nature COMMUNICATIONS

ARTICLE

DOI: 10.1038/s41467-018-06261-7 OPEN

Quantum majorization and a complete set of entropic conditions for quantum thermodynamics

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What does it mean for one quantum process to be more disordered than another? Interestingly, this apparently abstract question arises naturally in a wide range of areas such as information theory, thermodynamics, quantum reference frames, and the resource theory of asymmetry. Here we use a quantum-mechanical generalization of majorization to develop a framework for answering this question, in terms of single-shot entropies, or equivalently, in terms of semi-definite programs. We also investigate some of the applications of this framework, and remarkably find that, in the context of quantum thermodynamics it provides the first complete set of necessary and sufficient conditions for arbitrary quantum state transformations under thermodynamic processes, which rigorously accounts for quantum-mechanical properties, such as coherence. Our framework of generalized thermal processes extends thermal operations, and is based on natural physical principles, namely, energy conservation, the existence of equilibrium states, and the requirement that quantum coherence be accounted for thermodynamically.

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Irreversibility—the loss of order and the increase of disorder—is a fundamental and ubiquitous feature of physics that is typically described through thermodynamics and thermodynamic entropy. However, its scope goes above and beyond what one would ordinarily consider thermodynamic in nature. For example, the use of quantum entanglement within photonic quantum computing is subject to a form of irreversibility that need not be attached to either a particular energy scale or an equilibrium environment. Increasingly, a broader notion of irreversibility has been developed, that has been shown to include thermodynamic irreversibility as a special case, and has also allowed us to study intrinsically quantum mechanical order (such as entanglement or coherence) in contrast to classically ordered systems. Majorization is at the core of this development.

Majorization is a fundamental tool that finds application across a wide range of subjects from economics and statistics, to physics, chemistry, and pure mathematics[1]. At its core lies a notion of "deviations from uniformity", and the theory ties together mathematical techniques in convexity, combinatorics, and mathematical statistics.

An example of its use is in statistical mechanics of a physical system with N energy levels. If we assume, for the sake of discussion, that the system is fully degenerate in energy, its thermal equilibrium state is described by the uniform probability distribution $\boldsymbol{\gamma} = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$ over the energy levels. Given any two other probability distributions $\mathbf{p} = (p_1, \ldots, p_N)$ and $\mathbf{q} = (q_1, \ldots, q_N)$, one might wish to say whether one is more or less out of equilibrium than the other. Majorization provides a concrete way of stating this. The distribution $\mathbf{p}$ is more ordered than $\mathbf{q}$ (or "$\mathbf{p}$ majorizes $\mathbf{q}$", written $\mathbf{q} \prec \mathbf{p}$) if $\mathbf{q} = D\mathbf{p}$ for some doubly stochastic matrix D[1]. A crucial property of majorization is that it can be equivalently formulated in terms of a complete set of monotones. For example, it is well-known that $\mathbf{q} \prec \mathbf{p}$ if and only if $\sum_k f(p_k) \geq \sum_k f(q_k)$ for all continuous real-valued convex functions f. Therefore, the value of any continuous convex function f on statistical distributions can never increase under doubly stochastic transformations. Such functions are therefore monotones that quantify the deviation from equilibrium; moreover, they constitute a complete set of monotones because the comparison of their values provides a sufficient condition for the existence of a doubly stochastic transformation.

Majorization also finds extensive use in various parts of quantum information theory, such as in entanglement theory[2] and recent formulations of resource theories[3]. In particular, it has a central role in the recent thermodynamic frameworks using the quantum information theory[3–14]. In particular, it was found that state transformations with zero coherences in energy are fully characterized by thermo-majorization[5] (see also earlier works[15,16]), which is a natural generalization of majorization[4,17]. However, it was shown in ref. [10] that such thermo-majorization results are insufficient for describing quantum coherence under thermal operations, and that novel coherence measures are required. Low temperature coherence regimes were shown to admit solvable analysis[9], general coherence bounds were developed[11], and a framework for coherence based on the concept of asymmetry under time-translations was proposed[10,12]. However, a complete specification of the structure of non-equilibrium quantum states was still lacking.

A natural question is therefore whether there exists a generalization of majorization (or thermo-majorization) that can accommodate such intrinsically quantum-mechanical orderings. Several candidate generalizations exist[1,18,19], however, the one most relevant to our present work is called matrix majorization[16], which is a specialization to linear algebra of ideas coming from the theory of statistical comparison (see ref. [20] and references therein). Given two matrices of real numbers A and B, we say that A matrix-majorizes B, and write $B \prec_m A$, if and only if $B = AX$ for some row stochastic matrix X. It is easy to see that this is a generalization of majorization: for the two-row matrices $A = \begin{bmatrix} \mathbf{p} \\ \mathbf{e} \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{q} \\ \mathbf{e} \end{bmatrix}$, with $\mathbf{e} \equiv (1, 1, \ldots, 1)$, the relation $B \prec_m A$ is equivalent to $\mathbf{q} \prec \mathbf{p}$. Similarly, other variants of majorization, like thermo-majorization, are special cases of matrix majorization. However, such an ordering is inherently classical, being ultimately based on stochasticity, as opposed to coherent quantum processes.

A key component of our work is to generalize matrix majorization in a natural way into the quantum-mechanical setting, and to provide applications to a number of topics. Our first contribution to this is to provide a complete entropic description of a fully quantum-mechanical form of majorization. We then outline the core features of the solution and discuss the inclusion of quantum-mechanical symmetries. Our final contribution is to define a natural framework for quantum thermodynamics that is based on three physical assumptions, provide a complete set of entropic conditions and discuss limiting thermodynamic regimes of the theory.

Results

Definition of quantum majorization. Our generalization of matrix majorization, which we call quantum majorization, defines a relation on bipartite quantum states, and consequently, due to the channel-state duality property of quantum theory, also defines a relation on quantum processes, i.e., completely positive and trace-preserving (CPTP) maps. Notice that notions equivalent to quantum majorization have previously been considered in refs. [21–25] in the contexts of quantum statistics and quantum information theory.

Definition 1: Let $\rho^{\mathrm{AB}} \in \mathcal{B}(\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}})$ and $\sigma^{\mathrm{AC}} \in \mathcal{B}(\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{C}})$ be two bipartite quantum states. We say that ρ^{AB} quantum majorizes σ^{AC}, and write $\sigma^{\mathrm{AC}} \prec_q \rho^{\mathrm{AB}}$, if and only if there exists a CPTP map $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\mathrm{B}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{C}})$ such that $\mathrm{id} \otimes \mathcal{E}(\rho^{\mathrm{AB}}) = \sigma^{\mathrm{AC}}$.

Remark 1: The preorder $\sigma^{\mathrm{AC}} \prec_q \rho^{\mathrm{AB}}$ is not symmetric with respect to the action of ε. It means that ρ^{AB} quantum majorizes σ^{AC} on B. However, in the remaining of this paper, it will be clear from the text that the action of $\mathcal{E}$ is on system B.

It is clear from Definition 1 that $\rho^{\mathrm{A}} = \sigma^{\mathrm{A}}$ is a necessary condition, called the compatibility condition, for the ordering of states to hold since $\mathcal{E}$ is trace-preserving, and when it holds the two states are said to be compatible. Moreover, in the special case that the marginals satisfy $\rho^{\mathrm{A}} = \sigma^{\mathrm{A}} = \frac{1}{d_{\mathrm{A}}} 1^{\mathrm{A}}$, we can express the bipartite states as the Choi matrices $\rho^{\mathrm{AB}} = \mathrm{id} \otimes \mathcal{D}\left(\varphi_+^{\mathrm{AA}'}\right)$ and $\sigma^{\mathrm{AC}} = \mathrm{id} \otimes \mathcal{F}\left(\varphi_+^{\mathrm{AA}'}\right)$, where $\mathcal{D} : \mathcal{B}(\mathcal{H}_{\mathrm{A}'}) \to \mathcal{B}(\mathcal{H}_{\mathrm{B}})$ and $\mathcal{F} : \mathcal{B}(\mathcal{H}_{\mathrm{A}'}) \to \mathcal{B}(\mathcal{H}_{\mathrm{C}})$ are two quantum processes (CPTP maps), and $\phi_+^{\mathrm{AA}'}$ is the projection on the maximally entangled state $|\phi_+^{\mathrm{AA}'}\rangle = \frac{1}{\sqrt{d_{\mathrm{A}}}} \sum_{i=1}^{d_{\mathrm{A}}} |ii\rangle$, where $\{|i\rangle\}_{i=1}^{d_{\mathrm{A}}}$ is an orthonormal basis for A. Therefore, in this case the condition $\mathrm{id} \otimes \mathcal{E}(\rho^{\mathrm{AB}}) = \sigma^{\mathrm{AC}}$ becomes equivalent to the degradability of $\mathcal{D}$ into $\mathcal{F}$, that is, $\mathcal{F} = \mathcal{E} \circ \mathcal{D}$, and we denote it simply by $\mathcal{F} \prec_q \mathcal{D}$.

Quantum majorization hence generalizes classical stochasticity and captures the notion that the process $\mathcal{F}$ is in some sense "more disordered" than $\mathcal{D}$, since it can be obtained from $\mathcal{D}$ via $\mathcal{E}$. However, it does not say anything about $\mathcal{E}$, which can be a completely general quantum process. Typically, in resource theories, it is important to place some additional restrictions on allowed (or "free") processes, and demand that $\mathcal{E}$ is a free operation of the theory. Many resource theories, such as

entanglement theory, do not admit a simple specification, however, as we shall see shortly, in both the resource theories of asymmetry and thermodynamics, such a restriction of $\mathcal{E}$ to lie in a subset of free (symmetric or thermodynamic) processes can be made with a natural modification of our core result.

Characterization of quantum majorization. Given the two bipartite states ρ^{AB} and σ^{AC}, how can we determine whether ρ^{AB} quantum majorizes σ^{AC}? One simple and intuitive necessary condition, that follows from the data processing inequality, is that

$$S(A|B)_\rho \leq S(A|C)_\sigma \,, \tag{1}$$

where $S(A|B) = S(A,B) - S(B)$ is the conditional entropy, and $S(\rho) = -\mathrm{Tr}[\rho \log \rho]$ is the von-Neumann entropy. The intuition is that, if $\sigma^{AC} \prec_q \rho^{AB}$, then information about system A is more accessible from system B than from system C. Hence, the uncertainty of A given B, i.e., $S(A|B)$, can only be smaller than the uncertainty of A given C, i.e., $S(A|C)$. However, only one entropic condition is far from being sufficient to completely characterize quantum majorization.

In order to produce more necessary conditions, one can use a similar intuition to generate infinitely many necessary conditions that follows from the following observation (Fig. 1):

$$\sigma^{AC} \prec_q \rho^{AB} \Rightarrow \Phi \otimes \mathrm{id}\left(\sigma^{AC}\right) \prec_q \Phi \otimes \mathrm{id}\left(\rho^{AB}\right) \tag{2}$$

for any quantum process $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{A'})$. Note that Φ is acting on system A while $\mathcal{E}$ in Definition 1 is acting on system B. We therefore conclude that if $\sigma^{AC} \prec_q \rho^{AB}$ then, for any quantum process Φ, we must have:

$$S(A'|B)_{\Phi\otimes\mathrm{id}(\rho^{AB})} \leq S(A'|C)_{\Phi\otimes\mathrm{id}(\sigma^{AC})} \,. \tag{3}$$

While the conditions above are necessary, again they are not sufficient, and even in the purely classical case: there exist classical states ρ^{AB} and σ^{AB} such that $\sigma^{AC} \not\prec_q \rho^{AB}$, even though the above equation holds of all Φ (and any dimensions of A')[26,27].

On the other hand, in the following central result of our paper, we show that if one replaces the conditional (von-Neumann) entropy in (3) with the conditional min-entropy[28], then the inequalities in (3) indeed provide, if all simultaneously satisfied, a sufficient condition for quantum majorization. Moreover, we can restrict Φ to be an entanglement breaking channel, and bound the dimension of system A' to be no greater than the dimension of system C. Similar results, dubbed "reverse data-processing theorems," have been obtained before[18,25,27,29], although in a different framework involving extra ancillas and a classical reference system, while the present relations are fully quantum and do not need additional external systems.

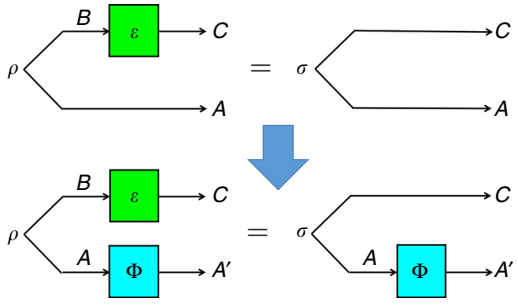

Fig. 1 Quantum majorization. The condition of quantum majorization $\sigma^{AC} \prec_q \rho^{AB}$ implies the infinite set of relations $(\Phi \otimes \mathrm{id})(\sigma^{AC}) \prec_q (\Phi \otimes \mathrm{id})(\rho^{AB})$, where Φ is any CPTP map acting on system A (cfr. Eq. (2) in the main text). Theorem 1 provides a complete set of monotones for quantum majorization, expressed as entropic functions of the bipartite state and the channel Φ acting on it

The conditional min-entropy, $H_{\min}(A|B)_\Omega$, of a bipartite state Ω^{AB}, is defined as[28]

$$H_{\min}(A|B)_\Omega := -\log \inf_{\tau^B \geq 0} \{\mathrm{Tr}[\tau^B] : 1^A \otimes \tau^B \geq \Omega^{AB}\}. \tag{4}$$

It is known to be a single-shot analog of the conditional (von-Neumann) entropy. This analogy is particularly motivated by the fully quantum asymptotic equipartition property[30], which states that in the asymptotic limit of many copies of Ω^{AB}, the smooth version of $H_{\min}(A|B)$ approaches the conditional (von-Neumann) entropy. The conditional min-entropy has numerous applications in single-shot quantum information (e.g., ref. [30] and references therein), quantum hypothesis testing (e.g., refs. [19,27] and references therein), and quantum resource theories[31].

Theorem 1: Let $\rho^{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\sigma^{AC} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_C)$ be two compatible bipartite quantum states. Let $\{M_j^A\}$ be an arbitrary, but fixed, informationally complete POVM on system A. Denote the dimension of any system X as $d_X \in \mathbb{N}$. The following are equivalent:

1. The state ρ^{AB} quantum majorizes σ^{AC},

$$\sigma^{AC} \prec_q \rho^{AB}. \tag{5}$$

2. For any quantum process (CPTP linear map) $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{A'})$, with $d_{A'} = d_C$,

$$H_{\min}(A'|B)_{\Phi\otimes\mathrm{id}(\rho^{AB})} \leq H_{\min}(A'|C)_{\Phi\otimes\mathrm{id}(\sigma^{AC})} \tag{6}$$

3. Eq. (6) holds for any measure-and-prepare quantum channel $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{A'})$ of the form:

$$\Phi\left(\eta^A\right) = \sum_{j=1}^{d_A^2} \mathrm{Tr}\left[M_j^A \eta^A\right] \omega_j^{A'} \,, \tag{7}$$

while the states $\{\omega_j^{A'}\}$ can freely vary.

4. $g(\rho^{AB}, \sigma^{AC}) \geq 1$, where the the function g is defined by the following semidefinite programming:

$$g(\rho^{AB}, \sigma^{AC}) = \max$$

$$\left\{ y \mid \forall j \; y\sigma_j^T \leq \mathrm{Tr}_B\left[\tau^{CB}(I \otimes \rho_j)\right],\; \tau^{CB} \geq 0,\; \tau^B \leq I \right\} \tag{8}$$

where

$$\rho_j \equiv \frac{\mathrm{Tr}_A\left[\left(M_j^A \otimes 1^B\right)\rho^{AB}\right]}{\mathrm{Tr}\left[M_j^A \rho^A\right]} \text{ and}$$

$$\sigma_j \equiv \frac{\mathrm{Tr}_A\left[\left(M_j^A \otimes 1^C\right)\sigma^{AC}\right]}{\mathrm{Tr}\left[M_j^A \sigma^A\right]}. \tag{9}$$

The proof of the above theorem is postponed to Supplementary Note 1.

Remark 2: In the classical case, both $\rho^{AB} = \sum_{x,y} p_{xy} |x\rangle\langle x| \otimes |y\rangle\langle y| \equiv P$ and $\sigma^{AC} = \sum_{x,z} q_{xz} |x\rangle\langle x| \otimes |z\rangle\langle z| \equiv Q$ are diagonal, where P (and Q) is the matrix whose components are the probabilities p_{xy} (q_{xz}). Therefore, the relation $\sigma^{AC} = \mathrm{id} \otimes \mathcal{E}(\rho^{AB})$ can be expressed as $Q = SP$, where S is a column stochastic matrix, so that $Q^T \prec_m P^T$ (The relation $Q = SP$ is equivalent to $Q^T = P^T S^T$, with S^T being a row stochastic matrix.). Dahl

obtained in ref. [16] that P matrix-majorizes Q if and only if for all sub-linear functionals f, that can be written as a maximum of a finite number of linear functionals, the following holds:

$$\sum_j f(\mathbf{p}_j) \geq \sum_k f(\mathbf{q}_k)\ , \tag{10}$$

where $\mathbf{p}_j$ and $\mathbf{q}_k$ are the rows of P and Q, respectively. Since classically $2^{-H_{\min}(A|B)}$ is a sub-linear functional (see more details in the Supplementary Notes 1 and 4), our theorem above provides the same result for the classical case, with a slight improvement that f can be restricted to sub-linear functionals that can be written as a maximum of at most d_C linear functionals.

Remark 3: The conditions in Eq. (6) are given in a form of monotones; i.e., functions that behave monotonically under certain operations (in our case under quantum majorization). In quantum resource theories monotones quantify resources as they do not increase under free operations. As we will see below, the conditional min-entropies that appear in Theorem 1 can be used to quantify asymmetry in the resource theory of quantum reference frames[32], and athermality in quantum thermodynamics. Since Eq. (6) has to hold for any CPTP map $\Phi : A \to A'$ (or measurement-prepare channels with any set of density matrices $\{\omega_j^{A'}\}$), quantum majorization is characterized in Theorem 1 by means of an infinite number of monotones. This can be related with the fact that here we consider exact transformations, and typically an exact (algebraic) solution to such an SDP feasibility problem is NP-hard. However, part 4 of the theorem demonstrates that the question of weather or not ρ^{AB} quantum majorizes ρ^{AC} can be solved efficiently using semidefinite programming. A discussion comparing the two formulations, i.e., one SDP versus infinite monotones, is presented the supplementary material Note 3.

If only system A is classical, that is the states $\rho^{AB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i$ and $\sigma^{AC} = \sum_i p_i |i\rangle\langle i| \otimes \sigma_i$ are classical-quantum states, we get that (5) is equivalent to

$$\sigma_i = \mathcal{E}(\rho_i) \tag{11}$$

for all i such that $p_i > 0$. This is a classic problem in quantum hypothesis testing, and the results presented here complement previous results in the same direction[22,23,25,27,33–35]. In particular, it can be shown (see Lemma 1 in the Supplementary Note 1) that Theorem 1 above implies the following corollary:

Corollary 1: There exists $\mathcal{E}$ satisfying (11) if and only if for any set of n density matrices $\{\omega_i^A\}_{i=1}^n$, we have $H_{\min}(A|B)_\Omega \leq H_{\min}(A|C)_\Omega$, where

$$\Omega^{ABC} = \frac{1}{n}\sum_{i=1}^{n} \omega_i^A \otimes \rho_i^B \otimes \sigma_i^C\ . \tag{12}$$

The same relation holds if the uniform distribution $1/n$ is replaced with any other arbitrary distribution q_i, with the only condition that $q_i > 0$.

