Editors' Suggestion

Type-Independent Characterization of Spacelike Separated Resources

Denis Rosset
Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada

David Schmid
Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada and Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L3G1, Canada

Francesco Buscemi
Graduate School of Informatics, Nagoya University, Chikusa-ku, Nagoya 464-8601, Japan

(Received 9 February 2020; accepted 3 September 2020; published 17 November 2020)

Quantum theory describes multipartite objects of various types: quantum states, nonlocal boxes, steering assemblages, teleportages, distributed measurements, channels, and so on. Such objects describe, for example, the resources shared in quantum networks. Not all such objects are useful, however. In the context of spacelike separated parties, devices which can be simulated using local operations and shared randomness are useless, and it is of paramount importance to be able to practically distinguish useful from useless quantum resources. Accordingly, a body of literature has arisen to provide tools for witnessing and quantifying the nonclassicality of objects of each specific type. In the present Letter, we provide a framework which subsumes and generalizes all of these resources, as well as the tools for witnessing and quantifying their nonclassicality.

DOI: 10.1103/PhysRevLett.125.210402

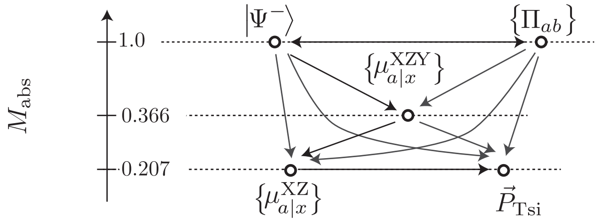
Spacelike separated resources of various types are studied in quantum information. In the bipartite setting, at least ten types of objects have been considered as resources in computation or communication tasks: density matrices [1], shared randomness [2], nonlocal boxes [3], steering assemblages [4], teleportages [5,6], distributed POVMs (or "semiquantum" channels) [7], device-independent steering channels [8], channel assemblages [9], Bob-with-input steering assemblages [10], and bipartite quantum channels [11]. Heterogenous objects appear in quantum networks [12]; for example, depending on the local operations available, various schemes of quantum key distribution have been proposed [13]. With the partial exception of Ref. [6], no unified framework has been given to describe and characterize [14] all these different multipartite spacelike separated resource types.

Here, we work in the context of spacelike separation, where local operations and shared randomness (LOSR operations) are free, but no signaling forbids classical communication, and we provide a framework which unifies the study of nonclassicality of arbitrary types of resources. We introduce a common notation for all resource types, distinguished by the nature (trivial, classical, or quantum) of the input and output systems, and define a unified notion of nonclassicality which subsumes the natural notions of nonclassicality for every type of resource.

That is, we here (together with Ref. [15]) demonstrate that entangled states, nonlocal boxes, unsteerable assemblages, nonclassical teleportages, and indeed resources of *all* the types just listed can be viewed as instances of a single notion of resourcefulness within a unified resource theory [16]: that of nonclassicality of common cause processes. This phrase was first introduced in Ref. [17] in the context of a resource theory of nonlocal boxes, and it is also apt in the broader type-independent context considered here. This unified view based on LOSR operations also resolves some long-standing confusions [18].

While [15] defines and studies this resource theory more abstractly, this paper proposes tools to quantify the nonclassicality of resources in a type-independent manner, and defines nonclassicality measures that transcend types. For example, we will here quantitatively compare resources of various different types that have previously been studied only separately in the literature. We also discuss how these measures can be computed or approximated using off-the-shelf software. We base our discussion on the resources presented in Fig. 1, whose features are elaborated on below. In other words, we show that one can rigorously compare the quantitative degree of nonclassicality inherent in distributed resources of arbitrary types.

A unified notation for all types of resources.—First, we introduce a notation which is capable of describing a wide variety of resources that arise naturally in Bell scenarios.



