measurement sharpness and incompatibility as quantum resources

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Probability theory is a measure theory with a soul. (Mark Kac)

Quantum foundations is a linear algebra with a soul. (anon.)

references

- F.B., K. Kobayashi, S. Minagawa:
  A complete and operational resource theory of measurement sharpness.
  Arxiv:2303.07737

- F.B., K. Kobayashi, S. Minagawa, P. Perinotti, A. Tosini:
  Unifying different notions of quantum incompatibility into a strict hierarchy of resource theories of communication.
  Quantum 7, 1035 (2023)

POVMs and instruments

in this talk: all sets ($\mathbb{X}, \mathbb{Y}$ etc.) are finite, all spaces ($\mathscr{H}_A, \mathscr{H}_B$ etc.) are finite-dimensional

POVM: family **P** of positive semidefinite operators on $\mathscr{H}$ labeled by set $\mathbb{X}$, i.e., $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$, with $P_x \geqslant 0$ and $\sum_x P_x = \mathbb{1}$

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\varrho\ P_x]$

instrument: family $\{\mathcal{I}_x : A \to B\}_{x\in\mathbb{X}}$ of completely positive (CP) linear maps from $\mathscr{B}(\mathscr{H}_A)$ to $\mathscr{B}(\mathscr{H}_B)$, such that $\sum_x \mathcal{I}_x$ is trace-preserving (TP)

interpretation: expected probability of outcome x is $p(x) = \mathrm{Tr}[\mathcal{I}_x(\varrho)]$, and corresponding post-measurement state is $\frac{1}{p(x)}\mathcal{I}_x(\varrho)$

first part:
the problem with measurement sharpness

sharp POVMs: conventional definition

definition (folklore): a POVM $\mathbf{P} = \{P_x\}_{x \in \mathbb{X}}$ is called sharp whenever all its elements are projectors, i.e., $P_x P_{x'} = \delta_{x,x'} P_x$ for all $x, x' \in \mathbb{X}$

intuition: sharp POVMs are "sharp" because

- orthogonal projectors are "pointed"
- they can be measured in a repeatable, "clear-cut" way

sharpness as a resource: Paul Busch already in 2005 envisioned a "resource theory of sharpness" proposing a class of sharpness measures; most recent work is by Liu and Luo (2022), and by Mitra (2022)

question: how can sharpness be "processed"?

POVM processing

POVMs can be transformed using

- a quantum preprocessing, i.e., a CPTP linear map $\mathcal{E}$ such that $\{P_x\}_{x\in\mathbb{X}} \mapsto \{Q_x\}_{x\in\mathbb{X}}$ with $Q_x = \mathcal{E}^\dagger(P_x)$

- a classical postprocessing, i.e., a conditional distribution $\mu(y|x)$ such that $\{P_x\}_{x\in\mathbb{X}} \mapsto \{Q_y\}_{y\in\mathbb{Y}}$ with $Q_y = \sum_x \mu(y|x)P_x$

- a convex mixture with another fixed POVM $\mathbf{T} = \{T_x\}_{x\in\mathbb{X}}$, i.e., $\{P_x\}_{x\in\mathbb{X}} \mapsto \{\lambda P_x + (1-\lambda)T_x\}_{x\in\mathbb{X}}$, with $\lambda \in [0,1]$

- a composition of the above

the problem

Which processings are sharpness-non-increasing?

- quantum preprocessings: can turn non-sharp into sharp $\rightsquigarrow$ **ILLEGAL**

- classical postprocessings: can turn non-sharp into sharp $\rightsquigarrow$ **ILLEGAL**

- convex mixtures: legal if $\mathbf{T}$ is "maximally dull", but we need to characterize maximally dull POVMs first

new definition: sharp$^\sharp$ POVMs

Definition

A given POVM $\mathbf{P} = \{P_x\}_{x \in \mathbb{X}}$ is sharp$^\sharp$ whenever the set

$$\mathrm{range}\,\mathbf{P} := \left\{ \mathbf{p} \in \mathbb{R}_+^{|\mathbb{X}|} : \exists \varrho \text{ state}, p_x = \mathrm{Tr}[\varrho\ P_x]\,, \forall x \right\}$$

