incompatible incompatibilities

and how to make them compatible again

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references

- F.B., E. Chitambar, W. Zhou:
  A complete resource theory of quantum (POVMs) incompatibility as quantum programmability.
  Physical Review Letters 124, 120401 (2020)

- F.B., K. Kobayashi, S. Minagawa, P. Perinotti, A. Tosini:
  Unifying different notions of quantum (instruments) incompatibility into a strict hierarchy of resource theories of communication.
  Quantum 7, 1035 (2023)

POVMs and instruments

in this talk: all sets ($\mathbb{X}, \mathbb{Y}$ etc.) are finite, all spaces ($\mathscr{H}_A, \mathscr{H}_B$ etc.) are finite-dimensional

POVM: family P of positive semidefinite operators on $\mathscr{H}$ labeled by set $\mathbb{X}$, i.e., $\mathsf{P} = \{P_x\}_{x\in\mathbb{X}}$, with $P_x \geqslant 0$ and $\sum_x P_x = \mathbb{1}$

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\varrho\ P_x]$

instrument: family $\{\mathcal{I}_x : A \to B\}_{x\in\mathbb{X}}$ of completely positive (CP) linear maps from $\mathscr{B}(\mathscr{H}_A)$ to $\mathscr{B}(\mathscr{H}_B)$, such that $\sum_x \mathcal{I}_x$ is trace-preserving (TP)

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\mathcal{I}_x(\varrho)]$, and corresponding post-measurement state is $\frac{1}{p(x)}\mathcal{I}_x(\varrho)$

incompatibility

In quantum theory, some measurements necessarily exclude others.

If all measurements were compatible, we would not have QKD, violation of Bell's inequalities, quantum speedups, etc.

Various formalizations:

- preparation uncertainty relations
- measurement (e.g., error-disturbance) uncertainty relations
- incompatibility

compatible POVMs 1/2

Definition

given a family $\{\mathbf{P}^{(i)}\}_{i\in\mathbb{I}} \equiv \{P_x^{(i)}\}_{x\in\mathbb{X},i\in\mathbb{I}}$ of POVMs, all defined on the same system A, we say that the family is **compatible**, whenever there exists

- a "mother" POVM $\mathbf{O} = \{O_w\}_{w\in\mathbb{W}}$ on system A
- a conditional probability distribution $\mu(x|w,i)$

such that

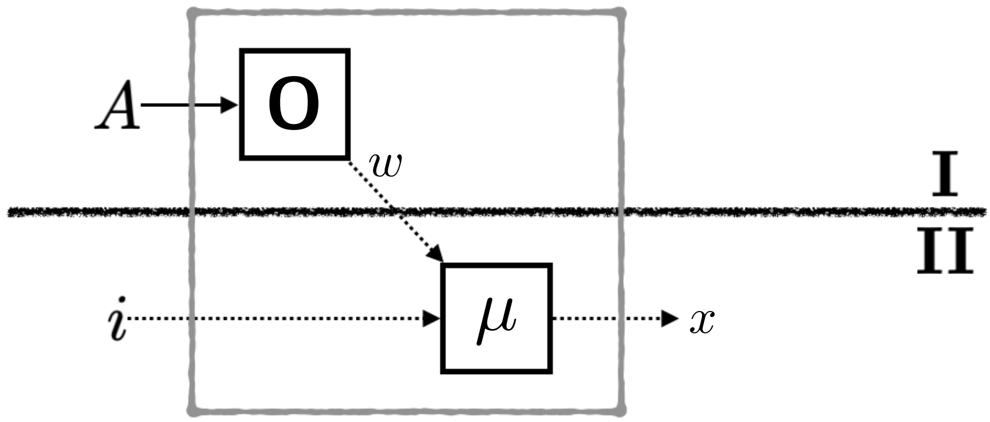
$$P_x^{(i)} = \sum_w \mu(x|w,i) O_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