Notation	Resource R	Type $\mathsf{T}[R]$	$M_{\mathrm{abs}}(R)$
$\|\Psi^-\rangle$	Singlet	$\mathsf{II} \to \mathsf{QQ}$	1
$\{\Pi_{ab}\}$	BM^2 distributed POVM	$\mathsf{QQ} \to \mathsf{CC}$	1
$\{\mu^{\mathrm{XZY}}_{a\|x}\}$	XZY-singlet assemblage	$\mathsf{CI} \to \mathsf{CQ}$	≈ 0.366
$\{\mu^{\mathrm{XZ}}_{a\|x}\}$	XZ-singlet assemblage	$\mathsf{CI} \to \mathsf{CQ}$	≈ 0.207
$\vec{P}_{\mathrm{Tsi}}$	Tsirelson box	$\mathsf{CC} \to \mathsf{CC}$	≈ 0.207

FIG. 1. We exhaustively characterize the nonclassicality of five resources of four different types by their convertibility relations under LOSR operations (figure). We also characterize their nonclassicality using a type-independent absolute robustness monotone M_{abs} (table) that we introduce below. Any single monotone is only partially informative; e.g., M_{abs} assigns the same value to $\{\mu^{XZ}_{a|x}\}$ and $\vec{P}_{\mathrm{Tsi}}$, even though the former is strictly more nonclassical than the latter.

In our framework, a "resource" is a quantum device distributed over multiple spacelike separated parties which receives inputs and produces outputs. Viewed broadly as a channel, this subsumes a wide variety of special cases in the literature, as we will show. For simplicity, we restrict ourselves to the two-party case and to finite-dimensional Hilbert spaces: we denote by $\mathcal{X}$, $\mathcal{Y}$ the input systems and by $\mathcal{A}$, $\mathcal{B}$ the output systems of the first (Alice) and second (Bob) parties, respectively, as shown in Fig. 2. We also use $\mathcal{H}$ as an auxiliary system or placeholder.

To each system, say $\mathcal{H}$, we associate two values: the *dimension* $\mathsf{d}[\mathcal{H}]$ and the *type* $\mathsf{T}[\mathcal{H}]$ so that $\mathcal{H} = (\mathsf{d}[\mathcal{H}], \mathsf{T}[\mathcal{H}])$. The dimension $\mathsf{d}[\mathcal{H}]$ describes the associated Hilbert space $\mathbb{C}^{\mathsf{d}[\mathcal{H}]}$, equipped with the computational basis $\{|i_{\mathcal{H}}\rangle\}_{i=1}^{\mathsf{d}[\mathcal{H}]}$. We write the set of Hermitian operators on $\mathbb{C}^{\mathsf{d}[\mathcal{H}]}$ as $\mathsf{H}(\mathcal{H})$, the set of positive semidefinite operators as $\mathsf{H}_+(\mathcal{H}) = \{\rho \in \mathsf{H}(\mathcal{H}) : \rho \geq 0\}$, and the set of density matrices as $\mathsf{D}(\mathcal{H}) = \{\rho \in \mathsf{H}_+(\mathcal{H}) : \mathrm{tr}(\rho) = 1\}$. The type $\mathsf{T}[\mathcal{H}] \in \{\mathsf{1}, \mathsf{C}, \mathsf{Q}\}$ describes whether the system is trivial, classical, or quantum. A system is trivial ($\mathsf{T}[\mathcal{H}] = \mathsf{1}$) if and only if $\mathsf{d}[\mathcal{H}] = 1$. A classical system ($\mathsf{T}[\mathcal{H}] = \mathsf{C}$) restricts the operators in $\mathsf{H}(\mathcal{H})$ to a commutative subalgebra; without loss of generality, we take these operators to be diagonal in the computational basis $|i_{\mathcal{H}}\rangle$. Finally, a quantum system ($\mathsf{T}[\mathcal{H}] = \mathsf{Q}$) has no such restriction. We sometimes omit the tensor product symbol: e.g., $\mathsf{D}(\mathcal{XY})$ denotes $\mathsf{D}(\mathcal{X} \otimes \mathcal{Y})$, and we omit subscripts labeling systems when convenient and unambiguous.

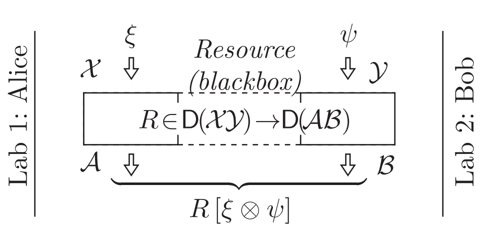


FIG. 2. A two-party resource $R \in \mathsf{D}(\mathcal{XY}) \to \mathsf{D}(\mathcal{AB})$ and its action on product input $\xi \otimes \psi$; by allowing $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{A}$, $\mathcal{B}$ to be trivial, classical, or quantum, we unify all the resource types shown in Table I.