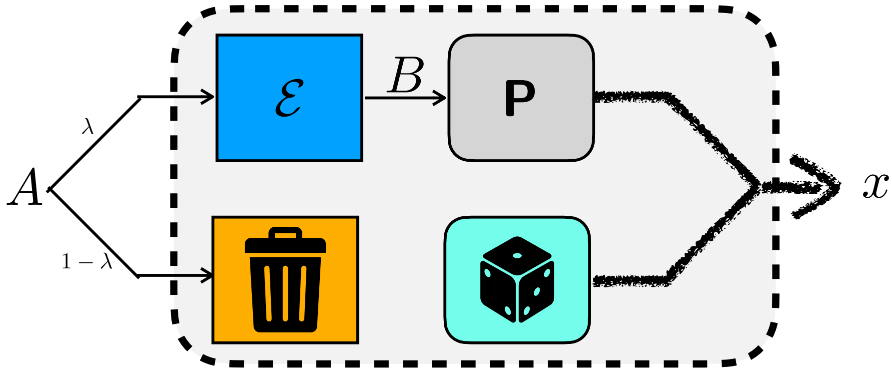
coincides with the entire probability simplex ("sharp"!) on $\mathbb{X}$.
It is dull$^\sharp$ whenever $\mathrm{range}\,\mathbf{P}$ is a singleton.

- sharp$^\sharp$ $\implies$ $\dim \mathscr{H} \geqslant |\mathbb{X}|$ $\implies$ nondegenerate observables are "canonical"
- sharp$^\sharp$ $\iff$ $\forall x, \exists |\psi_x\rangle : P_x|\psi_x\rangle = |\psi_x\rangle$ ($\implies$ $P_x|\psi_{x'}\rangle = 0$ for $x \neq x'$)
- excluding null POVM elements, sharp $\underset{\not\Leftarrow}{\Rightarrow}$ sharp$^\sharp$ $\underset{\Leftarrow}{\Rightarrow}$ repeatably measurable
- dull$^\sharp$ $\iff$ $P_x \propto \mathbb{1}$, for all $x \in \mathbb{X}$

fuzzifying operations

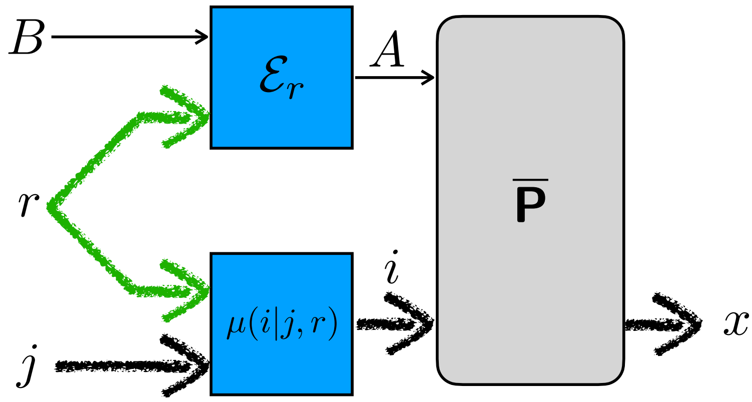
for sharpness$^\sharp$:

- quantum preprocessing: **LEGAL**
- convex mixture with any dull$^\sharp$ POVM: **LEGAL**



$$P_x \longmapsto \lambda \mathcal{E}^\dagger(P_x) + (1-\lambda)p(x)\mathbb{1}\,, \qquad \forall x \in \mathbb{X}$$

fuzzifying operations as LOSR preprocessings of programmable POVMs



where

- $i, j \in \{0, 1, 2, \ldots, |\mathbb{X}|\}$ label all possible programs
- $\overline{\mathbf{P}} = (\mathbf{P}, \mathbf{T}^{(1)}, \mathbf{T}^{(2)} \ldots, \mathbf{T}^{|\mathbb{X}|})$ is a programmable POVM with $|\mathbb{X}| + 1$ program states, with $\mathbf{T}^{(i)} = \{T_x^{(i)}\}_{x\in\mathbb{X}}$ denoting the deterministic POVMs, i.e., $T_x^{(i)} = \delta_{i,x}\mathbb{1}$

