} = \{T_x^{(i)}\}_{x \in \mathbb{X}}$ denoting the deterministic POVMs, i.e., $T_x^{(i)} = \delta_{i,x}\mathbb{1}$

the sharpness$^\sharp$ preorder

Definition

given two POVMs $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$ and $\mathbf{Q} = \{Q_x\}_{x\in\mathbb{X}}$, we say that **P** is sharper$^\sharp$ than **Q** ($\mathbf{P} \succ^\sharp \mathbf{Q}$) whenever:

- there exists a fuzzifying operation transforming **P** into **Q**
- equivalently: there exists a CPTP linear map $\mathcal{E}$ such that

$$\mathbf{Q} \in \mathrm{conv}\{\mathcal{E}^\dagger(\mathbf{P}), \mathbf{T}^{(1)}, \mathbf{T}^{(2)} \ldots, \mathbf{T}^{|\mathbb{X}|}\}$$

maximal and minimal elements for $\succ^\sharp$

Theorem

A given POVM **P** *is*

- sharp$^\sharp$ $\iff$ $\succ^\sharp$*-maximal: if, for any* **Q** *such that* $\mathbf{Q} \succ^\sharp \mathbf{P}$, *then* $\mathbf{P} \succ^\sharp \mathbf{Q}$
- dull$^\sharp$ $\iff$ $\succ^\sharp$*-minimal: if, for any* **Q** *such that* $\mathbf{P} \succ^\sharp \mathbf{Q}$, *then* $\mathbf{Q} \succ^\sharp \mathbf{P}$

Hence, all sharp$^\sharp$ (dull$^\sharp$) POVMs are equivalent to each other under fuzzifying operations.

$\rightsquigarrow$ this has thus become a problem of statistical comparison, with its Blackwell–like theorem and a complete family of monotones.

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a solution for measurements incompatibility

bridging the two camps: q-compatibility

Definition

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **q-compatible**, whenever there exist

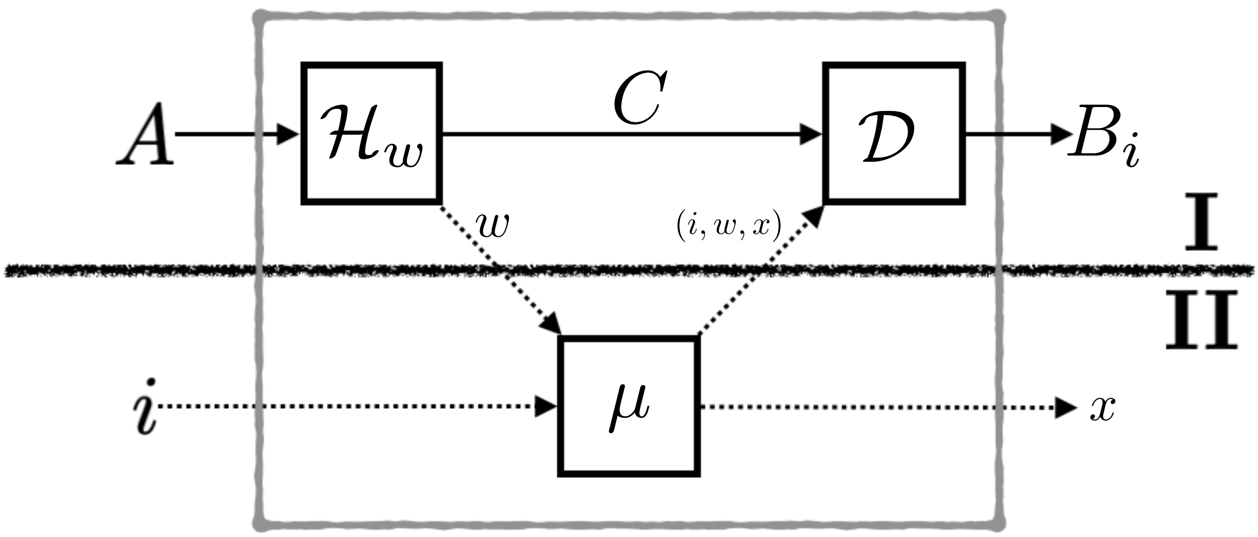
- a mother instrument $\{\mathcal{H}_w : A \to C\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$
- a family of postprocessing channels $\{\mathcal{D}^{(x,w,i)} : C \to B_i\}_{x\in\mathbb{X}, w\in\mathbb{W}, i\in\mathbb{I}}$

such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathcal{D}^{(x,w,i)} \circ \mathcal{H}_w] \ ,$$