But what does it mean, *operationally*, if I say that, e.g., a certain laboratory can only perform compatible measurement?

compatible POVMs 2/2

There's a bipartition hidden in the concept of (in)compatibility:



[F.B., E. Chitambar, W. Zhou; PRL 2020]

the problem

While there is consensus on a single notion of compatibility for POVMs, in the case of instruments, the situation is less clear...

classical compatibility 1/2

Definition (Heinosaari–Miyadera–Reitzner, 2014)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **classically compatible**, whenever there exist

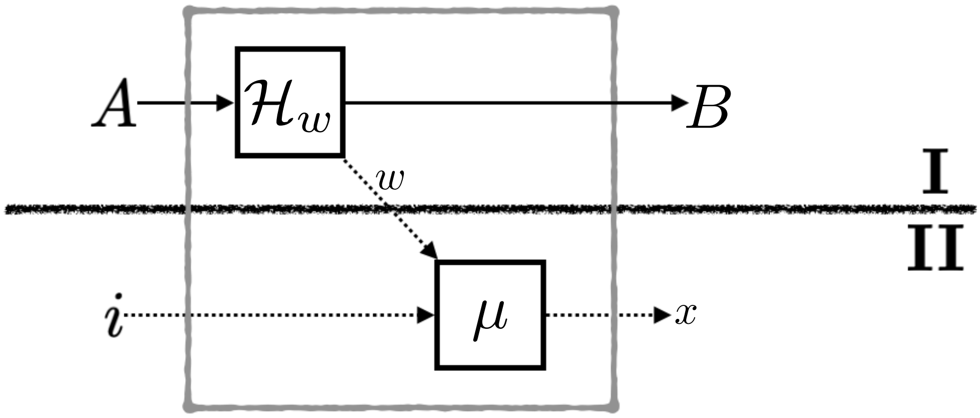
- a mother instrument $\{\mathcal{H}_w : A \to B\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$

such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)\mathcal{H}_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

classical compatibility 2/2



crucially:

- II is classical: no shared entanglement, communication is classical
- the box is II→I non-signaling:communication goes only from I to II; see [Ji and Chitambar; PRA 2021]

alternative: marginalizing the mother

- in the case of POVMs: without loss of generality (because classical information can be copied), one can consider only marginalizations, i.e.,

$$P_x^{(i)} = \sum_{x_j : j \neq i} O_{x_1, x_2, \ldots, x_n}$$

- the notion of "parallel compatibility" for instruments applies the same intuition to the quantum outputs too

parallel compatibility 1/2

Definition (Heinosaari–Miyadera–Ziman, 2015)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **parallelly compatible**, whenever there exist

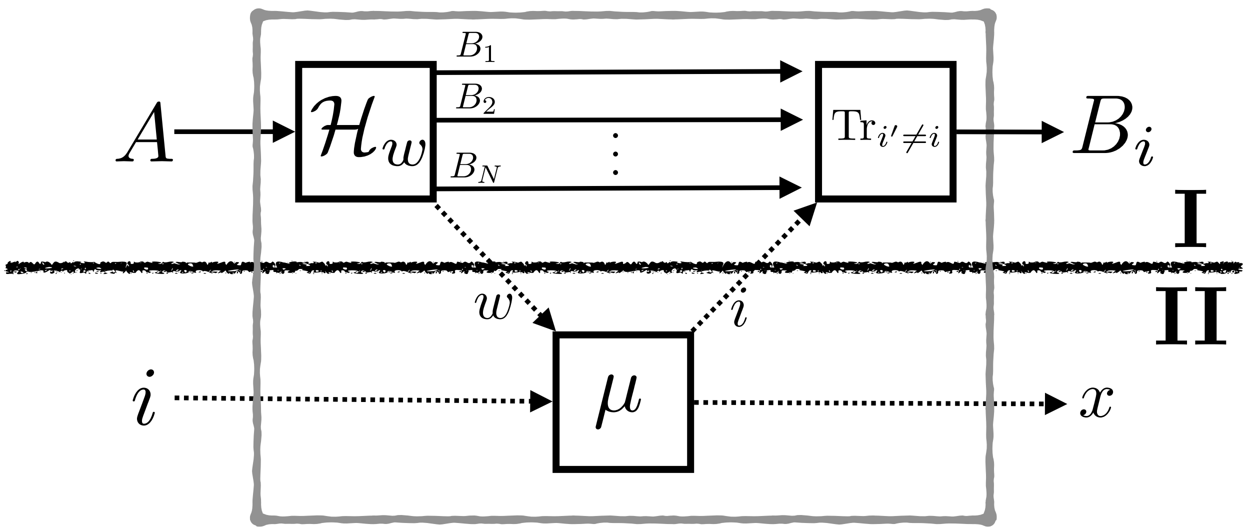
- a mother instrument $\{\mathcal{H}_w : A \to \bigotimes_{i\in\mathbb{I}} B_i\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$