A resource R is a map

$$R_{\mathcal{AB}|\mathcal{XY}} : \mathsf{D}(\mathcal{XY}) \to \mathsf{D}(\mathcal{AB}), \tag{1}$$

which is trace preserving

$$\mathrm{tr}(R[\xi \otimes \psi]) = 1 \quad \forall\ \xi \in \mathsf{D}(\mathcal{X}), \psi \in \mathsf{D}(\mathcal{Y}) \tag{2}$$

and completely positive [19,20]

$$(R \otimes \mathbb{I})[\nu_{\mathcal{XYH}'}] \geq 0 \quad \forall\ \nu_{\mathcal{XYH}'} \in \mathsf{D}(\mathcal{XYH}') \tag{3}$$

for any auxiliary space $\mathcal{H}'$. The subscript $\mathcal{AB}|\mathcal{XY}$ corresponds to the types and dimensions of all input and output systems. For a resource R as in (1), we define the resource type written

$$\mathsf{T}[R] = \mathsf{T}[\mathcal{X}]\mathsf{T}[\mathcal{Y}] \to \mathsf{T}[\mathcal{A}]\mathsf{T}[\mathcal{B}], \tag{4}$$

and the resource dimension

$$\mathsf{d}[R] = (\mathsf{d}[\mathcal{A}], \mathsf{d}[\mathcal{B}], \mathsf{d}[\mathcal{X}], \mathsf{d}[\mathcal{Y}]). \tag{5}$$

We consider only resources that are nonsignaling from every party to every other. No signaling from Alice to Bob implies that if one ignores Alice's output $\mathcal{A}$, then Alice's input $\mathcal{X}$ has no influence on Bob's output $\mathcal{B}$. That is, the reduced channel $R_{\mathcal{B}|\mathcal{XY}} \equiv \mathrm{tr}_{\mathcal{A}} R_{\mathcal{AB}|\mathcal{XY}}$ of such a resource R satisfies $R_{\mathcal{B}|\mathcal{XY}}[\xi \otimes \psi] = R_{\mathcal{B}|\mathcal{XY}}[\xi' \otimes \psi]$ for all inputs $\xi, \xi' \in \mathsf{D}(\mathcal{X})$ and $\psi \in \mathsf{D}(\mathcal{Y})$. Hence, one can pick an arbitrary ξ to construct the unique $R_{\mathcal{B}|\mathcal{Y}}$ such that

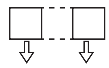
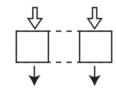
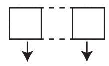
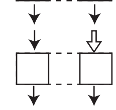
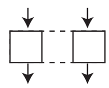
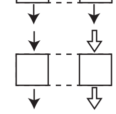
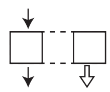
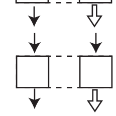
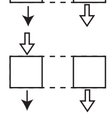
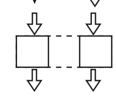
$$R_{\mathcal{B}|\mathcal{XY}}[\xi \otimes \psi] = \mathrm{tr}[\xi] R_{\mathcal{B}|\mathcal{Y}}[\psi] \tag{6}$$

for all ξ and ψ. No signaling from Bob to Alice can be encoded similarly as

$$R_{\mathcal{A}|\mathcal{XY}}[\xi \otimes \psi] = \mathrm{tr}[\psi] R_{\mathcal{A}|\mathcal{X}}[\xi]. \tag{7}$$

Input and output types.—Note that when an output $\mathcal{A}$ of a party is trivial, with $\mathsf{T}[\mathcal{A}] = \mathsf{I}$ (or equivalently $\mathsf{d}[\mathcal{A}] = \mathsf{I}$), no signaling guarantees that the party is irrelevant and can always be ignored. In contrast, when an input $\mathcal{X}$ of a party is trivial, with $\mathsf{T}[\mathcal{X}] = \mathsf{I}$, that party can nonetheless be nontrivial, as happens, e.g., with quantum states. When an output $\mathcal{A}$ is classical, with $\mathsf{T}[\mathcal{A}] = \mathsf{C}$, the channel satisfies (for all ξ and all ψ):

TABLE I. Types of resources studied in the literature. The arrow⇨ indicates a quantum input or output space, while the arrow ➔ indicates a classical input or output space.