the sharpness$^\sharp$ preorder

Definition

given two POVMs $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$ and $\mathbf{Q} = \{Q_x\}_{x\in\mathbb{X}}$, we say that **P** is sharper$^\sharp$ than **Q** ($\mathbf{P} \succ^\sharp \mathbf{Q}$) whenever:

- there exists a fuzzifying operation transforming **P** into **Q**
- equivalently: there exists a CPTP linear map $\mathcal{E}$ such that

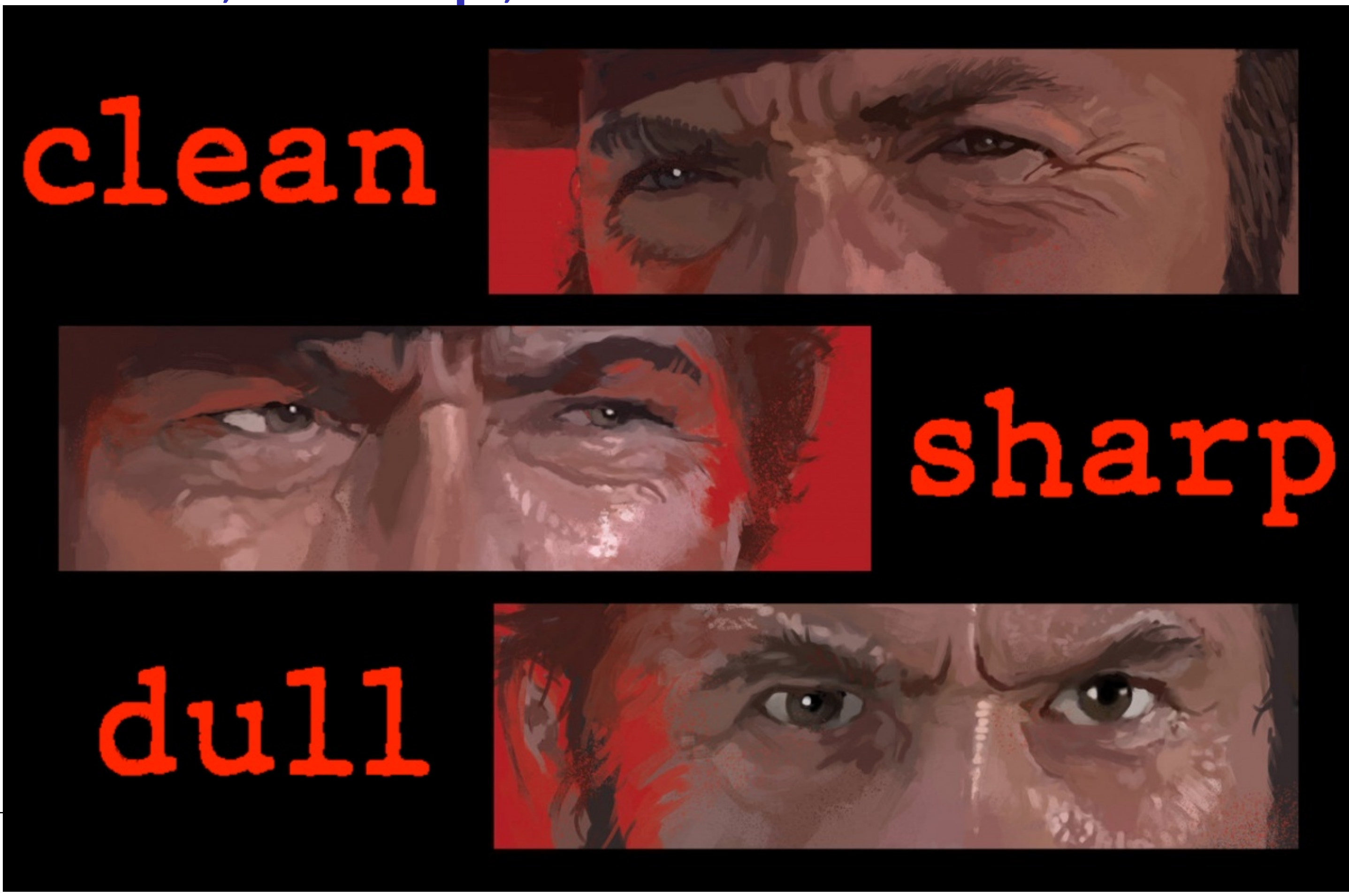
$$\mathbf{Q} \in \mathrm{conv}\{\mathcal{E}^\dagger(\mathbf{P}), \mathbf{T}^{(1)}, \mathbf{T}^{(2)} \ldots, \mathbf{T}^{|\mathbb{X}|}\}$$