A complete set of entropic conditions for the resource theory of asymmetry. So far we considered the relation $\sigma^{AC} = \mathrm{id} \otimes \mathcal{E}(\rho^{AB})$ with arbitrary CPTP map $\mathcal{E} : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_C)$. We now impose additional constraint on $\mathcal{E}$, requiring it to be G-covariant with respect to a compact group G. That is, $\mathcal{E}$ is G-covariant with respect to two unitary representations of G on systems B and C, denoted, respectively, by $\{V_g\}_{g\in G}$ and $\{U_g\}_{g\in G}$, if

$$U_g \mathcal{E}(\rho) U_{g^{-1}} = \mathcal{E}(V_g \rho V_{g^{-1}}) \quad \forall\, g \in G\ . \tag{13}$$

We write $\sigma^{AC} \prec_q^G \rho^{AB}$, if $\sigma^{AC} = \mathrm{id} \otimes \mathcal{E}(\rho^{AB})$ with a G-covariant CPTP map $\mathcal{E}$.

Theorem 1 can be easily upgraded to accommodate G-covariant maps: the formal statement is given as Theorem 2 in Supplementary Note 5. Particularly, it can be shown that $\sigma^{AC} \prec_q^G \rho^{AB}$ if and only if

$$H_{\min}(A'|B)_{\mathcal{G}[\Phi\otimes \mathrm{id}(\rho^{AB})]} \leq H_{\min}(A'|C)_{\mathcal{G}[\Phi\otimes \mathrm{id}(\sigma^{AC})]} \tag{14}$$

for all CPTP entanglement breaking maps $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{A'})$ of the form (7). Here $\mathcal{G} : \mathcal{B}(\mathcal{H}_{A'} \otimes \mathcal{H}_C) \to \mathcal{B}(\mathcal{H}_{A'} \otimes \mathcal{H}_C)$ is the bipartite G-twirling map given by

$$\mathcal{G}[\tau^{A'C}] = \int dg\ (\overline{U}_g \otimes U_g)\ \tau^{A'C}\ (\overline{U}_g^\dagger \otimes U_g^\dagger)\ , \tag{15}$$

where the over bar denotes the complex conjugation made with respect to an arbitrary but fixed orthonormal basis.

In the special case in which both $\rho^{AB} = |0\rangle\langle 0|^A \otimes \rho^B$ and $\sigma^{AC} = |0\rangle\langle 0|^A \otimes \sigma^C$ are product states, our theorem is simplified to the following statement: ρ^B can be converted to σ^C by a G-covariant map if and only if for any density matrix $\eta^{A'}$,

$$H_{\min}(A'|B)_{\mathcal{G}[\eta^{A'}\otimes\rho^B]} \leq H_{\min}(A'|C)_{\mathcal{G}[\eta^{A'}\otimes\sigma^C]}\ . \tag{16}$$

Therefore, the quantities $H_{\min}(A'|B)_{\mathcal{G}[\eta^{A'}\otimes\rho^B]}$, for varying reference state $\eta^{A'}$, provide a complete set of asymmetry monotones for the resource theory of asymmetry[36–40]. In other words, for any given state $\eta^{A'}$, the functions $H_{\min}(A'|B)_{\mathcal{G}[\eta^{A'}\otimes\rho^B]}$ provide a single-copy characterization of the G-asymmetry content of state ρ^B. That is to say, even though asymmetry is not a state function in itself (as it is not totally ordered), it can still be completely described in terms of a complete set of such state functions. We are now ready to discuss the application of this result to quantum thermodynamics.

A complete set of entropic conditions for quantum thermodynamics. While thermodynamics in macroscopic, equilibrium, and classical regimes is well understood[41], there is the fundamental question of how one can extend thermodynamic notions into non-equilibrium, finite-sized systems[42–44], and in particular systems displaying highly non-classical properties such as quantum coherence, contextuality, and entanglement[45–49]. One particular approach to this problem[3–14] has been to utilize tools and concepts developed in the study of entanglement, which is understood within the framework of resource theories. A resource theory provides a way to quantify physical characteristics that are not simply given by Hermitian observables, and is defined once we specify a set of free states, as those that do not have the properties one wishes to study, together with set of free operations, that are compatible with the set of free states in the sense that their action on any free state always yields another free state.

This approach of analyzing thermodynamics in terms of its process structure (instead of starting with problematic terms such as “heat” or “work” or “entropy”) turns out to have a long and successful history dating back to the 1909 seminal work of Carathéodory[50]. Other notable accounts were obtained in 1964 by Giles[51] and more recently in 1999 by Lieb and Yngvason[52], who provided a thorough analysis in terms of adiabatic accessibility. Moreover, it has recently been shown in[13] that the thermodynamic structure of incoherent quantum states obtained from an information-theoretic perspective coincides with the phenomenological analysis in ref. [52], which demonstrates the soundness of the resource theoretic approach.

In thermodynamics, a preferred class of states are singled out as free states from the condition of complete passivity[53,54]. In the simplest case, the Gibbs state $\frac{1}{Z}e^{-\beta H}$, with $\beta = (kT)^{-1}$ and $Z = \mathrm{Tr}[e^{-\beta H}]$, is the only quantum state that can be freely admitted without trivializing the theory energetically. More

generally, in the presence of additional additive conserved charges $\{X_1, \ldots X_n\}$, such as angular momenta and particle numbers, this can be extended (under certain assumptions on external constraints[41,55–61]) to the generalized Gibbs state

$$\gamma^{\mathrm{A}} = \frac{1}{\mathcal{Z}} e^{-\beta(H^{\mathrm{A}} - \sum_k \mu_k X_k^{\mathrm{A}})}, \tag{17}$$

with $\{\mu_k\}$ being Lagrange multiplier constants for the conserved quantities and $\mathcal{Z} = \mathrm{Tr}[e^{-\beta(H^{\mathrm{A}} - \sum_k \mu_k X_k^{\mathrm{A}})}]$. In the case that we just have a single additional number operator N, the constant is the usual chemical potential[41].

Generalized thermal processes. Our thermodynamic framework is an extension of the resource theory of thermal operations (TOs)[4,5,7] to a set of transformations that contains TOs as a proper subset. It is an extension in two ways: firstly, it makes a weaker assumption about the underlying microscopic process, and secondly it is defined in terms of a collection of distinguished thermodynamic observables, such as those in the generalized Gibbs ensemble, and not just in terms of energy. We elaborate on the relation between the two classes in Supplementary Note 8. We shall refer to these free transformations as generalized thermal processes (abbreviated to TPs), and they are specified by the following three physical assumptions:

- *A*1. Microscopic conservation: Each input quantum system and output quantum system has a Hamiltonian H, and a collection of distinguished observables $X_1, \ldots X_n$. The total energy and the observables $\{X_k\}$ are conserved microscopically in any free process, and moreover $[H, X_k] = 0$ for all $k = 1, \ldots, n$.
- *A*2. Equilibrium preservation: For every (input or output) system A, an equilibrium free state exists that is stable under the class of free processes.
- *A*3. Incoherence: The free processes do not require any sources of quantum coherence between eigenbases of conserved quantities.

The microscopic conservation assumption ensures that every quantum system A has a well-defined Hamiltonian H^{A} at the initial time and some other Hamiltonian $H^{\mathrm{A}'}$ at the final time. It also allows for an arbitrary set of additional conserved charges, as discussed. More precisely, any TP map $\mathcal{E}$ on A admits a Stinespring dilation onto some larger system B such that

$$\mathcal{E}(\rho^{\mathrm{A}}) = \mathrm{Tr}_{\mathrm{C}} V(\rho^{\mathrm{A}} \otimes \sigma^{\mathrm{B}}) V^{\dagger} \tag{18}$$

where B is some other quantum system defining the thermal environment. The microscopic conservation assumption implies that the isometry V obeys

$$V(H^{\mathrm{A}} \otimes 1^{\mathrm{B}} + 1^{\mathrm{A}} \otimes H^{\mathrm{B}}) = (H^{\mathrm{A}'} \otimes 1^{\mathrm{C}} + 1^{\mathrm{A}'} \otimes H^{\mathrm{C}}) V$$

$$V(X_k^{\mathrm{A}} \otimes 1^{\mathrm{B}} + 1^{\mathrm{A}} \otimes X_k^{\mathrm{B}}) = (X_k^{\mathrm{A}'} \otimes 1^{\mathrm{C}} + 1^{\mathrm{A}'} \otimes X_k^{\mathrm{C}}) V \tag{19}$$

for all $k = 1, \ldots n$, which defines the microscopic energy conservation and the conservation of the charges. Note that we also allow the input system and output to differ, which may occur due to the presence of strong-couplings that affect factorizability into independent subsystems. It is also important to emphasize that we do not assume or require microscopic control of V. It is only the total process $\mathcal{E}$ that is experimentally relevant. The particular set of observables are determined by the physical context and we shall refer to them as the thermodynamic observables for the system.

The equilibrium preservation assumption says that for every system A there is a state ρ_*^{A}, such that $\mathcal{E}(\rho_*^{\mathrm{A}}) = \rho_*^{\mathrm{A}}$ for all TPs $\mathcal{E}$. However (A1) singles out a set of distinguished observables $\{H^{\mathrm{A}}, X_1^{\mathrm{A}}, \ldots, X_n^{\mathrm{A}}\}$ that microscopically are additively conserved. The fact that ρ_*^{A} is a free state of the theory implies[58–60] that the only form of ρ_*^{A} that can yield a non-trivial resource theory in these observables is one for which $\log \rho_*^{\mathrm{A}}$ is a linear combination of the observables—namely it must be a generalized Gibbs state γ^{A} as defined in (17), at some fixed temperature $T = (k\beta)^{-1}$ and Lagrange multipliers $\mu_1, \ldots, \mu_n$. Therefore the free states of the theory are defined uniquely by these parameters.

The final assumption on incoherence is a statement of non-classicality within the theory and requires us to provide an explicit accounting for coherence resources. It is known for thermal operations that if the only coherences present are within energy eigenspaces then the resultant theory is essentially classical, and is described by thermo-majorization[5]. However, coherences between energy eigenspaces behave differently and do not have such a classical description[10]. Therefore one must carefully account for these coherences thermodynamically. The precise formulation of this requirement in the case of energy is that if any free process $\mathcal{E}$ is discussed in Supplementary Note 6, and has the consequence that any $\mathcal{E}$ can be represented as

$$\mathcal{E}(\rho) = \mathrm{Tr}_{\mathrm{C}} V(\rho^{\mathrm{A}} \otimes \sigma^{\mathrm{B}}) V^{\dagger} \tag{20}$$

where V is a conserving interaction, and σ^{B} is an external state incoherent in the energy eigenbases and obeys $\sigma^{\mathrm{B}} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \mathrm{d}t U_{\mathrm{B}}(t) \sigma^{\mathrm{B}} U_{\mathrm{B}}^{\dagger}(t)$. This captures the notion that $\mathcal{E}$ is realized without consuming any coherent resources from the external degrees of freedom in B. At the level of quantum operations on S, this implies that we have the following symmetry property for all free operations

$$U'(t) \mathcal{E}(\rho^{\mathrm{A}}) U'(t)^{\dagger} = \mathcal{E}(U(t) \rho^{\mathrm{A}} U(t)^{\dagger}) \tag{21}$$

where $U(t) = \exp[-itH^{\mathrm{A}}]$ and $U'(t) = \exp[-itH^{\mathrm{A}'}]$ are respectively free evolution of the input/output system for an interval of time t. The operation $\mathcal{E}$ is said to be covariant under time-translation. The more general case of multiple conserved charges is discussed below.

The three physical assumptions specify the set of generalized thermal processes, and it is readily seen that it contains the set of thermal operations. In the case when the only conserved quantity is H, there is no particular physical reason to choose one set of operations over the other. However, in the case of multiple conserved charges $X_1, \ldots, X_n$, the use of TPs has an advantage in that it allows one to handle generalized Gibbs ensemble scenarios more easily. The details of system B are, in general, not observed thermodynamical degrees of freedom, and with an explicit microscopic specification, such as with thermal operations, subtleties arise in the case of additional charges. Particularly, subtleties arise if one wishes to have non-trivial μ_k Lagrange multipliers in the generalized Gibbs ensemble (17) and also satisfy the microscopic conservation assumption. The formulation here simply avoids this by not demanding a specific form for the microscopic state $\tilde{\sigma}^{\mathrm{B}}$ in the definition of the free processes. The incoherence assumption only constrains the microscopic details to the extent that there are no observable effects of coherence at the level of the process $\mathcal{E}$.

In the Supplementary Note 7, we show that our core result on quantum majorization can be adapted to the setting of generalized thermal processes to fully describe the state interconversion structure. This is obtained by establishing the following lemma, which is proved in the Supplementary Note 6.

Lemma 1: Consider two sets of thermodynamic observables $\{H^{\mathrm{S}}, X_1^{\mathrm{S}}, \ldots, X_n^{\mathrm{S}}\}$ for quantum system $S = A$ and quantum

systems $S = A'$. Then, the set of all quantum processes from A into A' defined by (A1-A3) coincides with the set of all γ-preserving processes on A that are covariant under the group G generated by the thermodynamic observables on A and A'.

State conversions under thermal processes. Since TPs are G-covariant we may make use of our earlier results on G-covariant state interconversion of a collection of states $\{\rho_i^{\mathrm{A}}\}$ into $\{\sigma_i^{\mathrm{B}}\}$. We first consider the case where energy is the only distinguished thermodynamic observable that is conserved microscopically. Combining the G-covariant version of Theorem 1 with the above lemma we get the following theorem (see Supplementary Note 7 for more details).

Theorem 2: Let A and A' be two quantum systems, with the respective Hamiltonians H^{A} and $H^{\mathrm{A}'}$ being the only thermodynamic observables, and let $0 < q < 1$ be an arbitrary but fixed number. The state transformation $\rho^{\mathrm{A}} \to \sigma^{\mathrm{A}'}$ is possible under generalized thermal processes at a temperature $(k_{\mathrm{B}}\beta)^{-1}$ if and only if for all reference frame systems R with the same dimension as of A' and with Hamiltonian $H^{\mathrm{R}} = -(H^{\mathrm{A}'})^{\mathrm{T}}$, and for all pairs of states $\boldsymbol{\eta} = (\eta_1^{\mathrm{R}}, \eta_2^{\mathrm{R}})$, we have

$$S_\eta(\rho^{\mathrm{A}}) \le S_\eta(\sigma^{\mathrm{A}'}) \,, \tag{22}$$

where $S_\eta(\rho^{\mathrm{A}}) := H_{\min}(R|A)_\Omega$ and

$$\Omega^{\mathrm{RA}} = \langle q\eta_1^{\mathrm{R}} \otimes \rho^{\mathrm{A}} + (1-q)\eta_2^{\mathrm{R}} \otimes \gamma^{\mathrm{A}} \rangle \,. \tag{23}$$

Here, $\gamma^{\mathrm{A}} = \exp[-\beta H^{\mathrm{A}}]/Z$ is the Gibbs state on A, and $\langle \omega^{\mathrm{RA}} \rangle \equiv \lim_{\tau\to\infty} 1/\tau \int_0^\tau \mathrm{d}t\; U(t)\omega^{\mathrm{RA}}U^\dagger(t)$ is the channel that maps any state of RA to its time-averaged version, and $U(t) = \exp[-it(H^{\mathrm{R}} \otimes 1^{\mathrm{A}} + 1^{\mathrm{R}} \otimes H^{\mathrm{A}})]$ is the unitary time-evolution under the Hamiltonian for the composite system RA.

It is important to note that these conditions can be greatly reduced. In particular one can simply consider $q = \frac{1}{2}$ alone, however, in some cases it is useful to choose different values and so we give the general case here. Also, it readily seen that the state η_2^{R} can be chosen to be block-diagonal in the energy eigenbasis, while η_1^{R} can be restricted to reference frame states that have the same modes of coherence as ρ^{A}[12,37].

Time-energy constraints on state conversions. Next, we show that the necessary and sufficient condition found in Theorem 2 has an interesting physical interpretation; loosely speaking, it implies that a state conversion is possible in quantum thermodynamics, if and only if it does not lead to any net increase in work or time-information.

A key obstacle in quantum thermodynamics is that to determine the existence of the transformation $\rho^{\mathrm{A}} \to \sigma^{\mathrm{A}}$, one needs to consider two different types of physical properties of states: (i) properties related to their energy distribution, which leads to conditions such as thermo-majorization[62], and (ii) properties related to the coherence in the energy eigen-basis. Roughly speaking, one needs to check that the initial state ρ^{A} has (at least) as much as free energy and coherence as the desired final state $\sigma^{\mathrm{A}'}$.

It is not possible in general to quantify both of these simultaneously in a measurement scheme. Coherences in energy are precisely the time-dependent components of a quantum system and thus one encounters an obstacle of complementarity between time and energy measurements. Physically these two aspects can be viewed as "clock" and "work" regimes of a quantum system. Theorem 2 gets around this complementarity by allowing the reference system R to act simultaneously as a "clock/work reference". In other words, one can interpolate smoothly between the two regimes via the different choices of quantum states η^{R}. This is illustrated schematically in Fig. 2.

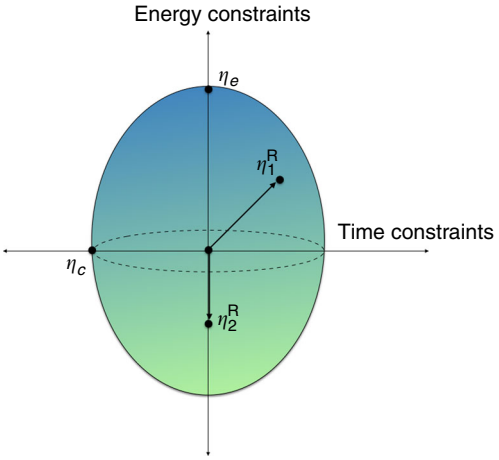

Fig. 2 Time-energy constraints for thermal processes. The entropic conditions for a state transformation $\rho^{\mathrm{A}} \to \sigma^{\mathrm{A}'}$ under TPs are defined with respect to a quantum reference frame R and two states η_1^{R} and η_2^{R}. The schematic vertical axis denotes states block-diagonal in energy (e.g., an energy eigenstate $\eta_e = |E\rangle\langle E|$), while the horizontal axis denotes states with maximal time-dependent oscillations—'clock' states η_{C} of R. When η_1^{R} is confined to being incoherent (the vertical axis) we recover thermo-majorization. For R being macroscopic and $\eta_1^{\mathrm{R}} = \eta_{\mathrm{C}}$, we obtain a Page–Wootters clock constraint on the thermodynamic transformation. Varying η_1^{R} smoothly interpolates between the time constraints and energy constraints

To see this better, we first consider the case where either the input or output state is incoherent in the energy eigenbasis. This regime is described by an essentially classical stochastic energy condition. The following result is shown in the Supplementary Note 4.

Corollary 2: Let A and A' be two quantum systems, with respective Hamiltonians H^{A} and $H^{\mathrm{A}'}$ being the only thermodynamic observables. Let ρ^{A} and $\sigma^{\mathrm{A}'}$ be quantum states on the input and output systems, respectively. If either $[\rho^{\mathrm{A}}, H^{\mathrm{A}}] = 0$ or $[\sigma^{\mathrm{A}'}, H^{\mathrm{A}'}] = 0$, then the state transformation $\rho^{\mathrm{A}} \to \sigma^{\mathrm{A}'}$ is possible under generalized thermal processes at a temperature $T = (k\beta)^{-1}$ if and only if $[\sigma^{\mathrm{A}'}, H^{\mathrm{A}'}] = 0$ and ρ^{A} thermo-majorizes $\sigma^{\mathrm{A}'}$.

This recovers previous results[5] on quantum thermodynamics for the case of one of the states having no coherences between energy eigenspaces. Moreover, in the case of incoherent input ρ^{A}, the use of a coherent reference state η^{R} does not yield any additional constraint. Specifically, $\Omega^{\mathrm{RA}} = q\langle \eta_1^{\mathrm{R}} \rangle \otimes \rho^{\mathrm{A}} + (1-q)\langle \eta_2^{\mathrm{R}} \rangle \otimes \gamma^{\mathrm{A}}$, and so the coherence of states η_1^{R} and η_2^{R} is irrelevant. The only relevant constraints in state transformation $\rho^{\mathrm{A}} \to \sigma^{\mathrm{A}'}$ are constraints related to the energy distribution of states.