Name	Type $\mathsf{T}[R]$	Drawing	Name	Type $\mathsf{T}[R]$	Drawing
Quantum state [1,18]	$\mathsf{II} \to \mathsf{QQ}$		Distributed measurement (see semiquantum games [7])	$\mathsf{QQ} \to \mathsf{CC}$	
Shared randomness [2]	$\mathsf{II} \to \mathsf{CC}$		MDI steering assemblage [8]	$\mathsf{CQ} \to \mathsf{CC}$	
Nonlocal box [3]	$\mathsf{CC} \to \mathsf{CC}$		Channel assemblage [9]	$\mathsf{CQ} \to \mathsf{CQ}$	
Steering assemblage [4]	$\mathsf{CI} \to \mathsf{CQ}$		Bob-with-input steering assemblage [10]	$\mathsf{CC} \to \mathsf{CQ}$	
Teleportage [6] (in teleportation exps. [5])	$\mathsf{QI} \to \mathsf{CQ}$		General bipartite channel [11]	$\mathsf{QQ} \to \mathsf{QQ}$	

$$\forall\, i_{\mathcal{A}} \neq j_{\mathcal{A}}, \qquad \langle i_{\mathcal{A}}|R[\xi \otimes \psi]|j_{\mathcal{A}}\rangle = 0. \tag{8}$$

When an input $\mathcal{X}$ is classical, with $\mathsf{T}[\mathcal{X}] = \mathsf{C}$, the channel acts on the diagonal subspace, and thus satisfies (for all ψ)

$$\forall\, i_{\mathcal{X}} \neq j_{\mathcal{X}}, \qquad R[|i_{\mathcal{X}}\rangle\langle j_{\mathcal{X}}| \otimes \psi] = 0. \tag{9}$$

Definition 1: The set $\mathbf{R}_{\mathcal{AB}|\mathcal{XY}}$ of all nonsignaling resources of given types $\mathsf{T}[R]$ and dimensions $\mathsf{d}[R]$ is defined as those channels which satisfy Eqs. (2)–(7) and (when applicable) (8), (9).

For any given type and dimensionalities, this set is representable via a semidefinite program (SDP), using the Choi matrix [21,22] representation of resources; see Sec. III of [23] for an explicit proof.

We show ten distinct types of resources which our framework subsumes and which have been previously studied in the literature in Table I.

Examples of resources.—We now define the examples from Fig. 1.

The *singlet* $|\Psi^-\rangle$ is a quantum state (type $\mathsf{II} \to \mathsf{QQ}$), written as a channel acting on a trivial input, denoted 1:

$$R_{|\Psi^-\rangle}[1] = |\Psi^-\rangle\langle\Psi^-| = (|01\rangle - |10\rangle)(\langle 01| - \langle 10|)/2. \tag{10}$$

The BM^2 distributed POVM $\{\Pi_{ab}\}$ (type $\mathsf{QQ} \to \mathsf{CC}$) is inspired by semiquantum tests of entanglement [15,50,51]. It can be constructed by two parties who share a singlet state $|\Psi^-\rangle$; Alice jointly measures system $\mathcal{X}$ together with her half of the singlet using a Bell measurement in the basis $\{(\sigma_a \otimes \mathbb{1})|\Psi^-\rangle\}_{a=0,1,2,3}$, where a labels her outcome, and Bob proceeds similarly on system $\mathcal{Y}$ together with his half of the singlet, with $b = 0,\ 1,\ 2,\ 3$ as his outcome. (Here, $\sigma_0 = \mathbb{1}$ and $\sigma_{1,2,3}$ are the Pauli matrices.) The action of the resulting distributed POVM with quantum inputs $\mathcal{X}$ and $\mathcal{Y}$ on input quantum states ξ and ψ can be written

$$R_{\{\Pi_{ab}\}}[\xi \otimes \psi] = \sum_{ab} |ab\rangle\langle ab| \langle\Psi^-|(\sigma_a \xi \sigma_a) \otimes (\sigma_b \psi \sigma_b)|\Psi^-\rangle. \tag{11}$$