☞ note: $\mathcal{E}$ trace-preserving $\implies$ $\mathcal{E}^\dagger$ unital $\implies$ $\mathcal{E}^\dagger(\mathbf{T}^{(i)}) = \mathbf{T}^{(i)}$

☞ this means that all dull$^\sharp$ are equivalent: given any two dull$^\sharp$ POVMs $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$, both $\mathbf{D}^{(1)} \succ^\sharp \mathbf{D}^{(2)}$ and $\mathbf{D}^{(2)} \succ^\sharp \mathbf{D}^{(1)}$ hold

☞ **question**: what about sharp$^\sharp$ POVMs? are they also all equivalent?

the clean, the sharp$^\sharp$, and the dull$^\sharp$



the clean, the sharp$^\sharp$, and the dull$^\sharp$

Definition

given two POVMs $\mathbf{P} = \{P_x\}_{x\in\mathbb{X}}$ and $\mathbf{Q} = \{Q_x\}_{x\in\mathbb{X}}$, if a CPTP linear map $\mathcal{E}$ exists such that $Q_x = \mathcal{E}^\dagger(P_x)\ \forall x$, we say that $\mathbf{P}$ is cleaner than $\mathbf{Q}$ ($\mathbf{P} \succ \mathbf{Q}$)

- POVMs $\mathbf{P}$ such that, if $\mathbf{Q} \succ \mathbf{P}$, then also $\mathbf{P} \succ \mathbf{Q}$, are called clean
- **theorem**: $\mathbf{P}$ clean $\iff$ $\mathbf{P} \succ \mathbf{Q}$, $\forall\mathbf{Q}$ (with same outcome set)
- **theorem**: clean $\iff$ sharp$^\sharp$
- hence, all sharp$^\sharp$ are also equivalent
- by the way: a POVM is dull$^\sharp$ if and only if it is minimal for $\succ$, that is, if it satisfies $\mathbf{P} \succ \mathbf{Q} \implies \mathbf{Q} \succ \mathbf{P}$

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second part: the problem with instruments (in)compatibility

incompatibility

In quantum theory, some measurements necessarily exclude others.

This feature is what enables quantum algorithms, QKD protocols, violations of Bell's inequalities, etc.

Various formalizations:

- preparation uncertainty relations (e.g., Robertson)
- measurement uncertainty relations (e.g., Ozawa)
- incompatibility

compatible POVMs 1/2

Definition

given a family $\{\mathbf{P}^{(i)}\}_{i\in\mathbb{I}} \equiv \{P_x^{(i)}\}_{x\in\mathbb{X},i\in\mathbb{I}}$ of POVMs, all defined on the same system A, we say that the family is *compatible*, whenever there exists