On the other hand, if both the input-output states ρ^{A} and $\sigma^{\mathrm{A}'}$ contain coherence, then by choosing reference states η_1^{R} which contain coherence, we obtain new additional coherence constraints, i.e., constraints independent of thermo-majorization. Note that coherence with respect to energy eigenbasis is equivalent to symmetry-breaking (asymmetry) with respect to time-translations generated by the system Hamiltonian. In other words, coherence of states ρ^{A} and $\sigma^{\mathrm{A}'}$ is related to how well time t can be estimated from states $\rho^{\mathrm{A}}(t) = e^{-iH_{\mathrm{A}}t}\rho^{\mathrm{A}}e^{iH_{\mathrm{A}}t}$ and $\sigma^{\mathrm{A}'}(t) = e^{-iH_{\mathrm{A}'}t}\sigma^{\mathrm{A}'}e^{iH_{\mathrm{A}'}t}$.

The TPs are both covariant under time-translation and preserve the Gibbs state. In the Supplementary Note 6, we will show that the converse is also true (i.e. a covariant Gibbs preserving map is a TP). Therefore, previously discussed

measures, such as those that are based on Renyi divergences of the form $A_\alpha(\rho^{\rm A}) = S_\alpha(\rho^{\rm A}||\langle\rho^{\rm A}\rangle)$, behaves monotonically under TPs, and provide independent thermodynamic constraints beyond thermo-majorization[10]. One can also use constraints on modes of coherence[12,37] and the Fisher Information[63], to derive other independent measures of athermality. However, the set of conditional min-entropy measures obtained here is complete and therefore sufficient to imply the monotonicity of all of these measures.

In the Supplementary Note 4, we show that the entropic conditions with $\eta_1^{\rm R}$ being incoherent in energy leads to thermo-majorization, and captures the degree to which the system A is ordered in energy. Since in quantum systems one has complementarity between time and energy one might expect that the case of $\eta_1^{\rm R}$ being highly coherent in energy might therefore capture the degree to which A is ordered in some temporal sense.

This turns out to be the case, although since time forms a continuous one-parameter group there are technical obstacles to making this statement precise. However, as we show in the Supplementary Note 9, one can in general make finite precision approximations and model time evolution for any finite dimensional quantum system (which can be assumed to have an energy spectrum of rational numbers and thus has periodic dynamics under its Hamiltonian) with the discrete group $\mathbb{Z}_N$, for some sufficiently large N and with $t = n\varepsilon$. Here $\varepsilon > 0$ is the minimal time interval that can be resolved. The representation of this discrete group on A is given by $n \mapsto U_\varepsilon^{\rm A}(n) := \exp[-{\rm i}n\varepsilon H^{\rm A}]$ and so the system is modeled as evolving in discrete time steps. Under these approximations, one can replace microscopic conservation assumption with a slightly weaker version described in Supplementary Note 9, and the interconversion conditions can be repeated for $G = \mathbb{Z}_N$ instead.

We define clock-times as the discrete instances $t = 0, \varepsilon, \ldots, n\varepsilon, \ldots, (N-1)\varepsilon$ for the joint system $\mathcal{H}_R \otimes \mathcal{H}_{\rm A}$. As shown in the Supplementary Note 10, there exist reference frame systems R that can provide a perfect classical encoding of the clock times into quantum states $\{|0^{\rm R}\rangle, |1^{\rm R}\rangle, \ldots, |N-1^{\rm R}\rangle\}$, and for which $U_\varepsilon^{\rm R}(n)|0^{\rm R}\rangle = |n^{\rm R}\rangle$ for any n. Moreover, these clock states are built from uniform superpositions in the energy eigenstates of R, and so are in a sense "maximally" coherent in energy. Given this, we can now demonstrate the claimed complementarity between time and energy and how it relates to the state of the reference R. We choose $\eta_1^{\rm R} = |0\rangle^{\rm R}$ and consider the limit $q \to 1$, which corresponds to the condition of time-translation covariance alone. For this one can show that

$$\Omega^{\rm RA} = \frac{1}{N}\sum_{k=0}^{N-1} |k\rangle\langle k|^{\rm R} \otimes \rho^{\rm A}(n), \tag{24}$$

where $\rho^{\rm A}(n) := U_\varepsilon^{\rm A}(n)\rho^{\rm A}(U_\varepsilon^{\rm A}(n))^\dagger$ is the state of the system A at the nth clock time for the joint system. Now, since $\Omega^{\rm RA}$ is a classical-quantum state, we have that[64,65]

$$H_{\min}(R|A)_\Omega = -\log p_{\rm guess}, \tag{25}$$

where $p_{\rm guess}$ is the optimal Helstrom guessing probability for the ensemble of states $\{(\frac{1}{N}, \rho^{\rm A}(n))\}_{n=0}^{N-1}$ on A. This implies that $2^{-H_{\min}(R|A)_\Omega}$ is the optimal guessing probability of the clock time $t = n\varepsilon$ for the joint system, given the single copy of $\rho^{\rm A}$. Monotonicity of $H_{\min}(R|A)_\Omega$ under the thermal processes therefore implies monotonicity of the clock time guessing probability for the system. Phrased differently, the time-translation covariance property of thermal processes implies that the ability of the thermodynamic system A to act a quantum clock[66] can never increase. This demonstrates how the reference frame system R functions to define both time and energy constraints on the state interconversion for the system A.

We note that this result connects with foundational work by Page and Wootters[67], who considered how one can have dynamics in a universe that is covariant in time. They proposed a conditional probability formalism, which mirrors our present set up and relies on covariant measurements with $P(X^{\rm R} = x|Y^{\rm A} = y)$, the probability that some observable $X^{\rm R}$ has a sharp value given a measurement of $Y^{\rm A}$ yielding a particular result. These relational expressions were shown to describe dynamics within the time-translation invariant global state, such as $\Omega^{\rm RA}$ here.

We note that the condition in Theorem 2 that $H^{\rm R} = -(H^{\rm A})^{\rm T}$ can be understood in the context of global time-translation covariance. Formally, it says that the free evolution of the reference system R is via the representation dual to the time-translation action on A. More physically, it says that the joint Hamiltonian $H^{\rm RA} = H^{\rm R} \otimes 1^{\rm A} + 1^{\rm R} \otimes H^{\rm A}$ admits a non-trivial eigenspace with zero total energy in which coherent dynamics on A can be fully encoded. This should be compared with the Wheeler de Witt equation $H_{\rm tot}|\Psi\rangle = 0$ in quantum gravity, which also ensures global covariance under time-translations for the universe[68].

Multiple conserved charges. Finally, we can state the necessary and sufficient conditions for the case of having additional, additively conserved observables $\{X_1, \ldots, X_n\}$. In this case assumption (A3) follows a similar argument to the one for energy, and the auxiliary system can be assumed to be in a state $\sigma^{\rm B}$ for which $\sigma^{\rm B} = e^{-{\rm i}sX_k^{\rm B}}\sigma^{\rm B}e^{{\rm i}sX_k^{\rm B}}$, for all $s \in \mathbb{R}$ and for any thermodynamic observable $X_k^{\rm B}$. Ranging over all the observables, this condition can be expressed more compactly as $\sigma^{\rm B} = U(g)\sigma^{\rm B}U^\dagger(g)$, for all unitary transformations $U(g)$ in the Lie group G generated by the observables $\{H^{\rm B}, X_1^{\rm B}, \ldots, X_n^{\rm B}\}$. Note that this condition is equivalent to $\sigma^{\rm B} = \int_G dg\, U(g)\sigma^{\rm B}U^\dagger(g)$, where dg is the uniform (Haar) measure over this group. Therefore, this assumption, together with (A1) imply that the process is covariant with respect to group G, i.e., $\mathcal{U}_g \circ \mathcal{E} = \mathcal{E} \circ \mathcal{U}_g$, where $\mathcal{U}_g(\rho^{\rm A}) := U(g)\rho^{\rm A}U^\dagger(g)$. In other words, the process is covariant under the symmetry group action generated by the thermodynamic observables on the input/output systems. Our main result on the thermodynamic structure of states under TPs is as follows.

Theorem 3: [Generalized thermal processes] Let A and A' be two quantum systems, with thermodynamic observables $\{H^{\rm A}, X_1^{\rm A} \ldots, X_n^{\rm A}\}$ and $\{H^{\rm A'}, X_1^{\rm A'}, \ldots X_n^{\rm A'}\}$, respectively, and fix $0 < q < 1$. The state transformation $\rho^{\rm A} \to \sigma^{\rm A'}$ is possible under generalized thermal processes at a temperature $(k_{\rm B}\beta)^{-1}$ and at fixed Lagrange multipliers $\mu_1, \ldots \mu_n$, if and only if for all reference frame systems R of equal dimension to A' with thermodynamic observables $H^{\rm R} = -(H^{\rm A'})^{\rm T}$ and $\{X_k^{\rm R} = -(X_k^{\rm A'})^{\rm T}\}_{k=1}^n$, and for all pairs of states $\boldsymbol{\eta} = (\eta_1, \eta_2)$ we have $S_\eta(\rho^{\rm A}) \le S_\eta(\sigma^{\rm A'})$, where $S_\beta(\rho^{\rm A}) := H_{\min}(R|A)_\Omega$ and

$$\Omega^{\rm RA} = \int_G dg\, U(g)(q\eta_1^{\rm R} \otimes \rho^{\rm A} + (1-q)\eta_2^{\rm R} \otimes \gamma^{\rm A})U(g)^\dagger \tag{26}$$

where $\{U(g)\}$ is the symmetry group generated by the additively conserved observables $\{H^{\rm R} \otimes 1^{\rm A} + 1^{\rm R} \otimes H^{\rm A}, X_k^{\rm R} \otimes 1^{\rm A} + 1^{\rm R} \otimes X_k^{\rm A}; k = 1, \ldots, n\}$ on the composite system RA, with group parameters g, and $\gamma^{\rm A} = \exp[-\beta(H^{\rm A} - \sum_k \mu_k X_k^{\rm A})]/\mathcal{Z}$, being the generalized Gibbs ensemble on A.

This result is a fully covariant statement that is based on minimal assumptions, namely microscopic conservation, equilibrium preservation and incoherence, which reduces to Theorem 2 in the case of no additional thermodynamic observables beyond the system's energy.

Discussion

In this work, we have considered a generalization of majorization for quantum processes, found a necessary and sufficient condition for this notion of majorization in terms of entropic quantities, and demonstrated some of its applications in the context of the resource theories of asymmetry and quantum thermodynamics. In particular, we derived a complete set of entropic conditions for state transformations in both of these resource theories. In contrast to the previous results, which are only applicable to restricted families of states (such as incoherent states) our approach can be applied to all states. Furthermore, these results can be generalized to the case of approximate transformations in which we only require transformations up to an epsilon smoothing. However, the approximate case requires additional tools and is left for future work.

Since our entropic monotones provide a full characterization of the resource, it is interesting to study their operational interpretations. We discussed some of these interpretations in the context of clocks. Another possible interpretation could be provided by the results of ref. [69], which relates the smoothed entropy $H_{\max}^{\varepsilon}(R|A')$ to the minimal work cost to perform a quantum process. The duality relation between min and max entropies tells us that, where C purifies the state on RA', and so this suggests a potential interpretation of our results in terms of generalized work costs on a purifying environment.

We also introduced a new framework for quantum thermodynamics based on the notion of generalized thermal processes, which extends thermal operations, and is based on natural physical principles. This explicitly handles coherences and is the first framework of its kind for which a complete set of state conditions has been derived.

Data availability

The authors declare that all the data supporting the findings of this study are within the paper and its supplementary information files.

Received: 9 May 2018 Accepted: 20 August 2018
Published online: 17 December 2018

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Acknowledgements

G.G. would like to thank Rob Spekkens for pointing out ref. [34], and for many discussions on classical matrix majorization. We would also like to thank Eric Chitambar, Philippe Faist, Mark Girard, Kamil Korzekwa, and Matteo Lostaglio for useful discussions, and to Nicole Yunger Halpern for many comments on the first version. We also thanks for an anonymous referee who provided the simplified proof for Lemma 5 of the current version. F.B. acknowledges support from JSPS KAKENHI, Grant nos. 26247016 and 17K17796. G.G. acknowledges support from the Natural Sciences and Engineering Research Council of Canada (NSERC), D.J. is funded by the Royal Society.

Author contributions

G.G. and D.J. conceived and developed the presented idea. Some of these ideas were conceived independently by F.B. who later on joined the project and helped both in developing the theory further, and in the writing of the paper. R.D. conceived the idea that the problem of quantum majorization can be solved with semidefinite programming and helped with the initial computations. I.M. helped with many discussions and with the revisions of the manuscript. All authors discussed the results and contributed to the final manuscript.

Additional information

Supplementary Information accompanies this paper at https://doi.org/10.1038/s41467-018-06261-7.

Competing interests: The authors declare no competing interests.

Supplementary Material – Quantum majorization and a complete set of entropic conditions for quantum thermodynamics

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SUPPLEMENTARY NOTE 1: PROOF OF THEOREM 1

For completeness we repeat the statement of theorem 1.

Theorem 1 Let $\rho^{\rm AB}\in\mathcal{B}(\mathcal{H}_{\rm A}\otimes\mathcal{H}_{\rm B})$ and $\sigma^{\rm AC}\in\mathcal{B}(\mathcal{H}_{\rm A}\otimes\mathcal{H}_{\rm C})$ be two compatible bipartite quantum states. Let $\{M_j^{\rm A}\}$ be an arbitrary, but fixed, informationally complete POVM on system A. Denote the dimension of any system X as $d_X\in\mathbb{N}$. The following are equivalent:

1. The state $\rho^{\rm AB}$ quantum majorizes $\sigma^{\rm AC}$,
$$\sigma^{\rm AC}\prec_q\rho^{\rm AB}. \tag{1}$$

2. For any quantum process (CPTP linear map) $\Phi:\mathcal{B}(\mathcal{H}_{\rm A})\to\mathcal{B}(\mathcal{H}_{\rm A'})$, with $d_{\rm A'}=d_{\rm C}$,
$$H_{\min}(A'|B)_{\Phi\otimes\mathsf{id}(\rho^{\rm AB})}\leqslant H_{\min}(A'|C)_{\Phi\otimes\mathsf{id}(\sigma^{\rm AC})} \tag{2}$$

3. Supplementary Equation (2) holds for any measure-and-prepare quantum channel $\Phi:\mathcal{B}(\mathcal{H}_{\rm A})\to\mathcal{B}(\mathcal{H}_{\rm A'})$ of the form:
$$\Phi\left(\eta^{\rm A}\right)=\sum_{j=1}^{d_{\rm A}^2}{\rm Tr}\left[M_j^{\rm A}\eta^{\rm A}\right]\omega_j^{\rm A'}\ , \tag{3}$$
while the states $\{\omega_j^{\rm A'}\}$ can freely vary.

4. $g(\rho^{\rm AB},\sigma^{\rm AC})\geqslant 1$, where the the function g is defined by the following semidefinite programming:
$$g(\rho^{\rm AB},\sigma^{\rm AC})=\max \left\{\ y\ \middle|\ \forall j\ \ y\sigma_j^{\rm T}\leqslant{\rm Tr}_{\rm B}\left[\tau^{CB}(I\otimes\rho_j)\right],\ \tau^{CB}\geqslant 0,\ \tau^{\rm B}\leqslant I\right\} \tag{4}$$
where
$$\rho_j\equiv\frac{{\rm Tr}_{\rm A}\left[\left(M_j^{\rm A}\otimes\mathbb{1}^{\rm B}\right)\rho^{\rm AB}\right]}{{\rm Tr}\left[M_j^{\rm A}\rho^{\rm A}\right]}\ \text{and}$$
$$\sigma_j\equiv\frac{{\rm Tr}_{\rm A}\left[\left(M_j^{\rm A}\otimes\mathbb{1}^{\rm C}\right)\sigma^{\rm AC}\right]}{{\rm Tr}\left[M_j^{\rm A}\sigma^{\rm A}\right]}. \tag{5}$$

In order to prove Theorem 1, we will begin by proving the following lemma. Recall the definition of conditional min-entropy $H_{\min}(A|B)_\Omega$, of a bipartite state $\Omega^{\rm AB}$,
$$H_{\min}(A|B)_\Omega:=-\log\inf_{\tau^{\rm B}\geqslant 0}\{{\rm Tr}\left[\tau^{\rm B}\right]:\mathbb{1}^{\rm A}\otimes\tau^{\rm B}\geqslant\Omega^{\rm AB}\}. \tag{6}$$

Lemma 1 Let $\{\rho_i^{\rm B}\}_{i=1}^n$ and $\{\sigma_i^{\rm C}\}_{i=1}^n$ be two sets of n density matrices in $\mathcal{B}(\mathcal{H}_{\rm B})$ and $\mathcal{B}(\mathcal{H}_{\rm C})$, respectively. Let $\{q_i\}_{i=1}^n$ be some arbitrary but fixed probability distribution with $q_i>0$. For any set of n density matrices $\{\omega_i^{\rm A}\}_{i=1}^n$ in $\mathcal{B}(\mathcal{H}_{\rm A})$ (with $d_{\rm A}=d_{\rm C}$) define the following tripartite separable matrix:
$$\Omega^{ABC}\equiv\sum_{i=1}^n q_i\ \omega_i^{\rm A}\otimes\rho_i^{\rm B}\otimes\sigma_i^{\rm C}\ . \tag{7}$$

Then, the following are equivalent:

1. There exists a CPTP map $\mathcal{E}:\ \mathcal{B}(\mathcal{H}_{\rm B})\to\mathcal{B}(\mathcal{H}_{\rm C})$ such that
$$\mathcal{E}(\rho_i^{\rm B})=\sigma_i^{\rm C}\quad\forall\ i=1,...,n\ . \tag{8}$$

2. For any $\omega_1^{\rm A},...,\omega_n^{\rm A}\in\mathcal{B}(\mathcal{H}_{\rm A})$:
$$2^{-H_{\min}(A|B)_\Omega}\geqslant d_{\rm C}\langle\phi_+^{\rm AC}|\Omega^{\rm AC}|\phi_+^{\rm AC}\rangle\ , \tag{9}$$
where $|\phi_+^{\rm AC}\rangle$ denotes the maximally entangled state on $\mathcal{H}_{\rm A}\otimes\mathcal{H}_{\rm C}$.

3. For any $\omega_1^{\rm A},...,\omega_n^{\rm A}\in\mathcal{B}(\mathcal{H}_{\rm A})$:
$$H_{\min}(A|B)_\Omega\leqslant H_{\min}(A|C)_\Omega\ . \tag{10}$$

Proof. Consider two families of density matrices, $\{\rho_i^{\rm B}\}_{i=1}^n$ and $\{\sigma_i^{\rm C}\}_{i=1}^n$. We want to reformulate, in an equivalent way, the condition
$$\exists\ {\rm CPTP}\ \mathcal{E}:\mathcal{E}(\rho_i^{\rm B})=\sigma_i^{\rm C},\forall i. \tag{11}$$
By introducing a set of self-adjoint operators $\{X_j^{\rm C}\}$ forming a basis for $\mathcal{B}(\mathcal{H}_{\rm C})$, Supplementary Equation (8) can be written as
$$\exists\ {\rm CPTP}\ \mathcal{E}:{\rm Tr}\left[\mathcal{E}(\rho_i^{\rm B})\ X_j^{\rm C}\right]={\rm Tr}\left[\sigma_i^{\rm C}\ X_j^{\rm C}\right],\forall i,j. \tag{12}$$
Let us now consider the set of real vectors
$$\boldsymbol{r}_{\mathcal{E}}=(r_{ij}):r_{ij}={\rm Tr}\left[\mathcal{E}(\rho_i^{\rm B})\ X_j^{\rm C}\right] \tag{13}$$
obtained by letting $\mathcal{E}$ vary over all possible CPTP maps from system B to system C, while the ρ_i's and the X_j's are kept fixed. It is clear that the set
$$\mathcal{S}=\{\boldsymbol{r}_{\mathcal{E}}:\mathcal{E}\ {\rm CPTP}\} \tag{14}$$
is a closed and bounded convex set, as it is the image, under a linear map, of the set of CPTP maps from B to

C (that is a closed and bounded convex set). By writing $\boldsymbol{s} = (s_{ij})$ when $s_{ij} = \mathrm{Tr}\big[\sigma_i^{\mathrm{C}}\ X_j^{\mathrm{C}}\big]$, Supplementary Equation (12) becomes

$$\boldsymbol{s} \in \mathcal{S}\ . \tag{15}$$

At this point, we invoke the separation theorem for convex sets (see, e.g., Ref. [1]), which in particular implies the following:

Lemma [Separation theorem] Let $\mathcal{S} \subset \mathbb{R}^n$ be a closed and bounded convex set. The vector $y \in \mathbb{R}^n$ belongs to $\mathcal{S}$, i.e. $y \in \mathcal{S}$, if and only if, for any vector $k \in \mathbb{R}^n$, $\max_{x\in\mathcal{S}} k\cdot x \geqslant k\cdot y$.