The resources $\{\mu_{a|x}^{XZY}\}$ and $\{\mu_{a|x}^{XZ}\}$ are steering assemblages (type $\mathsf{CI} \to \mathsf{CQ}$). The *XZY*-singlet assemblage $\{\mu_{a|x}^{XZY}\}$ can be constructed by two parties sharing a singlet, and one of them performing the measurement $\{|+\rangle\langle +|, |-\rangle\langle -|\}$, $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ or $\{|+i\rangle\langle +i|, |-i\rangle\langle -i|\}$, depending on whether the classical input x is 0, 1, or 2, respectively, and where $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$ and $|\pm i\rangle = (1/\sqrt{2})(|0\rangle \pm i|1\rangle)$. One can write the resulting assemblage in terms of its action on the three possible classical inputs as

$$\begin{aligned}
R_{\{\mu_{a|x}^{XZY}\}}[|0\rangle\langle 0|] &= (|0-\rangle\langle 0-| + |1+\rangle\langle 1+|)/2,\\
R_{\{\mu_{a|x}^{XZY}\}}[|1\rangle\langle 1|] &= (|01\rangle\langle 01| + |10\rangle\langle 10|)/2,\\
R_{\{\mu_{a|x}^{XZY}\}}[|2\rangle\langle 2|] &= (|0-i\rangle\langle 0-i| + |1+i\rangle\langle 1+i|)/2.
\end{aligned} \tag{12}$$

The *XZ*-singlet assemblage $\{\mu_{a|x}^{XZ}\}$ is defined similarly, but where the classical inputs are $x = 0,\ 1$, corresponding to the first two POVMs above, respectively.

The Tsirelson box $P_{\mathrm{Tsi}}(ab|xy)$ (type $\mathsf{CC} \to \mathsf{CC}$) of Ref. [52] is the quantum-realizable box which maximally violates the CHSH inequality; it is obtained from $|\Psi^-\rangle$ by projective measurements. Alice uses the same measurements as in the preparation of $R_{\{\mu_{a|x}^{XZ}\}}$, while Bob measures either $\{c|0\rangle + s|1\rangle, s|0\rangle - c|1\rangle\}$ or $\{s|0\rangle + c|1\rangle, c|0\rangle - s|1\rangle\}$, depending on whether $y = 0$ or $y = 1$, respectively, and where $c = \cos\pi/8$ and $s = \sin\pi/8$. Then with $a,\ b,\ x,\ y = 0,\ 1$, we have $R_{\mathrm{Tsi}}[|xy\rangle\langle xy|] = \sum_{ab} |ab\rangle\langle ab| P_{\mathrm{Psi}}(ab|xy)$ with

$$P_{\mathrm{Tsi}}(ab|xy) = \frac{1 + (-1)^{a+b+xy}/\sqrt{2}}{4}. \tag{13}$$

A unified resource theory.—Next, we introduce a single resource theory [16] which captures the relevant notion of nonclassicality for *all* resources of all of the types described above. Within this resource theory (which is expanded upon in the companion article [15]), free resources are the ones that are obtained using only LOSR operations [17,18]).

Definition 2 A resource $R_{\mathcal{AB}|\mathcal{XY}}$ is LOSR free if it admits of a convex decomposition into single party resources $R^i_{\mathcal{A}|\mathcal{X}}$ and $R^i_{\mathcal{B}|\mathcal{Y}}$ according to probability distribution p_i:

$$R_{\mathcal{AB}|\mathcal{XY}} = \sum_i p_i (R^i_{\mathcal{A}|\mathcal{X}} \otimes R^i_{\mathcal{B}|\mathcal{Y}}). \tag{14}$$

We denote the set of *all* free resources as $\mathbf{R}^{\mathrm{free}} \coloneqq \cup_{\mathcal{ABXY}} \mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}}$, and the set of *all* no-signaling resources (free or nonfree) as $\mathbf{R} \coloneqq \cup_{\mathcal{ABXY}} \mathbf{R}_{\mathcal{AB}|\mathcal{XY}}$; note that the union runs over all types and dimensions.

For some types, we have $\mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}} = \mathbf{R}_{\mathcal{AB}|\mathcal{XY}}$.

Definition 3 A resource type is T-trivial if every resource of that type is necessarily free.

Proposition 1 Any type $\mathsf{T}[\mathcal{X}]\mathsf{T}[\mathcal{Y}] \to \mathsf{T}[\mathcal{A}]\mathsf{T}[\mathcal{B}]$ with $\mathsf{T}[X] = \mathsf{I}$ and $\mathsf{T}[A] = \mathsf{C}$ is T-trivial.