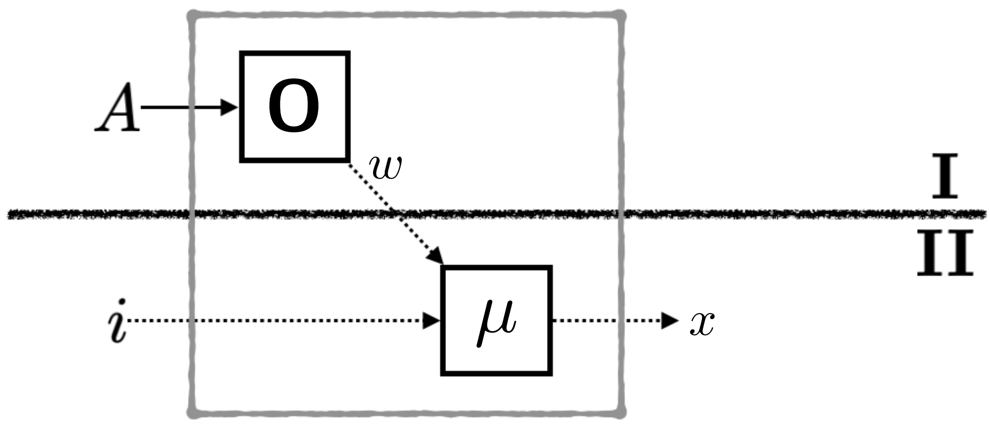
- a "mother" POVM $\mathbf{O} = \{O_w\}_{w\in\mathbb{W}}$ on system A
- a conditional probability distribution $\mu(x|w,i)$

such that

$$P_x^{(i)} = \sum_w \mu(x|w,i) O_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

compatible POVMs 2/2



[F.B., E. Chitambar, W. Zhou; PRL 2020]

the first problem

While there is consensus on a single notion of compatibility for POVMs, the situation is less clear for instruments...

classical compatibility 1/2

Definition (Heinosaari–Miyadera–Reitzner, 2014)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is *classically compatible*, whenever there exist

- a mother instrument $\{\mathcal{H}_w : A \to B\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$

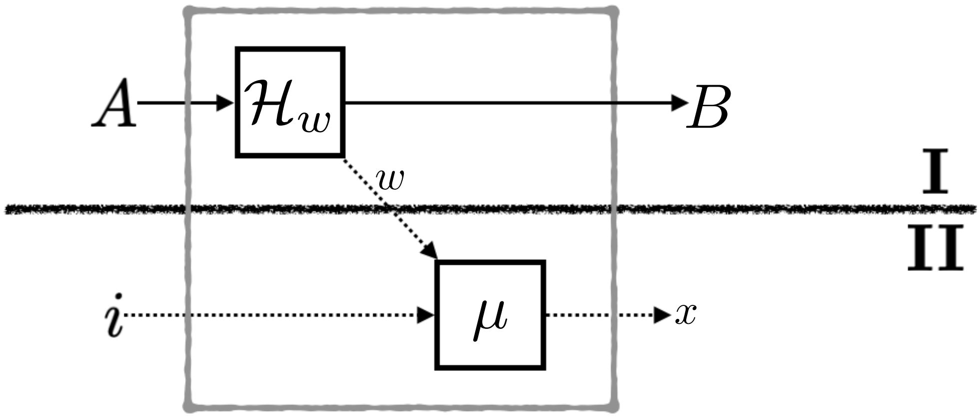
such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)\mathcal{H}_w \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

we call this "classical" because it involves only classical post-processings, but it is also called "traditional" [Mitra and Farkas; PRA 2022].

classical compatibility 2/2



crucially:

- no shared entanglement and communication is classical
- communication goes only from **I** to **II**, i.e., the above is necessarily **II**→**I** non-signaling, see [Ji and Chitambar; PRA 2021]

marginalizing the mother

- without loss of generality (classical labels can be copied), compatible POVMs may be assumed to be recovered by marginalization, i.e.,

$$P_x^{(i)} = \sum_{x_j : j \neq i} O_{x_1, x_2, \ldots, x_n}$$

- the notion of "parallel compatibility" for instruments lifts the above insight to the quantum outputs

parallel compatibility

Definition (Heinosaari–Miyadera–Ziman, 2015)

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is *parallelly compatible*, whenever there exist