Applied to our case, it yields that condition (8) is equivalent to

$$\forall \boldsymbol{\lambda} = (\lambda_{ij}) \text{ with } \lambda_{ij} \in \mathbb{R}, \quad \max_{\boldsymbol{r}\in\mathcal{S}} \boldsymbol{r}\cdot\boldsymbol{\lambda} \geqslant \boldsymbol{s}\cdot\boldsymbol{\lambda}\ , \tag{16}$$

namely,

$$\begin{aligned}&\forall \boldsymbol{\lambda} = (\lambda_{ij}) \text{ with } \lambda_{ij} \in \mathbb{R},\\ &\max_{\mathcal{E}:\mathrm{CPTP}} \sum_{ij} \lambda_{ij}\,\mathrm{Tr}\big[\mathcal{E}(\rho_i^{\mathrm{B}})\ X_j^{\mathrm{C}}\big] \geqslant \sum_{ij}\lambda_{ij}\,\mathrm{Tr}\big[\sigma_i^{\mathrm{C}}\ X_j^{\mathrm{C}}\big]\,.\end{aligned} \tag{17}$$

Defining self-adjoint operators $Z_i^{\mathrm{C}} = \sum_j \lambda_{ij} X_j^{\mathrm{C}}$, we can reformulate the statement as follows:

$$\begin{aligned}&\forall \text{ self-adjoint } \{Z_i^{\mathrm{C}}\},\\ &\max_{\mathcal{E}:\mathrm{CPTP}} \sum_i \mathrm{Tr}\big[\mathcal{E}(\rho_i^{\mathrm{B}})\ Z_i^{\mathrm{C}}\big] \geqslant \sum_i \mathrm{Tr}\big[\sigma_i^{\mathrm{C}}\ Z_i^{\mathrm{C}}\big]\,.\end{aligned} \tag{18}$$

In this condition we should vary operators Z_i^{C} over all self-adjoint operators. However, it turns out that we can only restrict these operators to the set of density operators. In other words, this condition is equivalent to

$$\begin{aligned}&\forall \text{ states } \{\omega_i^{\mathrm{C}}\},\\ &\max_{\mathcal{E}:\mathrm{CPTP}} \sum_i \mathrm{Tr}\big[\mathcal{E}(\rho_i^{\mathrm{B}})\ \omega_i^{\mathrm{C}}\big] \geqslant \sum_i \mathrm{Tr}\big[\sigma_i^{\mathrm{C}}\ \omega_i^{\mathrm{C}}\big]\,.\end{aligned} \tag{19}$$

To show this note that for any bounded self-adjoint operator Z_i^{C} and positive number $z_i > \|Z_i^{\mathrm{C}}\|_\infty$, $Z_i^{\mathrm{C}} + z_i\mathbb{1}^{\mathrm{C}}$ is a positive operator, and therefore $\omega_i^{\mathrm{C}} \equiv (Z_i^{\mathrm{C}} + z_i\mathbb{1}^{\mathrm{C}})/\,\mathrm{Tr}\big[Z_i^{\mathrm{C}} + z_i\mathbb{1}^{\mathrm{C}}\big]$ is a density operator. Furthermore, since constants z_i can be chosen independent of each other, we can choose them such that $\mathrm{Tr}\big[Z_i^{\mathrm{C}} + z_i\mathbb{1}^{\mathrm{C}}\big]$ is a positive constant independent of i. Then, putting this set of states ω_i^{C} into Supplementary Eq. (19) and using the fact that $\mathrm{Tr}\big[\mathcal{E}(\rho_i^{\mathrm{B}})\big] = \mathrm{Tr}\big[\sigma_i^{\mathrm{C}}\big] = 1$ we can recover Supplementary Eq. (18).

In Supplementary Eq.(19) all the terms corresponding to different states ω_i have equal weights in the summations. Next, we show how this constraint can be relaxed: Consider an arbitrary fixed probability distribution q_i, with full support, such that $q_{\min} \equiv \min_i q_i > 0$. Then, it turns out that condition (19) can be reformulated as:

$$\begin{aligned}&\forall \text{ states } \{\omega_i^{\mathrm{C}}\},\\ &\max_{\mathcal{E}:\mathrm{CPTP}} \sum_i q_i\,\mathrm{Tr}\big[\mathcal{E}(\rho_i^{\mathrm{B}})\ \omega_i^{\mathrm{C}}\big] \geqslant \sum_i q_i\,\mathrm{Tr}\big[\sigma_i^{\mathrm{C}}\ \omega_i^{\mathrm{C}}\big]\ .\end{aligned} \tag{20}$$

To show this we use the fact that for any state ω_i^{C}, the convex combination $(q_{\min}/q_i)\omega_i^{\mathrm{C}} + (1 - q_{\min}/q_i)\mathbb{1}^{\mathrm{C}}/d_{\mathrm{C}}$ is also a valid state. Using this together with the fact that $\mathrm{Tr}\big[\mathcal{E}(\rho_i^{\mathrm{B}})\big] = \mathrm{Tr}\big[\sigma_i^{\mathrm{C}}\big] = 1$ we can derive condition (19) from condition (20) and vice versa.

The next step is to introduce an auxiliary system $A \cong C$ (i.e., $d_{\mathrm{A}} = d_{\mathrm{C}}$), choose two orthonormal bases $\{|i_{\mathrm{A}}\rangle\}$ and $\{|i_{\mathrm{C}}\rangle\}$, and define the maximally entangled state

$$|\phi_+^{\mathrm{AC}}\rangle \equiv d_{\mathrm{A}}^{-1/2} \sum_{i=1}^{d_{\mathrm{A}}} |i_{\mathrm{A}}\rangle|i_{\mathrm{C}}\rangle\ . \tag{21}$$

Noticing that $\mathrm{Tr}[XY] = d\,\mathrm{Tr}\big[X \otimes Y^{\mathrm{T}}\ \phi_+\big]$, where the superscript T denotes the transposition with respect to the basis in (21), and that ω_i are density matrices if and only if $(\omega_i)^{\mathrm{T}}$ are, we arrive at

$$\begin{aligned}&\forall \text{ states } \{\omega_i^{\mathrm{A}}\}, \\ &\qquad \max_{\mathcal{E}:\mathrm{CPTP}} \sum_i q_i\,\mathrm{Tr}\big[\big\{\omega_i^{\mathrm{A}} \otimes \mathcal{E}(\rho_i^{\mathrm{B}})\big\}\ \phi_+^{\mathrm{AC}}\big] \\ &\qquad \geqslant \sum_i q_i\,\mathrm{Tr}\big[\big\{\omega_i^{\mathrm{A}} \otimes \sigma_i^{\mathrm{C}}\big\}\ \phi_+^{\mathrm{AC}}\big]\,.\end{aligned} \tag{22}$$

As shown in Ref. [5], the quantity

$$\max_{\mathcal{E}:\mathrm{CPTP}} \sum_i q_i\,\mathrm{Tr}\big[\big\{\omega_i^{\mathrm{A}} \otimes \mathcal{E}(\rho_i^{\mathrm{B}})\big\}\ \phi_+^{\mathrm{AC}}\big] \tag{23}$$

$$= \max_{\mathcal{E}:\mathrm{CPTP}} \langle\phi_+^{\mathrm{AC}}|(\mathsf{id}\otimes\mathcal{E})(\Omega^{\mathrm{AB}})|\phi_+^{\mathrm{AC}}\rangle\ , \tag{24}$$

for $\Omega^{\mathrm{AB}} \equiv \sum_i q_i\ \omega_i^{\mathrm{A}} \otimes \rho_i^{\mathrm{B}}$, can be written in terms of the conditional min-entropy (6) as

$$\frac{1}{d_{\mathrm{A}}} 2^{-H_{\min}(A|B)_\Omega}\ . \tag{25}$$

We thus proved that statements (1) and (2) of Lemma 1 are indeed equivalent.

Moreover, a *sufficient* condition for (8) is that

$$\begin{aligned}&\forall \text{ states } \{\omega_i^{\mathrm{A}}\}, \\ &\qquad \max_{\mathcal{E}:\mathrm{CPTP}} \sum_i q_i\,\mathrm{Tr}\big[\big\{\omega_i^{\mathrm{A}} \otimes \mathcal{E}(\rho_i^{\mathrm{B}})\big\}\ \phi_+^{\mathrm{AC}}\big] \\ &\qquad \geqslant \max_{\mathcal{F}:\mathrm{CPTP}} \sum_i q_i\,\mathrm{Tr}\big[\big\{\omega_i^{\mathrm{A}} \otimes \mathcal{F}(\sigma_i^{\mathrm{C}})\big\}\ \phi_+^{\mathrm{AC}}\big]\ ,\end{aligned} \tag{26}$$

namely

$$2^{-H_{\min}(A|B)_\Omega} \geqslant 2^{-H_{\min}(A|C)_\Omega}\ , \tag{27}$$

where now $\Omega^{\rm AB}$ and $\Omega^{\rm AC}$ are meant as the marginals of the same tripartite extension $\Omega^{ABC} = \sum_i q_i\ \omega_i^{\rm A} \otimes \rho_i^{\rm B} \otimes \sigma_i^{\rm C}$. However, it is easy to verify that the above condition is also necessary: indeed, if (8) holds, due to the data-processing theorem applied to the conditional min-entropy (see, e.g., Ref. [6] and [7]), $H_{\min}(A|B)_\Omega \leqslant H_{\min}(A|C)_\Omega$. We thus have that statements (1) and (3) are also logically equivalent, and hence the proof is complete. ∎

We are now ready to prove the main theorem.

Proof of Theorem 1:

Let $\{Q_k^{\rm A}\}_{k=1}^{d_{\rm A}^2}$ be the dual basis of $\{M_j^{\rm A}\}$ in $\mathcal{B}(\mathcal{H}_{\rm A})$, that is, $\mathrm{Tr}\left[M_j^{\rm A} Q_k^{\rm A}\right] = \delta_{jk}$. Then, since $\{Q_k^{\rm A}\}$ is itself a basis, we can write

$$\rho^{\rm AB} = \sum_{k=1}^{d_{\rm A}^2} Q_k^{\rm A} \otimes \widetilde{\rho}_k^{\rm B} \quad \text{and} \quad \sigma^{\rm AC} = \sum_{k=1}^{d_{\rm A}^2} Q_k^{\rm A} \otimes \widetilde{\sigma}_k^{\rm C} \tag{28}$$

where

$$\begin{aligned} \widetilde{\rho}_j^{\rm B} &\equiv \mathrm{Tr}_{\rm A}\left[\left(M_j^{\rm A} \otimes \mathbb{1}^{\rm B}\right)\rho^{\rm AB}\right] \\ \widetilde{\sigma}_j^{\rm C} &\equiv \mathrm{Tr}_{\rm A}\left[\left(M_j^{\rm A} \otimes \mathbb{1}^{\rm C}\right)\sigma^{\rm AC}\right] \end{aligned} \tag{29}$$

are sub-normalized quantum states (i.e. positive semi-definite matrices). Moreover, since $\rho^{\rm A} = \sigma^{\rm A}$ we have $\mathrm{Tr}\left[\widetilde{\rho}_j^{\rm B}\right] = \mathrm{Tr}\left[\widetilde{\sigma}_j^{\rm C}\right] \equiv p_j$. We therefore conclude that that there exists CPTP map $\mathcal{E}$ that satisfies Supplementary Equation (1) if and only if there exists a CPTP map $\mathcal{E}$ that satisfies

$$\sigma_j^{\rm C} = \mathcal{E}\left(\rho_j^{\rm B}\right) \tag{30}$$

where $\rho_j^{\rm B} \equiv \widetilde{\rho}_j^{\rm B}/p_j$ and $\sigma_j^{\rm C} \equiv \widetilde{\sigma}_j^{\rm C}/p_j$. To apply Lemma 1, we introduce a system A' with $d_{\rm A'} = d_{\rm C}$, we fix an arbitrary probability distribution $q_i > 0$, and define

$$\begin{aligned} &\Omega^{A'BC} \equiv \sum_{j=1}^{d_{\rm A}^2} q_j\ \omega_j^{\rm A'} \otimes \rho_j^{\rm B} \otimes \sigma_j^{\rm C} = \sum_{j=1}^{d_{\rm A}^2} \frac{q_j}{p_j^2} \times \\ &\omega_j^{\rm A'} \otimes \mathrm{Tr}_{\rm A}\left[\left(M_j^{\rm A} \otimes I^{\rm B}\right)\rho^{\rm AB}\right] \otimes \mathrm{Tr}_{\rm A}\left[\left(M_j^{\rm A} \otimes I^{\rm C}\right)\sigma^{\rm AC}\right]\ , \end{aligned} \tag{31}$$

where the states $\omega_i^{\rm A'}$ can vary. Then, taking $q_j = p_j$, we conclude that

$$\begin{aligned} \Omega^{A'B} &= \Phi \otimes \mathsf{id}\left(\rho^{\rm AB}\right)\ , \\ \Omega^{A'C} &= \Phi \otimes \mathsf{id}\left(\sigma^{\rm AC}\right)\ . \end{aligned} \tag{32}$$

Notice that, in case some $p_i = 0$, we can redefine the measurement operators $M_j^{\rm A} \to M_j^{\rm A} + \delta\mathbb{1}^{\rm A}$, in such a way that they still span the set $\mathcal{B}(\mathcal{H}_{\rm A})$ but have non-zero probability everywhere. With Supplementary Eq. (32) at hand, the proof of Theorem 1 follows now from Lemma 1. This completes the proof of theorem 1.

SUPPLEMENTARY NOTE 2: COMPLEXITY OF DECIDING QUANTUM MAJORIZATION

We will show now that the problem of whether there exists a CPTP map $\mathcal{E}$ such that $\mathcal{E}(\rho_i) = \sigma_i$ (see Lemma 1 above) can be formulated as a semidefinite programming. Following similar lines, also all the other versions of quantum majorization discussed in this paper can be shown to be equivalent to a semidefinite programming.

We start by noting that Supplementary Equation (9) can be written as

$$2^{-H_{\min}(A|B)_\Omega} \geqslant \sum_{i=1}^{n} q_i \mathrm{Tr}\left(\omega_i \sigma_i^{\rm T}\right) \tag{33}$$

where

$$\Omega^{\rm AB} = \sum_{i=1}^{n} q_i\ \omega_i \otimes \rho_i\ . \tag{34}$$

In the following we absorb the q_is into ω_is, so that the ω_is become subnormalized, satisfying $\sum_{i=1}^n \mathrm{Tr}[\omega_i] = 1$. We get that the above condition is equivalent to the condition $\alpha(t) \geqslant 1$ for all t, where

$$\begin{aligned} \alpha(t) \equiv &\frac{1}{t} \min \mathrm{Tr}[\tau] \\ &\text{subject to} \quad I^{\rm A} \otimes \tau \geqslant \sum_{i=1}^{n} \omega_i \otimes \rho_i, \\ &\qquad \sum_{i=1}^{n} \mathrm{Tr}\left(\sigma_i^{\rm T}\omega_i\right) = t\ ;\ \sum_{i=1}^{n} \mathrm{Tr}[\omega_i] = 1\ , \end{aligned} \tag{35}$$

with $\omega_i \geqslant 0$. After rescaling $\tau' \equiv \frac{1}{t}\tau$ and $\omega_i' \equiv \frac{1}{t}\omega_i$ we get

$$\begin{aligned} \alpha(t) \equiv &\min \mathrm{Tr}[\tau'] \\ &\text{subject to} \quad I^{\rm A} \otimes \tau' \geqslant \sum_{i=1}^{n} \omega_i' \otimes \rho_i, \\ &\qquad \sum_{i=1}^{n} \mathrm{Tr}\left(\sigma_i^{\rm T}\omega_i'\right) = 1\ ;\ \sum_{i=1}^{n} \mathrm{Tr}[\omega_i'] = 1/t\ , \end{aligned} \tag{36}$$

The condition $\alpha(t) \geqslant 1$ for all t is therefore equivalent to one condition, $\alpha \geqslant 1$ (more precisely, $\alpha = 1$ since it can be shown that α can never exceed 1), where

$$\begin{aligned} \alpha \equiv &\min \mathrm{Tr}[Z] \\ &\text{subject to} \quad I^{\rm A} \otimes Z \geqslant \sum_{i=1}^{n} X_i \otimes \rho_i, \\ &\qquad \sum_{i=1}^{n} \mathrm{Tr}\left(\sigma_i^{\rm T} X_i\right) = 1\ ;\ X_i \geqslant 0\ . \end{aligned} \tag{37}$$

We now show that the above minimization problem is an SDP. To see it, we define the following vector space, which is a direct sum of $n+2$ Hilbert spaces:

$$V_1 \equiv \mathrm{B}(\mathcal{H}^{\mathrm{A}}\otimes\mathcal{H}^{\mathrm{B}})\oplus\mathrm{B}(\mathcal{H}^{\mathrm{B}})\oplus\mathrm{B}(\mathcal{H}^{\mathrm{A}})\oplus\cdots\oplus\mathrm{B}(\mathcal{H}^{\mathrm{A}})\ . \quad (38)$$

The vector space V_1 is consisting of matrices $\zeta \in V_1$ of the form:

$$\zeta = (\eta, Z, X_1, ..., X_n) \quad (39)$$

where $\eta \in \mathrm{B}(\mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}})$, $Z \in \mathrm{B}(\mathcal{H}^{\mathrm{B}})$, and $X_i \in \mathrm{B}(\mathcal{H}^{\mathrm{A}})$ for each $i = 1, ..., n$. In addition, we define the vector space $V_2 \equiv \mathrm{B}(\mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}})$, and a linear transformation $\Gamma : V_1 \to V_2$ given by:

$$\Gamma\left(\zeta\right) = I \otimes Z - \sum_{i=1}^{n} X_i \otimes \rho_i - \eta\ . \quad (40)$$

Clearly, the map above is linear. Set $\sigma \equiv (\mathbf{0}, \mathbf{0}, \sigma_1^{\mathrm{T}}, ..., \sigma_n^{\mathrm{T}})$ so that $\mathrm{Tr}[\sigma\zeta] = \sum_{i=1}^{n} \mathrm{Tr}\left(\sigma_i^{\mathrm{T}} X_i\right)$. We also denote $C \equiv (\mathbf{0}, I, \mathbf{0}, ..., \mathbf{0})$. With these notations:

$$\alpha = \min\left\{ \mathrm{Tr}[C\zeta] \;\middle|\; \zeta \geqslant 0\ ;\ \Gamma(\zeta) = 0\ ,\ \mathrm{Tr}[\sigma\zeta] = 1 \right\} \quad (41)$$

To bring the above optimization problem to a canonical SDP form, we denote $H_j \equiv \Gamma^*(E_j)$, where $j = 1, ..., d_{\mathrm{A}}^2 d_{\mathrm{B}}^2$ and E_j is a basis of $\mathrm{B}(\mathcal{H}^{\mathrm{A}}\otimes\mathcal{H}^{\mathrm{B}})$. We also denote $H_0 \equiv \sigma$. With this notations we get

$$\begin{aligned} \alpha \equiv & \min \mathrm{Tr}[C\zeta] \\ & \text{subject to} \quad \zeta \geqslant 0 \\ & \qquad\qquad \mathrm{Tr}\left(\zeta H_j\right) = \delta_{0,j} \quad j = 0, 1, ..., d_{\mathrm{A}}^2 d_{\mathrm{B}}^2 \end{aligned} \quad (42)$$

It is interesting to note that the dual problem is given by

$$\begin{aligned} \beta \equiv & \max y \\ & \text{subject to} \quad y\sigma + \Gamma^*(\tau^{\mathrm{AB}}) \leqslant C \ \ , \tau^{\mathrm{AB}} \in V_2. \end{aligned} \quad (43)$$

where the dual map Γ^* is given by:

$$\begin{aligned} &\Gamma^*(\tau^{\mathrm{AB}}) = \\ &\left(-\tau^{\mathrm{AB}}, \tau^{\mathrm{B}}, -\mathrm{Tr}_{\mathrm{B}}\left[\tau^{\mathrm{AB}}(I\otimes\rho_1)\right], ..., -\mathrm{Tr}_{\mathrm{B}}\left[\tau^{\mathrm{AB}}(I\otimes\rho_n)\right]\right) \end{aligned} \quad (44)$$

Therefore, the dual problem can be expressed as

$$\begin{aligned} \beta = & \max\ y \\ & \text{subject to} \quad \tau^{\mathrm{AB}} \geqslant 0\ ;\ \tau^{\mathrm{B}} \leqslant I\ ;\ \text{and}\ \forall\ i = 1, ..., n \\ & \qquad\qquad y\sigma_i^{\mathrm{T}} \leqslant \mathrm{Tr}_{\mathrm{B}}\left[\tau^{\mathrm{AB}}(I\otimes\rho_i)\right] \end{aligned} \quad (45)$$

Note that α (or β) can be commuted efficiently using standard SDP algorithms.