such that

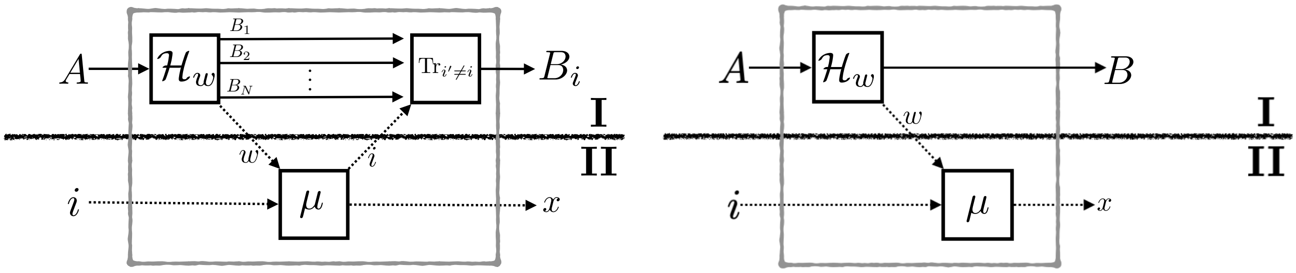
$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathrm{Tr}_{B_i^c} \circ \mathcal{H}_w] \;, \qquad B_i^c := \bigotimes_{i'\in\mathbb{I}:i'\neq i} B_{i'} \;,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

parallel compatibility 2/2



parallel compatibility VS classical compatibility



- parallel compatibility is both $\mathbf{I}\to\mathbf{II}$ and $\mathbf{II}\to\mathbf{I}$ signaling; therefore, parallel compatibility $\not\Longrightarrow$ classical compatibility
- non-disturbing instruments are never parallelly compatible; therefore, classical compatibility $\not\Longrightarrow$ parallel compatibility
- parallel compatibility is more closely related to quantum no-broadcasting than it is to measurement compatibility

bridging the two camps

q-compatibility

Definition

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is **q-compatible**, whenever there exist