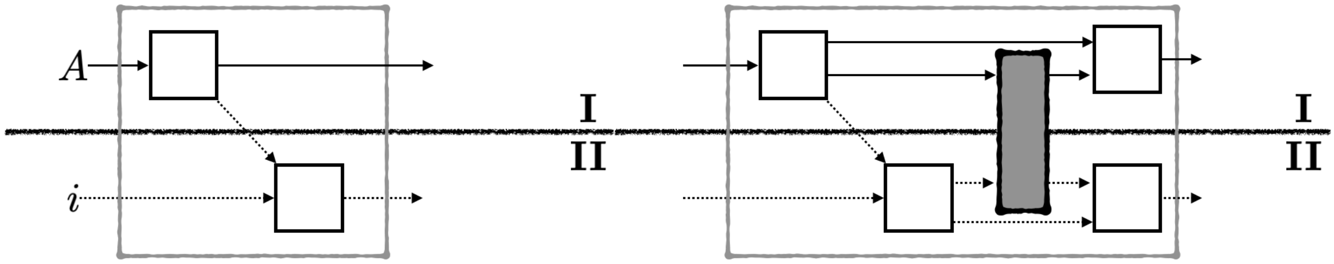
for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

q-compatibility as a circuit



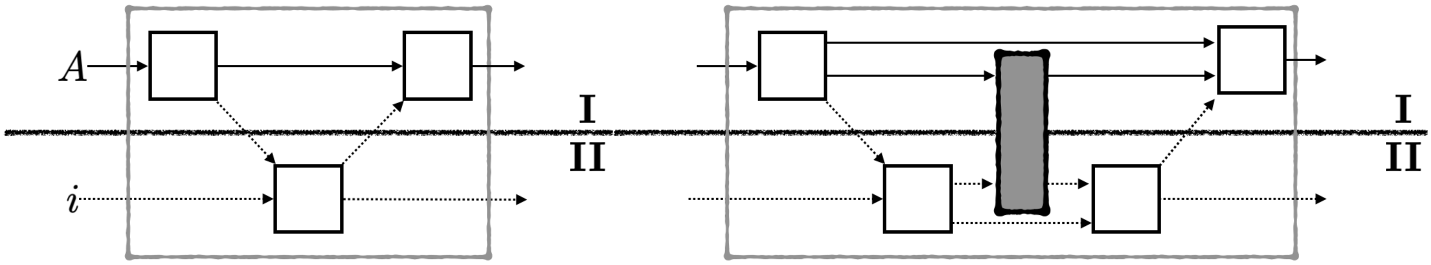
classical compatibility: $C \equiv B_i$ and $\mathcal{D}^{(x,w,i)} = \text{id}$
parallel compatibility: $C \equiv \bigotimes_i B_i$ and $\mathcal{D}^{(x,w,i)} = \mathrm{Tr}_{B_{i'}:i'\neq i}$

free operations for classical incompatibility



- all cassically compatible devices can be created for free
- if the initial device (the dark gray inner box) is classically compatible, the final device is also classically compatible

free operations for q-incompatibility



- all q-compatible devices can be created for free
- if the initial device (the dark gray inner box) is q-compatible, the final device is also q-compatible

the incompatibility preorder

given two families of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$ and $\{\mathcal{J}_y^{(j)} : C \to D_j\}_{y\in\mathbb{Y}, j\in\mathbb{J}}$, we say

"$\{\mathcal{I}_x^{(i)} : A \to B_i\}$ is more q-incompatible than $\{\mathcal{J}_y^{(j)} : C \to D_j\}$"

whenever the former can be transformed into the latter by means of a free operation

$\rightsquigarrow$ this is now an instance of statistical comparison: a Blackwell–like theorem can be proved, and a complete family of monotones obtained

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conclusion

take home messages

- about sharpness:
  - with the new definition, we now have a sound resource theory, complete with a family of sharpness-non-increasing operations and sharpness monotones
  - sharpness is essentially a measure of classical communication capacity (more precisely, signaling dimension)
- about incompatibility:
  - no need to argue about the "correct" definition of compatibility: q-compatibility provides an overarching framework
  - incompatibility is an essentially nonlocal concept, i.e., quantum information transmission, either in space (quantum channel) or in time (quantum memory)

thank you