SUPPLEMENTARY NOTE 3: TWO FORMULATIONS: SEMIDEFINITE PROGRAMMING VERSUS A COMPLETE SET OF MONOTONES

As mentioned in Remark 3 of the main text, and as shown in Note 2 above, all the instances of quantum majorization considered in this paper can be formulated as semidefinite programs. These are well known for being efficiently solvable. One may be left wondering, then, about the role and relevance of the alternative formulation of quantum majorization that we provide in Theorem 1, in terms of an infinite set of inequalities between state monotones. Clearly, if the problem is just to decide whether quantum majorization holds or not, one should run the corresponding semi-definite program. However, the SDP formulation does not provide us any further insight about why a solution exists or not, nor does it tell us anything about the resources at stake and the way to quantify them. In other words, it does not tell us much about the physics behind quantum majorization.

Ideally, a resource theory should not only provide an efficient way to check whether a free transformation exists between two states, but also a way to measure resources as state functions. While the SDP formulation fulfills the former requirement, the formulation in terms of a complete set of monotones fulfills the latter.

The fact that here we find a complete set of monotones comprising infinitely many such functions is not an artifact of the present approach, but it is something that appears in many other contexts too. For example, already in classical statistics, the majorization relation with catalytic transformations (i.e., the "trumping" relation) is known to be equivalent to an inequality that must hold for all (uncountably many) Rényi entropies [30], and no discrete set of equivalent conditions is known. Again in the classical case, catalytic thermal operations have also been characterized in terms of an infinite set of "second laws" involving free energy functionals [2]. In the quantum theory of entanglement, when the local dimension is four or higher, it is known that an infinite number of entanglement monotones is not only sufficient but also necessary, in order to determine state conversion [3].

The characterization of quantum majorization in terms of a complete set of monotones, as we show in what follows, is also able to completely capture the notion of quantum thermal processes, with respect to both energy and coherence. This is non-trivial (and perhaps even surprising), given that it was shown that no direct analogue, in terms of "simple" free energy functionals, would ever be able to capture the subtle interplay between energy and coherence appearing in genuinely quantum thermal processes between non-commuting states [4]. It is hence a merit of our approach to circumvent this obstacle providing, at the same time, a novel insight into the theory. Indeed, the monotones constructed here are able to go be-

yond free energy functionals, by explicitly bringing into the picture an external reference system, with respect to which information about energy and time (i.e., coherence) is measured. Such an insight, that suggests also an intriguing physical picture behind quantum majorization, cannot be gained by looking at the SDP formulation alone.

Finally, the characterization in terms of monotones has, with respect to the SDP formulation, another advantage, which is due to the fact that our monotones can be expressed as min-conditional entropies [5, 6]. This allows us to apply, in principle, the powerful tools developed for single-shot quantum information theory [7] in order to study their behavior in the asymptotic scenario, something that we leave open for future investigations.

SUPPLEMENTARY NOTE 4: RE-DERIVATION OF THERMO-MAJORIZATION AS THE CLASSICAL CASE

Thermo-majorization generalizes ordinary majorization in a natural way [8–13]. Given two probability distributions $\mathbf{p} = (p_i)$ and $\mathbf{q} = (q_i)$ together with the Gibbs distribution $\boldsymbol{\gamma} = (\gamma_i) = (\frac{1}{Z}e^{-\beta E_i})$ at temperature $T = (k\beta)^{-1}$, we say that $\mathbf{p}$ thermo-majorizes $\mathbf{q}$ and write $\mathbf{p} \succ_T \mathbf{q}$ exactly when the following holds

$$\sum_k |p_k - t\gamma_k| \geqslant \sum_k |q_k - t\gamma_k|, \tag{46}$$

for all $t \geqslant 0$. This can be shown to be equivalent [8–11, 13, 22] to the existence of a stochastic map S such that $S\mathbf{p} = \mathbf{q}$ and $S\boldsymbol{\gamma} = \boldsymbol{\gamma}$. In what follows, we show that quantum majorization reduces to Thermo-majorization in the classical case. In particular, we will show that the conditions in Theorem 1 (specifically, Supplementary Equation (10) of Lemma 1) reduces to Supplementary Eq. (46). We first start with the semi-classical case.

The semi-classical case

In this case, we assume that the n states, $\{\sigma_i^{\mathrm{C}}\}$, in Lemma 1 commute with each other. Therefore, we can assume that they are all diagonal with respect to a fixed basis. We show now that this immediately implies that the n states $\{\omega_i\}$ in Lemma 1 can also be taken to be diagonal in the same basis. In fact, in the following lemma we show that if $\{\sigma_i^{\mathrm{C}}\}$ are all symmetric with respect to some group, then the states $\{\omega_i\}$ also have the same symmetry.

Lemma 2 Using the same notations as in Lemma 1, let $\Delta : \mathrm{B}(\mathcal{H}^{\mathrm{C}}) \to \mathrm{B}(\mathcal{H}^{\mathrm{C}})$ be a CPTP map, and suppose $\Delta(\sigma_i^{\mathrm{T}}) = \sigma_i^{\mathrm{T}}$ for all $i = 1, ..., n$. Then, in all the statements of Lemma 1 we can replace the set $\{\omega_i^{\mathrm{A}}\}$ with the set $\Delta^\dagger(\omega_i^{\mathrm{A}})$.

Remark 1. The lemma above is particularly interesting if the map Δ corresponds to some symmetry. That is, suppose the states $\{\sigma_i^{\mathrm{T}}\}$ satisfy $U_g\sigma_i^{\mathrm{T}}U_g^\dagger = \sigma_i^{\mathrm{T}}$ for any $g \in G$, where $\{U_g\}$ is some unitary representation of a compact group G. In this case, one can take Δ to be the G-twirling, and thereby assume that all the ω_i^{A}s of Lemma 1 are also symmetric with respect to the same representation of G.

Proof. The proof follows from the two sides of Supplementary Eq. (9). On one hand,

$$\begin{aligned} d_{\mathrm{C}}\langle\phi_+^{\mathrm{AC}}|\Omega^{\mathrm{AC}}|\phi_+^{\mathrm{AC}}\rangle &= \sum_i q_i \operatorname{Tr}\left[\sigma_i^{\mathrm{T}}\omega_i\right] \\ &= \sum_i q_i \operatorname{Tr}\left[\Delta(\sigma_i^{\mathrm{T}})\omega_i\right] \\ &= \sum_i q_i \operatorname{Tr}\left[\sigma_i^{\mathrm{T}}\Delta^\dagger(\omega_i)\right] \ , \end{aligned} \tag{47}$$

where $\Delta^\dagger$ is the dual (adjoint) unital map of Δ. On the other hand, if

$$I \otimes \tau \geqslant \sum_{i=1}^n q_i \ \omega_i^{\mathrm{A}} \otimes \rho_i^{\mathrm{B}} \tag{48}$$

for some non-normalized state τ, then since $\Delta^\dagger$ is a unital CP map we get

$$I \otimes \tau \geqslant \sum_{i=1}^n q_i \ \Delta^\dagger(\omega_i^{\mathrm{A}}) \otimes \rho_i^{\mathrm{B}} \ . \tag{49}$$

That is,

$$2^{-H_{\min}(A|B)_\Omega} \geqslant 2^{-H_{\min}(A|B)_{\Delta^\dagger\otimes\mathrm{id}(\Omega)}} \ . \tag{50}$$

Combining (47) and (50) with (9) we conclude that if (9) holds for all states of the form $\{\Delta^\dagger(\omega_i^{\mathrm{A}})\}$ then it holds for any set of n states $\{\omega_i^{\mathrm{A}}\}$. This completes the proof of lemma 2.

The case that we are interested here is the one in which all the σ_is are diagonal with respect to some fixed basis. This is the case considered in Corollary 1 of Ref. [21]. In this case, we can take Δ to be the completely decohering map with respect to the fix basis. Since the set$\{\Delta(\omega_i)\}$ consists of diagonal matrices, we can assume w.l.o.g. that all the ω_is in Lemma 1 are diagonal. We can therefore write

$$\omega_i^{\mathrm{A}} \equiv \sum_{x=1}^{d_{\mathrm{A}}} r_{x|i}|x\rangle\langle x| \tag{51}$$

so that

$$\Omega^{\mathrm{AB}} = \sum_{x=1}^{d_{\mathrm{A}}} |x\rangle\langle x| \otimes \sum_{i=1}^n q_i r_{x|i}\rho_i^{\mathrm{B}} \tag{52}$$

is a classical quantum state. It is well known that for classical quantum states, the conditional min-entropy can

be expressed in terms of a guessing probability [5]. In the case that $d_{\rm A}=2$ the conditional-min entropy of $\Omega^{\rm AB}$ can be further simplified and we get

$$\begin{aligned}&2^{-H_{\min}(A|B)_\Omega}\\&=\min_\tau\left\{\mathrm{Tr}[\tau]:\ \tau\geqslant\sum_{i=1}^n q_i r_{x|i}\rho_i^{\rm B}\quad\forall x=1,2\right\}\\&=\frac{1}{2}+\frac{1}{2}\left\|\sum_{i=1}^n q_i(r_{1|i}-r_{2|i})\rho_i^{\rm B}\right\|_1\ .\end{aligned}\tag{53}$$

However, even if the σ_is all commute, it is not enough in general to restrict the comparison only to two-dimensional auxiliary states ω_i, if the goal is that of showing the existence of a CPTP map achieving $\rho_i\to\sigma_i$. If such a restriction is made, what one can show is the existence of a weaker map, namely, a 2-statistical morphism [14, 21], but counterexamples have been shown for which neither a CPTP nor a PTP map exists [23].

There are two very important exceptions to this. The first is the case in which there are only two commuting states $\{\rho_1,\rho_2\}$ and two commuting states $\{\sigma_1,\sigma_2\}$, namely, the case of two classical dichotomies. In this case, already Blackwell showed that two-dimensional commuting states ω_i suffice [11].

The second exception is that of two pairs of qubit density matrices $\{\rho_1,\rho_2\}$ and $\{\sigma_1,\sigma_2\}$: even if these do not commute, again, two-dimensional commuting states ω_i suffice [22].

Thermo-majorization

In the completely classical case, in addition to the ω_is, also the set $\{\rho_i^{\rm B}\}$ consists of diagonal matrices. Denoting

$$\rho_i^{\rm B}\equiv\sum_{y=1}^{d_{\rm B}}s_{y|i}|y\rangle\langle y|\tag{54}$$

we get that

$$\Omega^{\rm AB}=\sum_{x=1}^{d_{\rm A}}p_{xy}|x\rangle\langle x|\otimes|y\rangle\langle y|\ \ ;\ \ p_{xy}\equiv\sum_{i=1}^n q_i r_{x|i}s_{y|i}\ .\tag{55}$$

Now, in this case, the conditional min-entropy is given by

$$\begin{aligned}&2^{-H_{\min}(A|B)_\Omega}\\&=\min_\tau\left\{\mathrm{Tr}[\tau]:\ I^{\rm A}\otimes\tau\geqslant\sum_{x,y}p_{xy}|x\rangle\langle x|\otimes|y\rangle\langle y|\right\}\\&=\sum_y\max_x p_{xy}=\sum_y\max_x\ \boldsymbol{r}_x\cdot\boldsymbol{s}_y\end{aligned}\tag{56}$$

where for each x and y, $\boldsymbol{r}_x$ is the n-dimensional vector whose components are $\{q_i r_{x|i}\}_{i=1}^n$, and $\boldsymbol{s}_y$ is the n-dimensional probability vector whose components are $\{s_{y|i}\}_{i=1}^n$. Similarly, denoting by

$$\sigma_i^{\rm C}\equiv\sum_{z=1}^{d_{\rm B}}t_{z|i}|z\rangle\langle z|\ ,\tag{57}$$

we conclude that

$$2^{-H_{\min}(A|C)_\Omega}=\sum_z\max_x\ \boldsymbol{r}_x\cdot\boldsymbol{t}_z\ ,\tag{58}$$

where $\boldsymbol{t}_z$ is the probability vector whose components are $t_{z|i}$. Therefore, in the classical case, the condition in (10) is equivalent to

$$\sum_y f(\boldsymbol{s}_y)\geqslant\sum_z f(\boldsymbol{t}_z)\tag{59}$$

for any sub-linear functional f of the form $f(\boldsymbol{s})=\max_x\ \boldsymbol{r}_x\cdot\boldsymbol{s}$. Note that $\sum_y\boldsymbol{s}_y=\sum_z\boldsymbol{t}_z=(1,1,...,1)^{\rm T}$.

Finally, to obtain themo-majorization, we consider the case $n=2$. That is, we have two input states ρ_1 and ρ_2, and two output states σ_1 and σ_2. We can think of ρ_2 and σ_2 as Gibbs states. Note that all the vectors $\boldsymbol{r}_x$, $\boldsymbol{s}_y$, and $\boldsymbol{t}_z$ are two-dimensional since $n=2$. Therefore, in this case, it is sufficient to consider in (59) only sub-linear functionals with two elements; that is, of the form $f(\boldsymbol{s})=\max\{\boldsymbol{r}_1\cdot\boldsymbol{s},\ \boldsymbol{r}_2\cdot\boldsymbol{s}\}$ (see [20] for more details). We therefore conclude that the condition in (10) is equivalent to

$$\sum_y\max\{\boldsymbol{r}_1\cdot\boldsymbol{s}_y,\ \boldsymbol{r}_2\cdot\boldsymbol{s}_y\}\geqslant\sum_z\max\{\boldsymbol{r}_1\cdot\boldsymbol{t}_z,\ \boldsymbol{r}_2\cdot\boldsymbol{t}_z\}\tag{60}$$

for all $\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathbb{R}^2_+$. Using the relation $\max\{a,b\}=\frac{a+b}{2}+\frac{|a-b|}{2}$ for any two real numbers a and b, the equation above becomes equivalent to

$$\sum_y|(\boldsymbol{r}_1-\boldsymbol{r}_2)\cdot\boldsymbol{s}_y|\geqslant\sum_z|(\boldsymbol{r}_1-\boldsymbol{r}_2)\cdot\boldsymbol{t}_z|\tag{61}$$

where we used the fact that $\sum_y\boldsymbol{s}_z=\sum_z\boldsymbol{t}_z=(1,1,...,1)^{\rm T}$. Denoting by $\boldsymbol{r}_1-\boldsymbol{r}_2\equiv\begin{pmatrix}a\\b\end{pmatrix}\in\mathbb{R}^2$, the above equation becomes

$$\sum_y|as_{y|1}+bs_{y|2}|\geqslant\sum_z|at_{z|1}+bt_{z|2}|\tag{62}$$

Dividing by a and denoting $r\equiv-b/a$ we conclude that our condition in (10) reduces in the classical case to the thermo-majorization condition:

$$\sum_y|s_{y|1}-rs_{y|2}|\geqslant\sum_z|t_{z|1}-rt_{z|2}|\quad\forall r\geqslant0\ ,\tag{63}$$

Note that there is an equality above if $r<0$ so we assume w.l.o.g. that $r\geqslant0$.

Proof of Corollary 2

The proof of Corollary 2 can now be established. Suppose we are interested in the conversion of ρ^{A} into $\sigma^{\mathrm{A}'}$ under TPs. Moreover suppose that $[\rho^{\mathrm{A}}, \mathcal{H}^{\mathrm{A}}] = 0$, as explained in the main text one may restrict without loss of generality to η_1 and η_2 being incoherent in energy. Therefore the state Ω^{RA} is a classical state. Since TPs are covariant, and ρ^{A} is incoherent in energy it implies that the states accessible under this class must also be incoherent in energy and so $[\sigma^{\mathrm{A}'}, H^{\mathrm{A}'}] = 0$ is a necessary condition. Since both input and output states are incoherent the problem reduces to the interconversion of the distributions over energy under stochastic maps that preserve the Gibbs state. This coincides with the conditions for thermo-majorization as stated above.

On the other hand, suppose $[\sigma^{\mathrm{A}'}, H^{\mathrm{A}'}] = 0$. Now if there exists a a TP map $\mathcal{E}$ such that $\mathcal{E}(\rho^{\mathrm{A}}) = \sigma^{\mathrm{A}'}$ it is readily seen that $U'(t)\mathcal{E}(\rho^{\mathrm{A}})U'(t)^\dagger = \mathcal{E}(U'(t)\rho^{\mathrm{A}}U'(t)^\dagger) = \sigma^{\mathrm{A}'}$ for any t. Averaging over t gives that $\mathcal{E}(\langle\rho^{\mathrm{A}}\rangle) = \sigma^{\mathrm{A}'}$. Therefore $\rho^{\mathrm{A}} \to \sigma^{\mathrm{A}'}$ under TPs if and only if $\langle\rho^{\mathrm{A}}\rangle \to \sigma^{\mathrm{A}'}$ under TPs. Therefore such an interconversion is possible if and only if the distribution over energy of $\langle\rho^{\mathrm{A}}\rangle$ thermomajorizes the distribution over energy of $\sigma^{\mathrm{A}'}$.