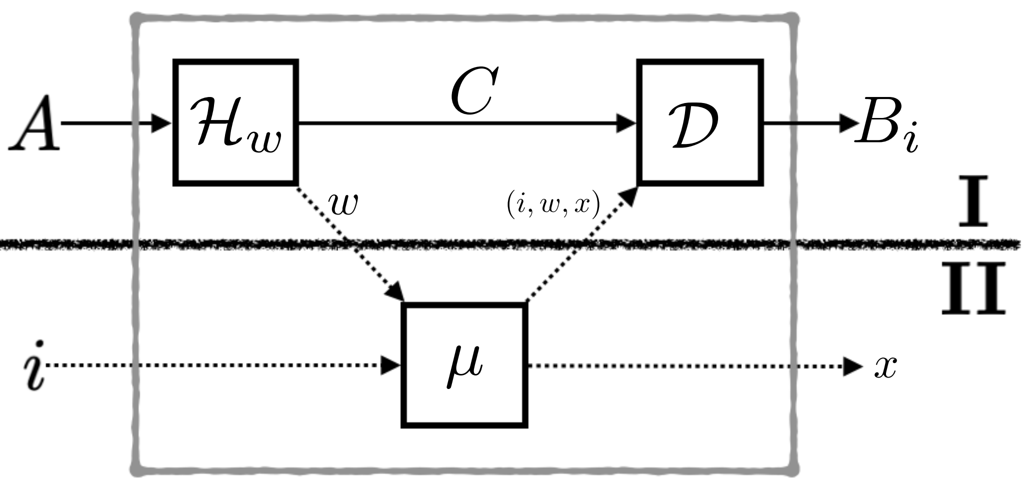
- a mother instrument $\{\mathcal{H}_w : A \to C\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$
- a family of postprocessing channels $\{\mathcal{D}^{(x,w,i)} : C \to B_i\}_{x\in\mathbb{X}, w\in\mathbb{W}, i\in\mathbb{I}}$

such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathcal{D}^{(x,w,i)} \circ \mathcal{H}_w] \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

q-compatibility as a circuit

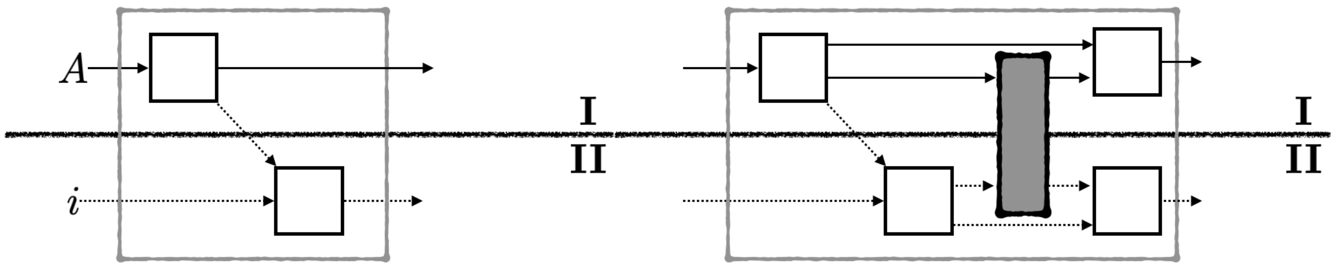


Special cases:

- classical compatibility: when $C \equiv B_i$ for all i and $\mathcal{D}^{(x,w,i)} = \text{id}$
- parallel compatibility: when $C \equiv \bigotimes_i B_i$ and $\mathcal{D}^{(x,w,i)} = \mathrm{Tr}_{B_i^c}$

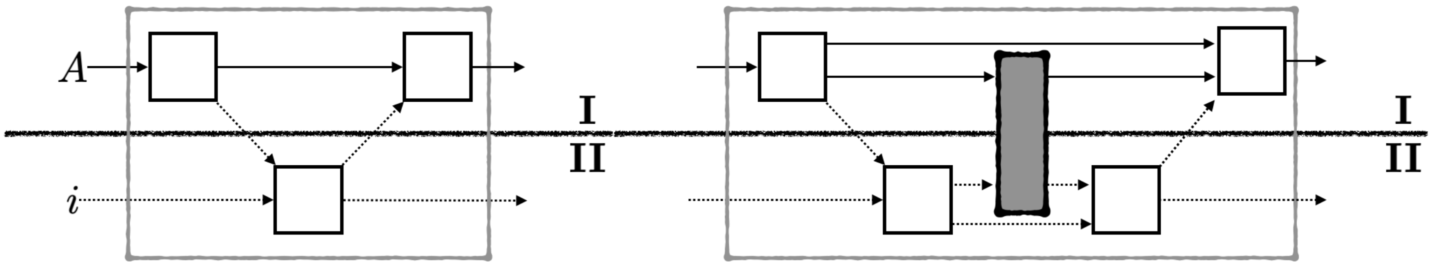
incompatibility-non-increasing operations