- a mother instrument $\{\mathcal{H}_w : A \to \otimes_{i\in\mathbb{I}} B_i\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$

such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathrm{Tr}_{B_{i':i'\neq i}} \circ \mathcal{H}_w] \ ,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

parallel compatibility VS classical compatibility

- parallel compatibility is able to go beyond no-signaling, hence, parallel compatibility $\not\Longrightarrow$ classical compatibility
- parallel compatibility has nothing to do with the "no information without disturbance" principle, because non-disturbing instruments are never parallelly compatible
- hence classical compatibility $\not\Longrightarrow$ parallel compatibility

bridging the two camps: q-compatibility

Definition

given a family of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x\in\mathbb{X}, i\in\mathbb{I}}$, we say that the family is *q-compatible*, whenever there exist

- a mother instrument $\{\mathcal{H}_w : A \to C\}_{w\in\mathbb{W}}$
- a conditional probability distribution $\mu(x|w,i)$
- a family of postprocessing channels $\{\mathcal{D}^{(x,w,i)} : C \to B_i\}_{x\in\mathbb{X}, w\in\mathbb{W}, i\in\mathbb{I}}$

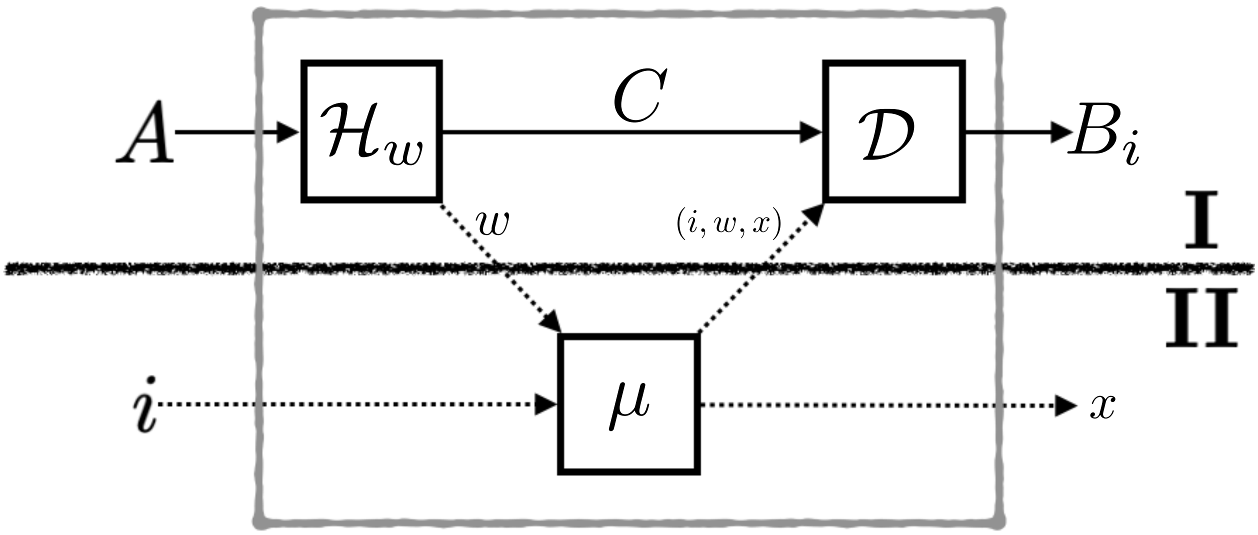
such that

$$\mathcal{I}_x^{(i)} = \sum_w \mu(x|w,i)[\mathcal{D}^{(x,w,i)} \circ \mathcal{H}_w] \,,$$

for all $x \in \mathbb{X}$ and all $i \in \mathbb{I}$.

classical compatibility: $C \equiv B_i$ and $\mathcal{D}^{(x,w,i)} = \mathrm{id}$
parallel compatibility: $C \equiv \bigotimes_i B_i$ and $\mathcal{D}^{(x,w,i)} = \mathrm{Tr}_{B_{i'}: i' \neq i}$