SUPPLEMENTARY NOTE 5: G-COVARIANT MAPS

Theorem 1 can also be specialized to G-covariant maps. In what follows, we consider three unitary representations $g \to U_g$ of the same compact group G on systems A, B, and C. We use the following notations: $\mathcal{U}_g(x) = U_g x U_g^\dagger$, $\overline{\mathcal{U}}_g(x) = U_g^* x U_g^{\mathrm{T}}$, $\mathcal{U}_g^{\mathrm{T}}(x) = U_g^{\mathrm{T}} x U_g^*$, and $\mathcal{U}_g^\dagger(x) = U_g^\dagger x U_g$, with obvious meaning of symbols. We also introduce the bipartite twirling operation

$$\mathcal{G}(x) = \int_G dg\, \overline{\mathcal{U}}_g \otimes \mathcal{U}_g(x) \tag{64}$$

G-covariant version of Lemma 1

Lemma 3 Let $\{\rho_i^{\mathrm{B}}\}_{i=1}^n$ and $\{\sigma_i^{\mathrm{C}}\}_{i=1}^n$ be two sets of n density matrices in $\mathcal{B}(\mathcal{H}_{\mathrm{B}})$ and $\mathcal{B}(\mathcal{H}_{\mathrm{C}})$, respectively. Let $\{q_i\}_{i=1}^n$ be some arbitrary but fixed probability distribution with $q_i > 0$. For any set of n density matrices $\{\omega_i^{\mathrm{A}}\}_{i=1}^n$ in $\mathcal{B}(\mathcal{H}_{\mathrm{A}})$ (with $d_{\mathrm{A}} = d_{\mathrm{C}}$) define the following tripartite separable matrix:

$$\Omega^{ABC} \equiv \sum_{i=1}^n q_i\, \omega_i^{\mathrm{A}} \otimes \rho_i^{\mathrm{B}} \otimes \sigma_i^{\mathrm{C}}\ , \tag{65}$$

and its twirled version

$$\widetilde{\Omega}^{ABC} = \int_G dg \sum_{i=1}^n q_i\, \overline{\mathcal{U}}_g(\omega_i^{\mathrm{A}}) \otimes \mathcal{U}_g(\rho_i^{\mathrm{B}}) \otimes \mathcal{U}_g(\sigma_i^{\mathrm{C}})\ . \tag{66}$$

Then, the following are equivalent:

1. There exists a covariant CPTP map $\mathcal{E}:\ \mathcal{B}(\mathcal{H}_{\mathrm{B}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{C}})$ such that

$$\mathcal{E}(\rho_i^{\mathrm{B}}) = \sigma_i^{\mathrm{C}} \quad \forall\, i = 1, ..., n\ . \tag{67}$$

2. For any $\omega_1^{\mathrm{A}}, ..., \omega_n^{\mathrm{A}} \in \mathcal{B}(\mathcal{H}_{\mathrm{A}})$:

$$2^{-H_{\min}(A|B)_{\widetilde{\Omega}}} \geqslant d_{\mathrm{C}} \langle\phi_+^{\mathrm{AC}}|\widetilde{\Omega}^{\mathrm{AC}}|\phi_+^{\mathrm{AC}}\rangle\ . \tag{68}$$

3. For any $\omega_1^{\mathrm{A}}, ..., \omega_n^{\mathrm{A}} \in \mathcal{B}(\mathcal{H}_{\mathrm{A}})$:

$$H_{\min}(A|B)_{\widetilde{\Omega}} \leqslant H_{\min}(A|C)_{\widetilde{\Omega}}\ . \tag{69}$$

Proof. The proof of Lemma 1 goes through unchanged, with the only difference being that we want to find a CPTP map $\mathcal{E}$ that is covariant, i.e., that satisfies the following property:

$$\mathcal{U}_g^{\mathrm{C}}[\mathcal{E}(\rho^{\mathrm{B}})] = \mathcal{E}(\mathcal{U}_g^{\mathrm{B}}[\rho^{\mathrm{B}}]) \quad \forall g \in G\ . \tag{70}$$

Hence, we can start from Supplementary Eq. (22), which in the covariant case becomes

$$\begin{aligned} &\forall \text{ states } \{\omega_i^{\mathrm{A}}\}, \\ &\max_{\mathcal{E}:\ \text{covar. CPTP}} \sum_i q_i \operatorname{Tr}\big[\{\omega_i^{\mathrm{A}} \otimes \mathcal{E}(\rho_i^{\mathrm{B}})\}\ \ \phi_+^{\mathrm{AC}}\big] \\ &\geqslant \sum_i q_i \operatorname{Tr}\big[\{\omega_i^{\mathrm{A}} \otimes \sigma_i^{\mathrm{C}}\}\ \ \phi_+^{\mathrm{AC}}\big]\ . \end{aligned} \tag{71}$$

Using the covariance of the channel Supplementary Eq. (70), and the so-called "ricochet property" of the maximally entangled state, that is, $(\mathbb{1}^{\mathrm{A}} \otimes X_{\mathrm{C}})|\phi_+^{\mathrm{AC}}\rangle = (X_{\mathrm{A}}^{\mathrm{T}} \otimes \mathbb{1}^{\mathrm{C}})|\phi_+^{\mathrm{AC}}\rangle)$, we can rewrite the left-hand side of the above inequality as follows:

$$\begin{aligned} &\sum_i q_i \operatorname{Tr}\big[\{\omega_i^{\mathrm{A}} \otimes \mathcal{E}(\rho_i^{\mathrm{B}})\}\ \ \phi_+^{\mathrm{AC}}\big] \\ &= \sum_i q_i \int_G dg \operatorname{Tr}\big[(\omega_i^{\mathrm{A}} \otimes \mathcal{E}(\rho_i^{\mathrm{B}}))\ (\mathcal{U}_g^{\mathrm{T}} \otimes \mathcal{U}_g^\dagger)(\phi_+^{\mathrm{AC}})\big] \quad (72) \\ &= \sum_i q_i \int_G dg \operatorname{Tr}\big[(\overline{\mathcal{U}}_g \otimes \mathcal{U}_g)(\omega_i^{\mathrm{A}} \otimes \mathcal{E}(\rho_i^{\mathrm{B}}))\ \phi_+^{\mathrm{AC}}\big] \quad (73) \\ &= \langle\phi_+^{\mathrm{AC}}|\ (\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}})(\widetilde{\Omega}^{\mathrm{AB}})\ |\phi_+^{\mathrm{AC}}\rangle\ , \quad (74) \end{aligned}$$

where, we recall, the channel $\mathcal{E}$ is assumed to be covariant.

Le us now consider the quantity

$$\max_{\mathcal{E}:\mathrm{CPTP}} \langle\phi_+^{\mathrm{AC}}|(\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}})(\widetilde{\Omega}^{\mathrm{AB}})|\phi_+^{\mathrm{AC}}\rangle\ , \tag{75}$$

where the maximization now is allowed to run over all possible CPTP maps, not only covariant ones. However,

since both $\widetilde{\Omega}^{\mathrm{AB}}$ and ϕ_+^{AC} are invariant for the action $\overline{\mathcal{U}}_g \otimes \mathcal{U}_g$, we immediately have that

$$\begin{aligned}
&\max_{\mathcal{E}:\mathrm{CPTP}} \langle\phi_+^{\mathrm{AC}}|(\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}})(\widetilde{\Omega}^{\mathrm{AB}})|\phi_+^{\mathrm{AC}}\rangle \\
&= \max_{\mathcal{E}:\mathrm{CPTP}} \mathrm{Tr}\Big[(\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}})(\widetilde{\Omega}^{\mathrm{AB}})\ \phi_+^{\mathrm{AC}}\Big] && (76)\\
&= \int_G dg\, \mathrm{Tr}\Big[(\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}}) \circ (\overline{\mathcal{U}}_g \otimes \mathcal{U}_g)(\widetilde{\Omega}^{\mathrm{AB}})\ (\overline{\mathcal{U}}_g \otimes \mathcal{U}_g)(\phi_+^{\mathrm{AC}})\Big] && (77)\\
&= \int_G dg\, \mathrm{Tr}\Big[(\mathcal{U}_g^{\mathrm{T}} \otimes \mathcal{U}_g^{\dagger}) \circ (\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}}) \circ (\overline{\mathcal{U}}_g \otimes \mathcal{U}_g)(\widetilde{\Omega}^{\mathrm{AB}})\ \phi_+^{\mathrm{AC}}\Big] && (78)\\
&= \int_G dg\, \mathrm{Tr}\Big[(\mathsf{id}_{\mathrm{A}} \otimes \mathcal{U}_g^{\dagger} \circ \mathcal{E}_{\mathrm{B}} \circ \mathcal{U}_g)(\widetilde{\Omega}^{\mathrm{AB}})\ \phi_+^{\mathrm{AC}}\Big] && (79)\\
&= \max_{\mathcal{E}:\text{ covar. CPTP}} \langle\phi_+^{\mathrm{AC}}|(\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}})(\widetilde{\Omega}^{\mathrm{AB}})|\phi_+^{\mathrm{AC}}\rangle\ , && (80)
\end{aligned}$$

and hence, using the conditional min-entropy,

$$\begin{aligned}
&\max_{\mathcal{E}:\text{ covar. CPTP}} \langle\phi_+^{\mathrm{AC}}|(\mathsf{id}_{\mathrm{A}} \otimes \mathcal{E}_{\mathrm{B}})(\widetilde{\Omega}^{\mathrm{AB}})|\phi_+^{\mathrm{AC}}\rangle && (81)\\
&= \frac{1}{d_{\mathrm{A}}} 2^{-H_{\min}(A|B)_{\widetilde{\Omega}}}\ . && (82)
\end{aligned}$$

Hence, statement (1) is equivalent to

$$\begin{aligned}
&2^{-H_{\min}(A|B)_{\widetilde{\Omega}}} && (83)\\
&\geqslant d_{\mathrm{A}} \sum_i q_i\, \mathrm{Tr}\big[\{\omega_i^{\mathrm{A}} \otimes \sigma_i^{\mathrm{C}}\}\ \phi_+^{\mathrm{AC}}\big] && (84)\\
&= d_{\mathrm{A}} \sum_i q_i\, \mathrm{Tr}\big[\mathcal{G}\{\omega_i^{\mathrm{A}} \otimes \sigma_i^{\mathrm{C}}\}\ \phi_+^{\mathrm{AC}}\big] && (85)\\
&= d_{\mathrm{A}} \langle\phi_+^{\mathrm{AC}}|\widetilde{\Omega}^{\mathrm{AC}}|\phi_+^{\mathrm{AC}}\rangle\ . && (86)
\end{aligned}$$

(Remember that $d_{\mathrm{A}} = d_{\mathrm{C}}$.) Following the same arguments used in the proof of Lemma 1, we also obtain the equivalence between statement (1) and statement (3). This completes the proof of lemma 3.

Remark 2. The existence of a covariant CPTP map achieving the transformation $\rho_i \to \sigma_i$ is of course a stronger requirement than the existence of a general CPTP map doing the same. Indeed, once we rewrite Supplementary Eq. (9) of Lemma 1 as

$$\begin{aligned}
2^{-H_{\min}(A|B)_{\Omega}} &\geqslant d_{\mathrm{C}} \langle\phi_+^{\mathrm{AC}}|\Omega^{\mathrm{AC}}|\phi_+^{\mathrm{AC}}\rangle \\
&= d_{\mathrm{C}} \langle\phi_+^{\mathrm{AC}}|\widetilde{\Omega}^{\mathrm{AC}}|\phi_+^{\mathrm{AC}}\rangle\ ,
\end{aligned}$$

and since, as a consequence of the data-processing inequality,

$$2^{-H_{\min}(A|B)_{\Omega}} \geqslant 2^{-H_{\min}(A|B)_{\widetilde{\Omega}}}\ ,$$

it is clear that it is in principle *harder* to satisfy condition (2) of the covariant Lemma 3, than its non-covariant counterpart (9).

G-covariant version of Theorem 1

As before we used Lemma 1 to prove Theorem 1, here we use Lemma 3 to prove Theorem 2.

Theorem 2 Let $\rho^{\mathrm{AB}} \in \mathcal{B}(\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}})$ and $\sigma^{\mathrm{AC}} \in \mathcal{B}(\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{C}})$ be two compatible bipartite quantum states. Denote the dimension of any system X as $d_X \in \mathbb{N}$. The following are equivalent:

1. There exists a G-covariant CPTP map $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{B}})$ such that
$$\sigma^{\mathrm{AC}} = \mathsf{id} \otimes \mathcal{E}\left(\rho^{\mathrm{AB}}\right) \qquad (87)$$

2. For any quantum process (CPTP linear map) $\Phi : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$, with $d_{\mathrm{A}'} = d_{\mathrm{C}}$,
$$\begin{aligned}
&H_{\min}(A'|B)_{\mathcal{G}[\Phi\otimes\mathsf{id}(\rho^{\mathrm{AB}})]} && (88)\\
&\leqslant H_{\min}(A'|C)_{\mathcal{G}[\Phi\otimes\mathsf{id}(\sigma^{\mathrm{AC}})]}\ .
\end{aligned}$$

3. Supplementary Eq. (88) holds for any measurement-prepare quantum channel $\Phi : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ of the form:
$$\Phi\left(\gamma^{\mathrm{A}}\right) = \sum_{j=1}^{d_{\mathrm{A}}^2} \mathrm{Tr}\left[M_j^{\mathrm{A}} \gamma^{\mathrm{A}}\right] \omega_j^{\mathrm{A}'}\ , \qquad (89)$$
where $\{M_j^{\mathrm{A}}\}$ is an arbitrary, but fixed, informationally complete POVM on system A, while the states $\{\omega_j^{\mathrm{A}'}\}$ can freely vary.

4. For any $\Phi : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ of the form (89) the following holds:
$$\begin{aligned}
&2^{-H_{\min}(A|B)_{\mathcal{G}[\Phi\otimes\mathsf{id}(\rho^{\mathrm{AB}})]}} && (90)\\
&\geqslant d_{\mathrm{A}}\ \langle\phi^+|\mathcal{G}\left[\Phi \otimes \mathsf{id}\left(\sigma^{\mathrm{AC}}\right)\right]|\phi^+\rangle\ , && (91)
\end{aligned}$$
where $|\phi_+^{A'C}\rangle$ is the maximally entangled state between systems A' and C.

We are now ready to prove the main theorem.

Proof of Theorem 2: Let $\{Q_k^{\mathrm{A}}\}_{k=1}^{d_{\mathrm{A}}^2}$ be the dual basis of $\{M_j^{\mathrm{A}}\}$ in $\mathcal{B}(\mathcal{H}_{\mathrm{A}})$, that is, $\mathrm{Tr}\left[M_j^{\mathrm{A}} Q_k^{\mathrm{A}}\right] = \delta_{jk}$. Then, since $\{Q_k^{\mathrm{A}}\}$ is itself a basis, we can write

$$\rho^{\mathrm{AB}} = \sum_{k=1}^{d_{\mathrm{A}}^2} Q_k^{\mathrm{A}} \otimes \widetilde{\rho}_k^{\mathrm{B}} \quad \text{and} \quad \sigma^{\mathrm{AC}} = \sum_{k=1}^{d_{\mathrm{A}}^2} Q_k^{\mathrm{A}} \otimes \widetilde{\sigma}_k^{\mathrm{C}} \qquad (92)$$

where

$$\begin{aligned}
\widetilde{\rho}_j^{\mathrm{B}} &\equiv \mathrm{Tr}_{\mathrm{A}}\left[\left(M_j^{\mathrm{A}} \otimes \mathbb{1}^{\mathrm{B}}\right)\rho^{\mathrm{AB}}\right] \\
\widetilde{\sigma}_j^{\mathrm{C}} &\equiv \mathrm{Tr}_{\mathrm{A}}\left[\left(M_j^{\mathrm{A}} \otimes \mathbb{1}^{\mathrm{C}}\right)\sigma^{\mathrm{AC}}\right] \qquad (93)
\end{aligned}$$

are sub-normalized quantum states (i.e. positive semi-definite matrices). Moreover, since $\rho^{\mathrm{A}} = \sigma^{\mathrm{A}}$ we have

$\mathrm{Tr}\left[\widetilde{\rho}_j^{\mathrm{B}}\right] = \mathrm{Tr}\left[\widetilde{\sigma}_j^{\mathrm{C}}\right] \equiv p_j$. We therefore conclude that that there exists a covariant CPTP map $\mathcal{E}$ that satisfies (87) if and only if there exists a covariant CPTP map $\mathcal{E}$ that satisfies

$$\sigma_j^{\mathrm{C}} = \mathcal{E}\left(\rho_j^{\mathrm{B}}\right) \tag{94}$$

where $\rho_j^{\mathrm{B}} \equiv \widetilde{\rho}_j^{\mathrm{B}}/p_j$ and $\sigma_j^{\mathrm{C}} \equiv \widetilde{\sigma}_j^{\mathrm{C}}/p_j$. To apply Lemma 3, we introduce a system A' with $d_{\mathrm{A}'} = d_{\mathrm{C}}$, we fix an arbitrary probability distribution $q_i > 0$, and define

$$\Omega^{A'BC} \equiv \sum_{j=1}^{d_{\mathrm{A}}^2} q_j\, \omega_j^{\mathrm{A}'} \otimes \rho_j^{\mathrm{B}} \otimes \sigma_j^{\mathrm{C}} = \sum_{j=1}^{d_{\mathrm{A}}^2} \frac{q_j}{p_j^2} \times$$
$$\omega_j^{\mathrm{A}'} \otimes \mathrm{Tr}_{\mathrm{A}}\left[\left(M_j^{\mathrm{A}} \otimes I^{\mathrm{B}}\right)\rho^{\mathrm{AB}}\right] \otimes \mathrm{Tr}_{\mathrm{A}}\left[\left(M_j^{\mathrm{A}} \otimes I^{\mathrm{C}}\right)\sigma^{\mathrm{AC}}\right] , \tag{95}$$

where the states $\omega_i^{\mathrm{A}'}$ can vary. The corresponding twirled state is

$$\widetilde{\Omega}^{A'BC} \equiv \int_G dg (\overline{\mathcal{U}}_g^{\mathrm{A}} \otimes \mathcal{U}_g^{\mathrm{B}} \otimes \mathcal{U}_g^{\mathrm{C}})(\Omega^{ABC}) .$$

Then, taking $q_j = p_j$, we conclude that

$$\widetilde{\Omega}^{A'B} = \mathcal{G}[\Phi \otimes \mathsf{id}\left(\rho^{\mathrm{AB}}\right)] ,$$
$$\widetilde{\Omega}^{A'C} = \mathcal{G}[\Phi \otimes \mathsf{id}\left(\sigma^{\mathrm{AC}}\right)] . \tag{96}$$

Notice that, in case some $p_i = 0$, we can redefine the measurement operators $M_j^{\mathrm{A}} \to M_j^{\mathrm{A}} + \delta \mathbb{1}^{\mathrm{A}}$, in such a way that they still span the set $\mathcal{B}(\mathcal{H}_{\mathrm{A}})$ but have non-zero probability everywhere. With Supplementary Eq. (96) at hand, the proof of Theorem 2 follows now from Lemma 3.

Covariant Stinespring dilations

Given systems A and A', with Hilbert spaces $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{A}'}$, we assume that each carry a unitary representation of a compact group G given by $U : G \to \mathcal{B}(H_{\mathrm{A}})$ and $U' : G \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ respectively. A quantum process $\mathcal{E} : \mathcal{B}(A_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ from A into A' is said to be *covariant* or *symmetric* if $\mathcal{E} \circ \mathcal{U}_g = \mathcal{U}'_g \circ \mathcal{E}$ for all $g \in G$. The following lemma was proved in [28], and we provide the proof here for convenience.

Lemma 4 [28] Given a covariant quantum process $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ there exists a Kraus decomposition

$$\mathcal{E}(\rho^{\mathrm{A}}) = \sum_{\lambda,m,k} K_{\lambda,m,k}\rho^{\mathrm{A}}K_{\lambda,m,k}^{\dagger}, \tag{97}$$

with Kraus operators $K_{\lambda,m,k} : \mathcal{H}_{\mathrm{A}} \to \mathcal{H}_{\mathrm{A}'}$ that transform as

$$U^{\mathrm{A}'}(g)K_{\lambda,m,k}U^{\mathrm{A}}(g)^{\dagger} = \sum_j v^{\lambda}(g)_{jk}K_{\lambda,m,j} \tag{98}$$

where $(v^{\lambda}(g)_{jk})$ are the matrix elements of the λ-irrep of G and m is a multiplicity label.

Proof Let $\{K_i\}$ be a set of linearly independent Kraus operators for $\mathcal{E}$. Since $\mathcal{E}$ is covariant we have that $\mathcal{U}_g^{\mathrm{A}'} \circ \mathcal{E} \circ (\mathcal{U}_g^{\mathrm{A}})^{\dagger} = \mathcal{E}$ for any $g \in G$, and so it follows that $\{U^{\mathrm{A}'}(g)K_iU^{\mathrm{A}}(g)^{\dagger}\}_i$ forms another set of Kraus operators for $\mathcal{E}$ for any fixed $g \in G$. Since the Kraus representation is unique up to unitary mixing this implies that $U^{\mathrm{A}'}(g)K_iU^{\mathrm{A}}(g)^{\dagger} = \sum_j V(g)_{ij}K_j$. Moreover, since the Kraus operators are linearly independent it follows that this unitary $V(g)$ is unique for any fixed g and so the matrices $V(g)$ form a non-projective unitary representation of G. Using the unitary freedom to choose the basis $\{K_i\}$ we can choose a basis for which $V(g)$ is block diagonal in terms of a sum of unitary irreps of G. We denote this basis $\{K_{\lambda,m,k}\}$, with $\{K_{\lambda,m,k}\}$ transforming as a λ irrep under G for each multiplicity m as in Equation (98), and k labels the basis vector of the irrep. This completes the proof.

Such Kraus operators are said to transform irreducibly under the group action, and are irreducible tensor operators.