free operations for classical incompatibility



- all cassically compatible devices can be created for free
- if the initial device (the dark gray inner box) is classically compatible, the final device is also classically compatible

free operations for q-incompatibility



- all q-compatible devices can be created for free
- if the initial device (the dark gray inner box) is q-compatible, the final device is also q-compatible

beyond q-compatibility

simultaneous VS sequential compatibility

compatibility of POVMs is always, w.l.o.g., simultaneous compatibility (again, because classical information can be copied)

for instruments, it is more subtle: for example, consider the following two instruments

$$\mathcal{I}_1(\bullet) = pU_1 \bullet U_1^\dagger \qquad \mathcal{J}_1(\bullet) = |0\rangle\langle 0| \bullet |0\rangle\langle 0|$$
$$\mathcal{I}_2(\bullet) = (1-p)U_2 \bullet U_2^\dagger \qquad \mathcal{J}_2(\bullet) = |1\rangle\langle 1| \bullet |1\rangle\langle 1|$$

the corresponding POVMs, i.e., $\{p\mathbb{1}, (1-p)\mathbb{1}\}$ and $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ are compatible, but the two instruments are not (they are not even q-compatible)

except that, in a sense, they actually *are* compatible!
just not simultaneously so: do $\mathcal{I}$, keep the outcome, undo the unitary, and finally do $\mathcal{J}$

remark: reversing the order (i.e., first $\mathcal{J}$, then $\mathcal{I}$) the same construction does not work

no-exclusivity

Definition

an instrument $\{\mathcal{I}_x : A \to B_1\}_{x\in\mathbb{X}}$ **does not exclude** another instrument $\{\mathcal{J}_y : A \to B_2\}_{y\in\mathbb{Y}}$, whenever there exist

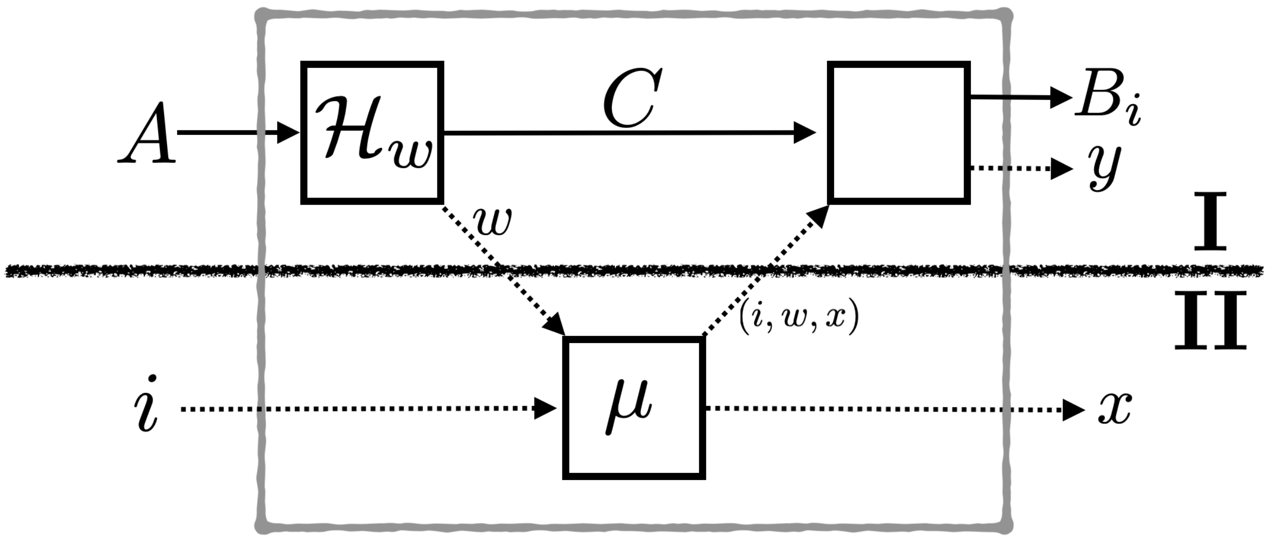
- a mother instrument $\{\mathcal{H}_w : A \to C\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w)$
- a family of postprocessing channels $\{\mathcal{D}^{(x,w)} : C \to B_1\}_{x\in\mathbb{X},w\in\mathbb{W}}$
- a family of instruments $\{\mathcal{K}_y^{(w)} : C \to B_2\}_{w\in\mathbb{W},y\in\mathbb{Y}}$