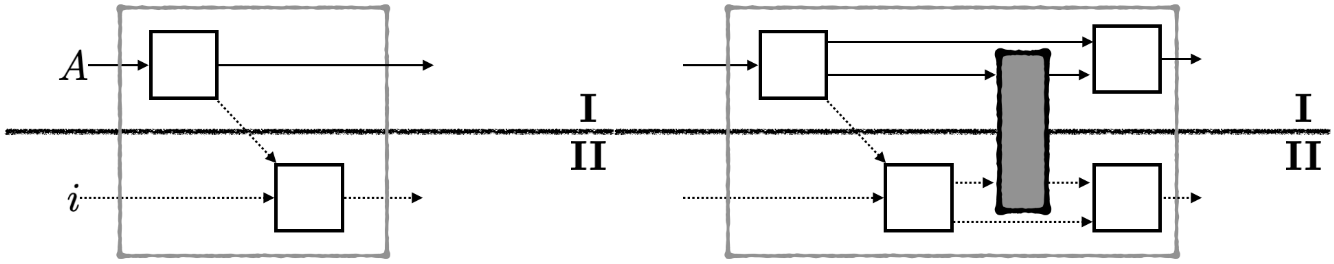
q-compatibility as a circuit



crucially:

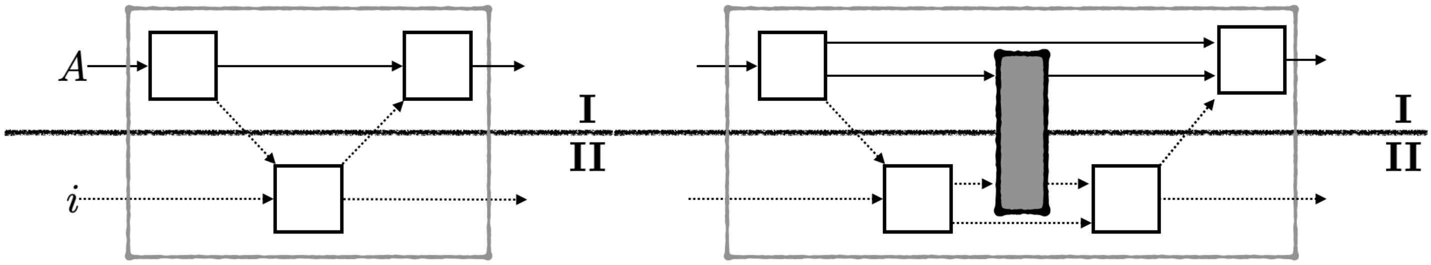
- no shared entanglement and communication is classical
- only one interactive round I→ II→I
- both classical and parallel compatibilities are special cases of q-compatibility

free operations for classical incompatibility



- all cassically compatible devices can be created for free
- if the initial device (the dark gray inner box) is classically compatible, the final device is also classically compatible

free operations for q-incompatibility



- all q-compatible devices can be created for free
- if the initial device (the dark gray inner box) is q-compatible, the final device is also q-compatible

the incompatibility preorder

given two families of instruments $\{\mathcal{I}_x^{(i)} : A \to B_i\}_{x \in \mathbb{X}, i \in \mathbb{I}}$ and $\{\mathcal{J}_y^{(j)} : C \to D_j\}_{y \in \mathbb{Y}, j \in \mathbb{J}}$, we say

"$\{\mathcal{I}_x^{(i)} : A \to B_i\}$ is more q-incompatible than $\{\mathcal{J}_y^{(j)} : C \to D_j\}$"

whenever the former can be transformed into the latter by means of a free operation

$\rightsquigarrow$ this is now an instance of statistical comparison: a Blackwell–like theorem can be proved, and a complete family of monotones obtained

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conclusion

take home messages

- **fuzzifying operations**: complete family of sharpness-non-increasing operations
- **sharpness is essentially a measure of classical communication capacity** (more precisely, signaling dimension)
- no need to argue about the "correct" definition of compatibility: **q-compatibility provides an overarching framework**
- **incompatibility is essentially quantum information transmission**, either in space (quantum channel) or in time (quantum memory)

thank you