Theorem 3 [Covariant Stinespring [29]] For any covariant quantum process $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ there exists a Stinespring dilation

$$\mathcal{E}(\rho^{\mathrm{A}}) = \mathrm{Tr}_{\mathrm{C}}V(\rho^{\mathrm{A}} \otimes |\sigma\rangle\langle\sigma|^{\mathrm{B}})V^{\dagger} \tag{99}$$

where $|\sigma\rangle^{\mathrm{B}} \in \mathcal{H}_{\mathrm{B}}$ is a symmetric state under the unitary representation U^{B} of G on system B, system C carries a unitary representation U^{C} of G, and $V : \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{A}'} \otimes \mathcal{H}_{\mathrm{C}}$ is an isometry such that

$$V(U^{\mathrm{A}}(g) \otimes U^{\mathrm{B}}(g)) = (U^{\mathrm{A}'}(g) \otimes U^{\mathrm{C}}(g))V, \tag{100}$$

for all $g \in G$.

Remark 3. This theorem was proved in [29] for the case $\mathcal{H}^{\mathrm{A}} = \mathcal{H}^{\mathrm{A}'}$. The proof of the general case is essentially identical, and we provide the proof below for convenience.

Proof. From the previous lemma, a covariant quantum process $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ always has a Kraus decomposition $\{K_{\lambda,m,k}\}$ such that

$$U^{\mathrm{A}'}(g)K_{\lambda,m,k}(U^{\mathrm{A}})^{\dagger}(g) = \sum_j v^{\lambda}(g)_{jk}K_{\lambda,m,j}, \tag{101}$$

where λ labels an irrep of G, m is a multiplicity label and k is the basis vector label of the irrep.

Let B be a system with Hilbert space $\mathcal{H}_{\mathrm{B}} = \mathrm{span}\{|\sigma\rangle\}$, with the state $|\sigma\rangle$ being symmetric under the action of G. For any pair (λ, m) appearing in the Kraus decomposition of $\mathcal{E}$, let $\mathcal{W}_{(\lambda^*,m)}$ be a Hilbert space isomorphic to the λ^*-irrep of G and for which we choose a basis $\{|\lambda^*, m, k\rangle\}_k$. We define $\mathcal{H}_{\mathrm{C}} := \mathcal{H}_{\mathrm{B}} \bigoplus_{(\lambda,m)} \mathcal{W}_{(\lambda^*,m)}$, where the direct sum ranges over all (λ, m) occurring in the Kraus decomposition of $\mathcal{E}$. The space $\mathcal{H}_{\mathrm{C}}$ carries the unitary group action

$$U^{\mathrm{C}}(g) = |\sigma\rangle\langle\sigma| \bigoplus_{(\lambda,m)} \sum_{j,k} (v^{\lambda}(g)_{jk})^* |\lambda^*, m, j\rangle\langle\lambda^*, m, k|, \tag{102}$$

where $(v^\lambda(g)_{jk})$ are the unitary matrix components of the irrep λ of G.

We define the operator $V : \mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm B} \to \mathcal{H}_{\rm A'} \otimes \mathcal{H}_{\rm C}$ as

$$V := \sum_{\lambda,m,k} K_{\lambda,m,k} \otimes |\lambda^*, m, k\rangle\langle\sigma|. \tag{103}$$

Using that the $\{K_{\lambda,m,k}\}_k$ transform irreducibly under the action of G, together with the fact that $(v^\lambda(g)_{jk})$ is a unitary matrix, it is readily verified that Equation (100) holds for all $g \in G$, and so V is covariant under the action of G. Moreover since $\sum_{\lambda,m,k} K^\dagger_{\lambda,m,k} K_{\lambda,m,k} = \mathbb{1}^{\rm A}$, and $\{|\lambda^*, m, k\rangle\}_{\lambda,m,k}$ is an orthonormal set of states we have that $V^\dagger V = \mathbb{1}^{\rm A} \otimes |\sigma\rangle\langle\sigma|^{\rm B}$ and so V is an isometry from $\mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm B}$ into $\mathcal{H}_{\rm A'} \otimes \mathcal{H}_{\rm C}$. Finally, we have that

$$\mathcal{E}(\rho^{\rm A}) = {\rm Tr}_{\rm C} V(\rho^{\rm A} \otimes |\sigma\rangle\langle\sigma|^{\rm B})V^\dagger, \tag{104}$$

and so have constructed the required dilation for the covariant quantum process $\mathcal{E}$. This completes the proof of theorem.

The following lemma clarifies that any mixed symmetric state can always be purified to a pure quantum state that is also symmetric under the group action.

Lemma 5 Consider a quantum system A, carrying a unitary representation $U^{\rm A} : G \to \mathcal{B}(\mathcal{H}^{\rm A})$, and a mixed quantum state $\sigma^{\rm A}$ for which $\mathcal{U}^{\rm A}_g(\sigma^{\rm A}) = \sigma^{\rm A}$ for all $g \in G$. Then, there exists a purification $|\psi^{\rm AB}\rangle$ of $\sigma^{\rm A}$ onto a composite system AB, and a unitary representation $V^{\rm B} : G \to \mathcal{B}(\mathcal{H}^{\rm B})$ such that $U^{\rm A}_g \otimes V^{\rm B}_g |\psi^{\rm AB}\rangle = |\psi^{\rm AB}\rangle$ for all $g \in G$.

Since any density operator on $\mathcal{H}^{\rm A}$ can be thought as a quantum process from a 1 dimensional input Hilbert space to $\mathcal{B}(\mathcal{H}^{\rm A})$, this lemma follows immediately from theorem 3 on Covariant Steinespring dilation. Here, we present a more direct proof.

Let $\{|j\rangle^{\rm A}\}_{j=1}^{\rm R}$ be an orthonormal basis of the support subspace of $\sigma^{\rm A}$, where r is the rank of $\sigma^{\rm A}$. Let

$$|\psi\rangle^{\rm AB} = (\sigma^{1/2} \otimes I^{\rm B})|\phi_+\rangle^{\rm AB} \quad ; \quad |\phi_+\rangle^{\rm AB} \equiv \sum_{j=1}^{\rm R} |j\rangle^{\rm A}|j\rangle^{\rm B} \tag{105}$$

be a purification of $\sigma^{\rm A}$. Then,

$$\begin{aligned}
&[U^{\rm A}_g \otimes \overline{U}^{\rm B}_g]|\psi\rangle^{\rm AB} \\
=&[U^{\rm A}_g \otimes \overline{U}^{\rm B}_g][\sqrt{\sigma^{\rm A}} \otimes {\sf id}_{\rm B}]|\phi_+\rangle^{\rm AB} \\
=&[U^{\rm A}_g \otimes {\sf id}^{\rm B}][\sqrt{\sigma^{\rm A}} \otimes {\sf id}_{\rm B}][{\sf id}^{\rm A} \otimes \overline{U}^{\rm B}_g]|\phi_+\rangle^{\rm AB} \\
=&[U^{\rm A}_g \otimes {\sf id}^{\rm B}][\sqrt{\sigma^{\rm A}} \otimes {\sf id}_{\rm B}][\overline{U}^{\rm A^{\rm T}}_g \otimes {\sf id}^{\rm B}]|\phi_+\rangle^{\rm AB} \\
=&[U^{\rm A}_g \sqrt{\sigma^{\rm A}} U^{\rm A^\dagger}_g \otimes {\sf id}^{\rm B}]|\phi_+\rangle^{\rm AB} \\
=&[\sqrt{U^{\rm A}_g \sigma^{\rm A} U^{\rm A^\dagger}_g} \otimes {\sf id}^{\rm B}]|\phi_+\rangle^{\rm AB} \\
=&[\sqrt{\sigma^{\rm A}} \otimes {\sf id}^{\rm B}]|\phi_+\rangle^{\rm AB} \\
=&|\psi\rangle^{\rm AB}.
\end{aligned} \tag{106}$$

Therefore, this completes the proof by taking $V_g \equiv \overline{U}^{\rm B}_g$.

SUPPLEMENTARY NOTE 6: GENERALIZED THERMAL PROCESSES

We prove the general result in the presence of thermodynamic observables $\{H^{\rm A}, X^{\rm A}_1, \ldots, X^{\rm A}_n\}$, which may have non-trivial commutation relations between them. The case on the Hamiltonian being the only thermodynamic observable follows as a special case of this result.

Assumptions (A1) and (A2), together with the requirement that the resource theory be non-trivial in these observables implies that the free state must take the form of the generalized Gibbs ensemble $\gamma^{\rm A}$,

$$\gamma^{\rm A} = \frac{1}{\mathcal{Z}} e^{-\beta(H^{\rm A} - \sum_k \mu_k X^{\rm A}_k)} \tag{107}$$

for constants $\beta, \mu_1, \ldots, \mu_n$. This is picked out in several different ways, for example perhaps the simplest to interpret is within the theory of equilibration. An alternative route is through a complete passivity argument in which one has additional access to an ordered macroscopic 'bath' for each observable that can give or take arbitrary amounts of that observable. Given an unbounded number of copies of the free state one wishes to know if one can trivialise the theory in terms of providing an arbitrary displacement for any of these observables. However in the presence of thermodynamic constraints, these are coupled in such a way that one must only consider an "effective" energy bath with Hamiltonian $\tilde{H} = H - \sum_k \mu_k X_k$. Complete passivity with respect to this observable implies the above generalized Gibbs state through standard arguments.

We now give a precise statement of assumption (A3) in the context of thermodynamic observables $\{H^{\rm S}, X^{\rm S}_1, \ldots, X^{\rm S}_n\}$ for any quantum system S. We first note there are two components to any TP process $\mathcal{E}$ at the microscopic level: the particular interactions between A and an auxiliary system B, and the state $\sigma^{\rm B}$ of the auxiliary system. Under assumption (A1) there are no couplings present between eigenspaces of different eigenvalues of the additively conserved observables, however this does not mean that coherence cannot be injected into A. Assumption (A3) places a minimal constraint on the use of coherence sources outside of A. The key idea is that while $\mathcal{E}$ may be realised through some specific interaction V between A and its environment B, and this environment may even contain quantum coherences in its state $\sigma^{\rm B}$, we can guarantee that $\mathcal{E}$ is not exploiting any of these coherences if it is the case that if we were to *remove* the coherences present in $\sigma^{\rm B}$ then the transformation $\mathcal{E}$ would still be possible through interactions with B. This motivates the following condition.

(A3) (Incoherence) Given a thermodynamically free process $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\rm A}) \to \mathcal{B}(\mathcal{H}_{\rm A'})$ there exists an interaction isometry W that obeys (133) and a quantum state $\eta^{\rm B}$ such that

$$\mathcal{E}(\rho^{\rm A}) = {\rm Tr}_{\rm C} W(\rho^{\rm A} \otimes \eta^{\rm B})W^\dagger, \tag{108}$$

and with $\eta^{\rm B} = \mathcal{G}(\eta^{\rm B})$ where

$$\mathcal{G}(\eta^{\rm B}) := \int dg U_{\rm B}(g)\eta^{\rm B}U_{\rm B}(g)^\dagger, \tag{109}$$

and where $U_{\rm B}(g)$ is the group representation on B generated by the observables $\{H^{\rm B}, X_1^{\rm B}, \ldots, X_n^{\rm B}\}$. Below in Lemma 7 we show that this assumption captures the demand that no coherence is being exploited from the environment, but before this we establish that the set TP has a compact formulation in terms of covariance.

Lemma 6 Given a set of thermodynamic observables $\{H^{\rm S}, X_1^{\rm S}, \ldots, X_n^{\rm S}\}$ for any quantum system S, the set TP of quantum processes from A into A' defined by (A1-A3) coincides with the set GPC of Gibbs-preserving processes on A that are covariant under the group G generated by the thermodynamic observables on A and A'.

Proof We first show that $TP \subset GPC$. Assumption (A2) ensures that the image of the Gibbs state $\gamma^{\rm A}$ under TP is the fixed point $\gamma^{\rm A'}$, so it suffices to establish covariance. For any system S we define $X_0^{\rm S} := H^{\rm S}$ so as to make notation compact. Given a process $\mathcal{E} \in TP$, assumption (A1) implies that

$$\mathcal{E}(\rho^{\rm A}) = {\rm Tr}_{\rm C} V(\rho^{\rm A} \otimes \sigma^{\rm B})V^\dagger, \tag{110}$$

for some V that obeys the conservation laws given by Equation (133). In particular, this implies that

$$V \exp\left[i\sum_{k=0}^{n}\theta_k X_k^{\rm AB}\right] = \exp\left[i\sum_{k=0}^{n}\theta_k X_k^{A'C}\right] V \tag{111}$$

$$X_k^{\rm AB} := X_k^{\rm A} \otimes \mathbb{1}^{\rm B} + \mathbb{1}^{\rm A} \otimes X_k^{\rm B} \tag{112}$$

$$X_k^{A'C} := X_k^{\rm A'} \otimes \mathbb{1}^{\rm C} + \mathbb{1}^{\rm A'} \otimes X_k^{\rm C} \tag{113}$$

for all $k = 0, \ldots d$ and for all $\theta_k \in \mathbb{R}$. Therefore the observables $\{X_k\}$ generate a representation of a group G, with elements g indexed by $(\theta_0, \ldots, \theta_d)$, and $U^{A'C}(g)V = VU^{\rm AB}(g)$ for all $g \in G$. Therefore the process sending any $\chi^{\rm AB} \to V\chi^{\rm AB}V^\dagger$ is G-covariant. As discussed, assumption (A3) says that the above $\sigma^{\rm B}$ can be taken to be symmetric under this group action: $\mathcal{U}_g^{\rm B}(\sigma^{\rm B}) = \sigma^{\rm B}$. Since discarding systems is G-covariant, and also composing of G-covariant processes results in a G-covariant process, we see that $\rho^{\rm A} \to \rho^{\rm A} \otimes \sigma^{\rm B} \to V(\rho^{\rm A} \otimes \sigma^{\rm B})V^\dagger \to {\rm Tr}_{\rm C} V(\rho^{\rm A} \otimes \sigma^{\rm B})V^\dagger = \mathcal{E}(\rho^{\rm A})$ is a G-covariant process for any $\mathcal{E}$ of the form (110). Therefore $TP \subset GPC$.

Conversely, let $\mathcal{E} \in GPC$. Since $\mathcal{E}(\gamma^{\rm A}) = \gamma^{\rm A'}$, assumption (A2) holds automatically. Since $\mathcal{E}$ is G-covariant with respect to the group generated by $\{X_k^{\rm A}\}$ as shown there exists a Stinespring dilation of the process $\mathcal{E}$ of the form

$$\mathcal{E}(\rho^{\rm A}) = {\rm Tr}_{\rm C} V(\rho^{\rm A} \otimes |\psi\rangle\langle\psi|^{\rm B})V^\dagger, \tag{114}$$

where V is a G-invariant isometry and $|\psi\rangle^{\rm B}$ is invariant under the group action on B. The invariance of V implies that assumption (A1) holds, while the symmetry of $|\psi\rangle$ implies that there are no coherences between eigenspaces of the distinguished observables and so (A3) holds. Therefore $\mathcal{E} \in TP$, and so the two sets of processes coincide as claimed. This completes the proof of lemma.

To summarize, the state interconversion under TPs is equivalent to the following requirement:

$$\mathcal{E}(\rho^{\rm A}) = \sigma^{\rm A'} \tag{115}$$

$$\mathcal{E}(\gamma^{\rm A}) = \gamma^{\rm A'}. \tag{116}$$

where $\mathcal{E}$ is required to be a G-covariant process.

We can now show that no coherences are exploited from the environment for any $\mathcal{E}$ in TP.

Lemma 7 Suppose $\mathcal{E}$ is in TP and realised as

$$\mathcal{E}(\rho^{\rm A}) = {\rm Tr}_{\rm C} V(\rho^{\rm A} \otimes \sigma^{\rm A})V^\dagger, \tag{117}$$

by some isometry V obeying (133 and interacting with a system B in a state $\sigma^{\rm B}$. Then (117) also holds with $\sigma^{\rm B}$ replaced by $\mathcal{G}(\sigma^{\rm B})$.

Proof Since $\mathcal{E}$ is a covariant map, we have that $\mathcal{U}_g[\mathcal{E}(\mathcal{U}_{g^{-1}}(\rho^{\rm A})] = \mathcal{E}(\rho^{\rm A})$ for any $g \in G$. Expressing $\mathcal{E}$ in terms of $(V, \sigma^{\rm B})$ and exploiting the fact that $U_{\rm A'}(g) \otimes \mathbb{1}_{\rm C} V = \mathbb{1}_{\rm A'} \otimes U_{\rm C}(g)^\dagger V U_{\rm A}(g) \otimes U_{\rm B}(g)$ we see that

$$\begin{aligned}\mathcal{U}_g({\rm Tr}_{\rm C} V(\mathcal{U}_{g^{-1}}(\rho^{\rm A}) \otimes \sigma^{\rm B})V^\dagger) &= {\rm Tr}_{\rm C} V(\rho^{\rm A} \otimes \mathcal{U}_g(\sigma^{\rm B}))V^\dagger \\ &= \mathcal{E}(\rho^{\rm A}),\end{aligned} \tag{118}$$

for any $g \in G$. Integrating over all g and using linearity we deduce that

$$\mathcal{E}(\rho^{\rm A}) = {\rm Tr}_{\rm C} V(\rho^{\rm A} \otimes \mathcal{G}(\sigma^{\rm B}))V^\dagger \tag{119}$$

as required.

Remark 4. Note that Lemma 5 shows that replacing any auxiliary $\sigma^{\rm B}$ with its dephased version $\mathcal{G}(\sigma^{\rm B})$ as discussed in the main text is consistent with the existence of a Stinespring form in which the auxiliary system is taken to be in a pure symmetric quantum state. Also note that that we implicitly assume that the group G generated by the thermodynamic observables on the input system coincides with the group generated by those on the output system, which is a basic physical requirement.

SDP solution for thermomajorization with coherence

Here we illustrate how to use SDP to solve the decision problem of determining if $\rho \in {\rm B}(\mathcal{H})$ can be converted to $\sigma \in {\rm B}(\mathcal{H})$ by a generalized thermal processes. For simplicity of the exposition, we consider the case of no charges with the same Hamiltonian for the input and output spaces. We want to know if there exists a thermal

process, that is, a CPTP map $\mathcal{E}:\mathrm{B}(\mathcal{H})\to\mathrm{B}(\mathcal{H})$ that is both Gibbs preserving and symmetric under time translation, such that $\sigma=\mathcal{E}(\rho)$. Denoting the Hamiltonian by H and the Gibbs state by $\gamma=\frac{1}{Z}e^{-\beta H}$, from (45) it follows that there exists such a Gibbs preserving symmetric map $\mathcal{E}$ if and only if $f(\rho,\sigma)\geqslant 1$, where $f(\rho,\sigma)$ is given by the SDP problem

$$\begin{aligned} f(\rho,\sigma)\equiv\max\ & y \\ \text{subject to: }\ & \mathbf{1.}\ \tau^{\mathrm{AB}}\geqslant 0\ ;\ \tau^{\mathrm{B}}\leqslant I \\ & \mathbf{2.}\ y\sigma^{\mathrm{T}}\leqslant \mathrm{Tr}_{\mathrm{B}}\left[\tau^{\mathrm{AB}}(I\otimes\rho)\right] \\ & \mathbf{3.}\ y\gamma^{\mathrm{T}}\leqslant \mathrm{Tr}_{\mathrm{B}}\left[\tau^{\mathrm{AB}}(I\otimes\gamma)\right] \\ & \mathbf{4.}\ \mathcal{G}(\tau^{AB})=\tau^{AB}\ . \end{aligned} \tag{120}$$

Here we added the condition $\mathcal{G}(\tau^{AB})=\tau^{AB}$ to (45) to ensure that $\mathcal{E}$ is symmetric under time translation with respect to the Hamiltonian H. The G-twirling is given as in (64) with g replaced by the time parameter t, and the group element $\mathcal{U}_t(\cdot)=e^{iHt}(\cdot)e^{-iHt}$.