such that

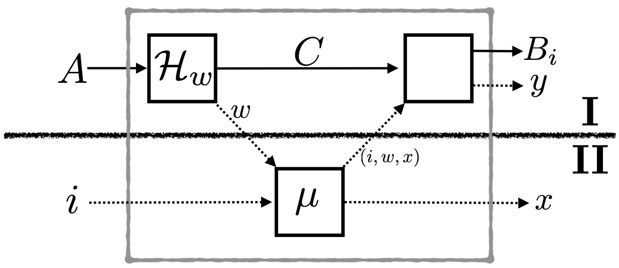
$$\mathcal{I}_x = \sum_w \mu(x|w)[\mathcal{D}^{(x,w)} \circ \mathcal{H}_w] , \qquad \mathcal{J}_y = \sum_w \mathcal{K}_y^{(w)} \circ \mathcal{H}_w ,$$

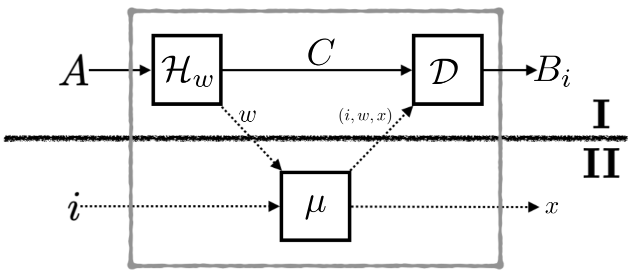
for all $x \in \mathbb{X}$ and all $y \in \mathbb{Y}$.

no-exclusivity as a circuit



no-exclusivity VS q-compatibility





in the definition of no-exclusivity, the post-processing box at **I** can be an instrument

intuition: the information necessary to reconstruct the result of one instrument (i.e., the non-exclud**ing** one) can be extracted with a disturbance "small enough" to not exclude the other instrument (i.e., the non-exclud**ed** one)

formulating resource theories of incompatibility

a hierarchy of incompatibility preorders

given two families of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x \in \mathbb{X}, i \in \mathbb{I}}$ and $\{\mathcal{J}_y^{(j)} : C \to D_j\}_{y \in \mathbb{Y}, j \in \mathbb{J}}$, we say

$\{\mathcal{I}_x^{(i)} : A \to B_i\}$ is **more incompatible/exclusive** than $\{\mathcal{J}_y^{(j)} : C \to D_j\}$

whenever the former can be transformed into the latter by means of a corresponding free operation

more classically incompatible $\Longrightarrow$ more q-incompatible $\Longrightarrow$ more exclusive

$\rightsquigarrow$ this is now an instance of *statistical comparison*: a complete family of monotones can be constructed and a Blackwell–like theorem proved

further details on arXiv:2211.09226

conclusion

take home messages

- no need to argue about the "correct" definition of compatibility for instruments: q-compatibility provides an overarching framework
- as such, we can think of it as a resource, and construct a whole hierarchy of complete resource theories of incompatibility, all of which fall back on the same notion of compatibility of POVMs in the case of instruments with trivial quantum output
- for instruments, simultaneous compatibility is not the end of the story: we also have a notion of sequential compatibility, i.e., no-exclusivity

thank you for your attention