The dual to this problem is very similar to (37) and the only difference is that one has to take the G-twirling on the term $\sum_{i=1}^{n}X_i\otimes\rho_i$. For the more specific case we consider here, we have:

$$\begin{aligned} & f(\rho,\sigma)=\min\mathrm{Tr}[Z] \\ & \text{subject to}\quad I\otimes Z\geqslant\mathcal{G}\left(X\otimes\rho\right)+Y\otimes\gamma, \\ & \qquad\qquad \mathrm{Tr}\left(\sigma^{\mathrm{T}}X\right)+\mathrm{Tr}\left(\gamma^{\mathrm{T}}Y\right)=1\ ;\ X,Y\geqslant 0 \end{aligned}$$

where we replaced $\mathcal{G}\left(Y\otimes\gamma\right)$ with $Y\otimes\gamma$ since γ is symmetric.

As before, the above minimization problem is an SDP. To see it, we follow now the same steps that led to Supplementary Eq. (42). We define the vector space V_1, consisting of all Hermitian matrices $\zeta\in V_1$ of the form:

$$\zeta=(\eta,Z,X,Y) \tag{121}$$

where $\eta\in\mathrm{B}(\mathcal{H}\otimes\mathcal{H})$, and $X,Y,Z\in\mathrm{B}(\mathcal{H})$ are all Hermitian. In addition, we define the vector space V_2 of all Hermitian matrices in $\mathrm{B}(\mathcal{H}\otimes\mathcal{H})$, and a linear transformation $\Gamma:V_1\to V_2$ given by:

$$\Gamma\left(\zeta\right)=I\otimes Z-\mathcal{G}\left(X\otimes\rho\right)-Y\otimes\gamma-\eta\ . \tag{122}$$

Clearly, the map above is linear. Set $\tau\equiv(\mathbf{0},\mathbf{0},\sigma^{\mathrm{T}},\gamma^{\mathrm{T}})$ so that $\mathrm{Tr}[\sigma\zeta]=\mathrm{Tr}\left(\sigma^{\mathrm{T}}X\right)+\mathrm{Tr}\left(\gamma^{\mathrm{T}}Y\right)$. We also denote $C\equiv(\mathbf{0},I,\mathbf{0},\mathbf{0})$. With these notations:

$$f(\rho,\sigma)=\min\left\{\mathrm{Tr}[C\zeta]\ \middle|\ \zeta\geqslant 0\ ;\ \Gamma(\zeta)=0\ ,\ \mathrm{Tr}[\sigma\zeta]=1\right\} \tag{123}$$

To bring the above optimization problem to a canonical SDP form, we denote as before $H_j\equiv\Gamma^*(E_j)$, where $j=1,...,d^4$, where $d\equiv\dim(\mathcal{H})$, and E_j is a basis of $\mathrm{B}(\mathcal{H}\otimes\mathcal{H})$. We also denote $H_0\equiv\tau$. With this notations we get

$$\begin{aligned} & f(\rho,\sigma)=\min\mathrm{Tr}[C\zeta] \\ & \text{subject to}\quad \zeta\geqslant 0\ ,\ \mathrm{Tr}\left(\zeta H_j\right)=\delta_{0,j}\quad\forall j\in\{0,1,...,d^4\}\ . \end{aligned} \tag{124}$$

The above optimization form is written in a canonical form and can be solved with SDP packages such as CVX.

SUPPLEMENTARY NOTE 7: PROOF OF THEOREMS 2 AND 3

Theorem 2 is a special case of theorem 3, where the only additive conserved observable is the Hamiltonian (See remark 5). Therefore, it suffices to prove theorem 3.

The proof is basically a corollary of lemma 3. For any system $S=R,A,A'$, let $\{U^{\mathrm{S}}(g)\}$ be the symmetry group generated by the additively conserved observables $\{H^{\mathrm{S}},X_k^{\mathrm{S}};\ k=1,\ldots,n\}$, where for system R we define

$$H^{\mathrm{R}}=-(H^{\mathrm{A}'})^{\mathrm{T}},\ \text{ and }\ X^{\mathrm{R}}=-(X^{\mathrm{A}'})^{\mathrm{T}}\ , \tag{125}$$

and the superscript T denotes the transpose. Note that this definition implies that for any group element g,

$$U^{\mathrm{R}}(g)=\overline{U}^{\mathrm{A}'}(g)\ . \tag{126}$$

We are interested to determine if there exists a CPTP map $\mathcal{E}:\ \mathcal{B}(\mathcal{H}^{\mathrm{A}})\to\mathcal{B}(\mathcal{H}^{\mathrm{A}'})$ which (i) satisfies

$$\mathcal{E}(\rho^{\mathrm{A}})=\sigma^{\mathrm{A}'},\ \text{and}\ \ \mathcal{E}(\gamma^{\mathrm{A}})=\gamma^{\mathrm{A}'}\ , \tag{127}$$

and (ii) is covariant, i.e. for all group elements g,

$$\mathcal{E}\circ\mathcal{U}_g^{\mathrm{A}}=\mathcal{U}_g^{\mathrm{A}'}\circ\mathcal{E}\ ,$$

where $\mathcal{U}_g^{\mathrm{S}}[\cdot]=U^{\mathrm{S}}(g)[\cdot]U^{\dagger}(g)$ for any system $S=A,A',R$.

Therefore, to apply lemma 3, we assume systems A, B and C in the statement of lemma 3 correspond, respectively, to systems R, A and A' in the statement of theorem 3. Furthermore, to apply the lemma we assume the set of input states $\{\rho_i\}$ has two elements, namely $\{\rho^{\mathrm{A}},\gamma^{\mathrm{A}}\}$, and the corresponding output states are $\{\sigma_i\}=\{\sigma^{\mathrm{A}'},\gamma^{\mathrm{A}'}\}$. Finally, in the statement of lemma 3 states $\{\omega_i\}$ denote possible states of the reference system. Here, we denote them by $\{\eta_1^{\mathrm{R}},\eta_2^{\mathrm{R}}\}$. Under these assignments state Ω^{ABC} in the statement of lemma becomes

$$\begin{aligned} & \widetilde{\Omega}^{RAA'} \\ & =\int_G dg\,\mathcal{U}_g^{RAA'}[q\eta_1^{\mathrm{R}}\otimes\rho^{\mathrm{A}}\otimes\sigma^{\mathrm{A}'}+(1-q)\eta_2^{\mathrm{R}}\otimes\gamma^{\mathrm{A}}\otimes\gamma^{\mathrm{A}'}]\ , \end{aligned}$$

Then, the equivalence of statements (1) and (3) in lemma 3 implies that the necessary and sufficient condition for existence of a G-covariant CPTP map which satisfies Supplementary Eq.(127) is

$$H_{\min}(R|A)_{\Omega}\leqslant H_{\min}(R|A')_{\Omega}\ . \tag{128}$$

This completes proof of theorem 3.

Remark 5. Note that in the case of theorem 2, where the only additive conserved observable is the Hamiltonian, the G-twirling operation $\int dg\, \mathcal{U}_g$ reduces to the time average

$$\int dg\, \mathcal{U}_g^{\mathrm{S}}[X^{\mathrm{S}}] = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt X^{\mathrm{S}}(t) := \langle X^{\mathrm{S}} \rangle \ , \quad (129)$$

where $X^{\mathrm{S}}(t)$ is the time evolved version of X^{S}, i.e. $X^{\mathrm{S}}(t) = e^{-iH^{\mathrm{S}}t} X^{\mathrm{S}} e^{iH^{\mathrm{S}}t}$ for any system S.

SUPPLEMENTARY NOTE 8: RELATION BETWEEN GENERALIZED THERMAL PROCESSES AND THERMAL OPERATIONS

Thermal operations (TO) are defined as quantum operations that take the form

$$\mathcal{E}(\rho^{\mathrm{A}}) = \mathrm{Tr}_{\mathrm{B}} V(\rho^{\mathrm{A}} \otimes \gamma^{\mathrm{B}}) V^\dagger \quad (130)$$

where V commutes with the total Hamiltonian of AB, and $\gamma^{\mathrm{B}} = \exp[-\beta H^{\mathrm{B}}]/Z$ is the Gibbs state on B. In contrast, the situation of thermal processes with just a Hamiltonian have a similar form, but now with γ^{B} replaced with a potentially non-Gibbsian incoherent state σ^{B} and with the condition that $\mathcal{E}(\gamma^{\mathrm{A}}) = \gamma^{\mathrm{A}}$ being imposed independently.

One might think that the two sets are equivalent and that every TP admits a form given by (130), but this turns out to not be the case. For example, if we consider the case of trivial Hamiltonians $H^{\mathrm{A}} = 0$ then TO is the set of noisy operations, whereas TP is the set of *unital* operations. It is well known that these two sets are not the same [18]. That said, the state interconversion conditions for noisy operations and unital operations are in fact the same – namely majorization on the eigenvalues of the states. One might then conjecture that TO and TP have the same "power" in the sense that $\rho \longrightarrow \sigma$ is possible under TO if and only if it is possible under TP. It can be seen that this is in fact the case if either state is incoherent in energy, since the necessary and sufficient conditions are simply thermo-majorization [17]. However for fully coherent states we do not have a clear understanding of how the two classes compare.

It is also clear that TO is a proper subset of TP, since every TO operation obeys (A1-A3), however there is at present no obvious coherent scenario that could be used to say that one of the two classes is preferred over the other.

In the case of multiple conserved quantities beyond the Hamiltonian, a similar comparison can be made. However in this case the formulation of TO faces subtleties since it relies on a notion of a *free state* which can be prepared in an unbounded number of copies, whereas in contrast TP only makes a statement on the existence of an *equilibrium state*. The free states in TO are singled out because they are completely passive and so are the only form of state admissible that does not trivialise the energetic degrees of freedom. However when introduces multiple conserved quantities $\{H^{\mathrm{B}}, X_1^{\mathrm{B}}, X_2^{\mathrm{B}}, \dots\}$ the question of complete passivity becomes more problematic (see [15, 24–27] for discussion). In general it is impossible for a state to be completely passive in all of the individual observables, and to circumvent this one must define an "effective Hamiltonian" and impose complete passivity solely on this. While this does lead to the generalized Gibbs ensemble, the effective Hamiltonian is not determined by the resource theory framework, but must be postulated independently. In contrast, TP does not require any complete passivity analysis for the bath, since it does not specify a set of free states in the sense described, but instead only demands the existence of an equilibrium state γ^{A} on the primary system. In these terms, the generalized Gibbs ensemble has been shown to arise naturally in the context of equilibration theory (see for example [19] and references therein).

SUPPLEMENTARY NOTE 9: FINITE PRECISION AND APPROXIMATE ENERGY INCOHERENCE

We can replace assumption (A3) with a slightly weaker version that takes into account that we only ever experimentally probe to some finite level of precision. The reason this is useful is that it avoids two technicalities: firstly that the time-translation group action is in general non-compact group $\mathbb{R}$, and secondly even if time-translation is the compact $U(1)$ Lie group no finite dimensional representations will exist in which one can encode all group elements into perfectly distinguishable quantum states. We can circumvent both of these technicalities with the following finite precision assumptions.

Firstly, we can always approximate any quantum system with one having finite dimension $d < \infty$, for d sufficiently large. Given this finite dimension, any spectrum $\{E_1, E_2, \dots, E_d\}$ for the system's Hamiltonian H^{A} can be approximated to an arbitrary precision by a set of rational numbers, $\{\tilde{E}_1 = \frac{a_1}{b_1}, \dots, \tilde{E}_d = \frac{a_d}{b_d}\}$ with $a_k, b_k \in \mathbb{Z}$ for each k and $\tilde{E}_k$ arbitrarily close to E_k. Thus, for simplicity we assume the Hamiltonian has a spectrum of rational numbers and so the resultant unitary dynamics $U^{\mathrm{A}}(t) = \exp[-itH^{\mathrm{A}}]$ is periodic for some finite period $\tau < \infty$.

The mapping $t \mapsto U^{\mathrm{A}}(t)$ is therefore a unitary representation of the continuous $U(1)$ group on the system A. We may further assume that we only ever resolve time intervals $[t_1, t_2]$ with $t_2 - t_1 \geqslant \epsilon$ for some small yet finite level of precision $\epsilon > 0$. More formally this means that we can replace the $U(1)$ group with the discrete $\mathbb{Z}_N$

action, where $N\epsilon = \tau$ and

$$n \mapsto U^{\mathrm{A}}(n\epsilon) = e^{-in\epsilon H^{\mathrm{A}}}, \tag{131}$$

with $n = 0, 1, \ldots N-1$. Therefore the dynamics of any single quantum system can always be approximated by such a discrete, finite action for some $N \in \mathbb{N}$ and sufficiently large.

In the case that we have multiple systems $A_1, A_2, \ldots, A_M$ with periods $\tau_1, \tau_2, \ldots, \tau_M$ respectively, we may choose $\tau = \prod_{k=1}^{M} \tau_k$ as the time-scale for the composite system. Therefore for multiple systems, there will always exist an $N \in \mathbb{N}$, sufficiently large so that the mapping $n \mapsto \exp[-in\epsilon H^{A_k}]$ is a unitary representation of $\mathbb{Z}_N$ on each $\mathcal{H}_{A_k}$, and which approximates the unitary dynamics of each A_k under its Hamiltonian to the specified level of precision. Given this, condition (A3) for incoherence of thermal processes can be replaced with the following.

(A3′) Approximate incoherence. Consider the case of the Hamiltonian being the only thermodynamic observable, and assume the finite precision approximations described above. If the thermodynamically free process $\mathcal{E} : \mathcal{B}(\mathcal{H}_{\mathrm{A}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{A}'})$ is realized microscopically as

$$\mathcal{E}(\rho^{\mathrm{A}}) = \mathrm{Tr}_{\mathrm{C}} V(\rho^{\mathrm{A}} \otimes \sigma^{\mathrm{B}})V^\dagger, \tag{132}$$

with V obeying Equation

$$\begin{aligned} V(H^{\mathrm{A}} \otimes \mathbb{1}^{\mathrm{B}} + \mathbb{1}^{\mathrm{A}} \otimes H^{\mathrm{B}}) &= (H^{\mathrm{A}'} \otimes \mathbb{1}^{\mathrm{C}} + \mathbb{1}^{\mathrm{A}'} \otimes H^{\mathrm{C}})V \\ V(X_k^{\mathrm{A}} \otimes \mathbb{1}^{\mathrm{B}} + \mathbb{1}^{\mathrm{A}} \otimes X_k^{\mathrm{B}}) &= (X_k^{\mathrm{A}'} \otimes \mathbb{1}^{\mathrm{C}} + \mathbb{1}^{\mathrm{A}'} \otimes X_k^{\mathrm{C}})V \end{aligned}, \tag{133}$$

then we also have

$$\mathcal{E}(\rho^{\mathrm{A}}) = \mathrm{Tr}_{\mathrm{C}} W(\rho^{\mathrm{A}} \otimes \mathcal{G}_\epsilon(\sigma^{\mathrm{B}}))W^\dagger. \tag{134}$$

with $\mathcal{G}_\epsilon(\sigma^{\mathrm{B}})$ being the group average over $\mathbb{Z}_N$ of the state σ^{B} given by

$$\mathcal{G}_\epsilon(\sigma^{\mathrm{B}}) := \frac{1}{N}\sum_{n=0}^{N-1} U_\epsilon^{\mathrm{B}}(n)\sigma^{\mathrm{B}}U_\epsilon^{\mathrm{B}}(n)^\dagger, \tag{135}$$

with $U_\epsilon^{\mathrm{B}}(n) := \exp[-in\epsilon H^{\mathrm{B}}]$ is the finite precision time evolution on B, and we interact this state with A through some potentially different isometry W that also obeys (133).

This implies that the constraint of time-translation covariance is replaced with $\mathbb{Z}_N$-covariance to this level of precision. Given this, the analysis for state interconversion may be repeated under $(A3')$ and results in the replacement of $\frac{1}{\tau}\int_0^\tau dt(\cdot)$ with $\frac{1}{N}\sum_{k=0}^{N-1}(\cdot)$ and $U^{\mathrm{R}}(t) \otimes U^{\mathrm{A}}(t)$ by the discrete approximation $U_\epsilon^{\mathrm{R}}(n) \otimes U_\epsilon^{\mathrm{A}}(n)$.

SUPPLEMENTARY NOTE 10: CLOCK TIMES AND GUESSING PROBABILITIES

As in the previous section, we may restrict our attention to a fully discrete setting with quantum systems of finite dimension and finite level of precision ϵ for time resolution. Covariance of the dynamics is now described with respect to the discrete group $\mathbb{Z}_N$ for some sufficiently large $N \in \mathbb{N}$.

For $q \to 1$ we obtain the $\mathbb{Z}_N$ covariance constraint alone, and the corresponding state Ω^{RA} takes the form

$$\Omega^{\mathrm{RA}} = \frac{1}{N}\sum_{k=0}^{N-1} U_\epsilon^{\mathrm{R}}(n)\eta_1^{\mathrm{R}}(U_\epsilon^{\mathrm{R}}(n))^\dagger \otimes U_\epsilon^{\mathrm{A}}(n)\rho^{\mathrm{A}}(U_\epsilon^{\mathrm{A}}(n))^\dagger. \tag{136}$$

For a sufficiently large reference frame R there exists a Hamiltonian H^{R} such that R allows a perfect encoding of the group elements of $G = \mathbb{Z}_N$. In particular for $\dim(\mathcal{H}_{\mathrm{R}}) = \mathrm{N}$ with orthonormal basis $\{|E_k\rangle^{\mathrm{R}}\}$, we can choose

$$U_\epsilon^{\mathrm{R}}(1) = \sum_{k=0}^{N-1} \omega^k |E_k\rangle\langle E_k|^{\mathrm{R}}, \tag{137}$$

where $\omega := e^{\frac{2\pi i}{N}}$ is an N^{th} root of unity. We then have that $(U_\epsilon^{\mathrm{R}}(1))^n = U_\epsilon^{\mathrm{R}}(n)$ for any $n = 1, 2, \ldots$ and $U_\epsilon^{\mathrm{R}}(N) = U_\epsilon^{\mathrm{R}}(0) = \mathbb{1}^{\mathrm{R}}$ as required.

Defining $|k\rangle^{\mathrm{R}} := F|E_k\rangle^{\mathrm{R}}$, with F being the discrete Fourier transform operator

$$F = \frac{1}{\sqrt{N}}\sum_{i,j=0}^{N-1} \omega^{ij}|E_i\rangle\langle E_j|^{\mathrm{R}}, \tag{138}$$

it is readily seen that

$$U_\epsilon^{\mathrm{R}}(n)|0\rangle^{\mathrm{R}} = |n\rangle^{\mathrm{R}}, \tag{139}$$

and $\langle n|m\rangle^{\mathrm{R}} = 0$ for $n \neq m$ and equal to 1 for $n = m$. Therefore the reference system R provides a perfect classical encoding of the group elements of $\mathbb{Z}_N$ in the pure states $\{|k\rangle^{\mathrm{R}}\}$.

Setting $\eta_1^{\mathrm{R}} = |0\rangle\langle 0|^{\mathrm{R}}$ in equation 136 gives the classical-quantum state

$$\Omega^{\mathrm{RA}} = \frac{1}{N}\sum_{k=0}^{N-1} |k\rangle\langle k|^{\mathrm{R}} \otimes \rho^{\mathrm{A}}(n). \tag{140}$$

where we define $\rho^{\mathrm{A}}(n) := U_\epsilon^{\mathrm{A}}(n)\rho^{\mathrm{A}}(U_\epsilon^{\mathrm{A}}(n))^\dagger$ for the state of A at time $t = n\epsilon$. These states fully encode the set of *clock times* $t = 0, \epsilon, \ldots, n\epsilon, \ldots, (N-1)\epsilon$ for the joint system.

Since Ω^{RA} is a classical-quantum state, we have that [5]

$$H_{\min}(R|A)_\Omega = -\log p_{\mathrm{guess}}, \tag{141}$$

where p_{guess} is the optimal Helstrom guessing probability for the ensemble of states $\{(\frac{1}{N}, \rho^{\mathrm{A}}(n))\}_{n=0}^{N-1}$ on A. This

implies that $2^{-H_{\min}(R|A)_\Omega}$ is the optimal guessing probability of the clock time $t = n\epsilon$ for the joint system, given the single copy of ρ^{A}. Monotonicity of $H_{\min}(R|A)_\Omega$ under the thermal processes implies monotonicity of the clock time guessing probability for the system.

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