Proof.—Let R have type $\mathsf{IT}[\mathcal{Y}] \to \mathsf{CT}[\mathcal{B}]$. Let $P(a) = \langle a|\mathrm{tr}_{\mathcal{B}} R[\psi]|a\rangle$ for an arbitrary $\psi \in \mathsf{D}(\mathcal{Y})$. By no signaling, $P(a)$ is independent of ψ and unique. For a with $P(a) > 0$, define the single-party channel $R_a[\_] = \langle a|R[\_]|a\rangle / P(a) \in \mathsf{D}(\mathcal{Y}) \to \mathsf{D}(\mathcal{B})$. Noting that $R[\psi] = \sum_a P(a)|a\rangle\langle a| \otimes R_a[\psi]$, it follows that R is LOSR free.

We now turn our attention towards transformations of resources: a generic transformation τ on resources is a completely positive, linear supermap [53]

$$\tau\colon [\mathsf{D}(\mathcal{XY}) \to \mathsf{D}(\mathcal{AB})] \to [(\mathsf{D}(\mathcal{X}'\mathcal{Y}') \to \mathsf{D}(\mathcal{A}'\mathcal{B}')] \tag{15}$$

that transforms a resource $R_{\mathcal{AB}|\mathcal{XY}}$ into a resource $R_{\mathcal{A}'\mathcal{B}'|\mathcal{X}'\mathcal{Y}'}$, possibly changing its type and dimensions. A transformation is LOSR free if it is obtainable by local operations and shared randomness, and hence admits of a convex decomposition into products of arbitrary supermaps [53] acting only on a single party. A generic (bipartite) free transformation is shown in Fig. 3(a) of Ref. [15]. The set of free transformations is closed under composition, and maps free resources to free resources.

The resourcefulness of resources is completely characterized by the conversions that are possible between them using free operations. For example, the relative nonclassicality of the resources in Fig. 1 are fully characterized by the conversion relations claimed therein, whose validity we now prove. When we defined the examples above, we described how the singlet $|\Psi^-\rangle$ can be transformed into the resources $\{\Pi_{ab}\}$, $\{\mu^{XZY}_{a|x}\}$, $\{\mu^{XZ}_{a|x}\}$, and $\vec{P}_{\mathrm{Tsi}}$ using local (and hence LOSR-free) transformations. Furthermore, $\{\Pi_{ab}\}$ can be freely transformed into the singlet $|\Psi^-\rangle$, as proved in [15], so the two are equally resourceful (and hence can be freely transformed into exactly the same resources). The assemblage $\{\mu^{XZY}_{a|x}\}$ can be freely transformed into assemblage $\{\mu^{XZ}_{a|x}\}$ simply by Alice ignoring the third input value, and then also into $P_{\mathrm{Tsi}}(ab|xy)$ by Bob further performing the measurements given above Eq. (13) on his quantum output. Finally, $\{\mu^{XZ}_{a|x}\}$ can be transformed into $P_{\mathrm{Tsi}}(ab|xy)$ by Bob performing these same measurements.

It remains to show that every arrow absent from the figure corresponds to a conversion that is impossible under LOSR-free transformations. The following proposition implies the nonconvertibility of P_{Tsi} into $\{\mu^{XZY}_{a|x}\}$, $\{\mu^{XZ}_{a|x}\}$, or $|\Psi^-\rangle$, as well as the nonconvertibility of either $\{\mu^{XZY}_{a|x}\}$ or $\{\mu^{XZ}_{a|x}\}$ into $|\Psi^-\rangle$. By transitivity, these imply the nonconvertibility of any of P_{Tsi}, $\{\mu^{XZY}_{a|x}\}$, and $\{\mu^{XZ}_{a|x}\}$ into $\{\Pi_{ab}\}$.

Proposition 2 For any transformation taking (i) boxes $(\mathsf{CC} \to \mathsf{CC})$ to states $(\mathsf{II} \to \mathsf{QQ})$, (ii) boxes $(\mathsf{CC} \to \mathsf{CC})$ to assemblages $(\mathsf{CI} \to \mathsf{CQ})$, or (iii) assemblages $(\mathsf{CI} \to \mathsf{CQ})$ to states $(\mathsf{II} \to \mathsf{QQ})$, the resulting resource is necessarily LOSR free.

Proof.—Consider first a generic LOSR transformation taking boxes $(\mathsf{CC} \to \mathsf{CC})$ to states $(\mathsf{II} \to \mathsf{QQ})$; the most general such transformation is depicted by the dashed operations in Fig. 3(a), since for any party with only a classical input and output, the side channels on their local supermaps may be taken to be classical systems without

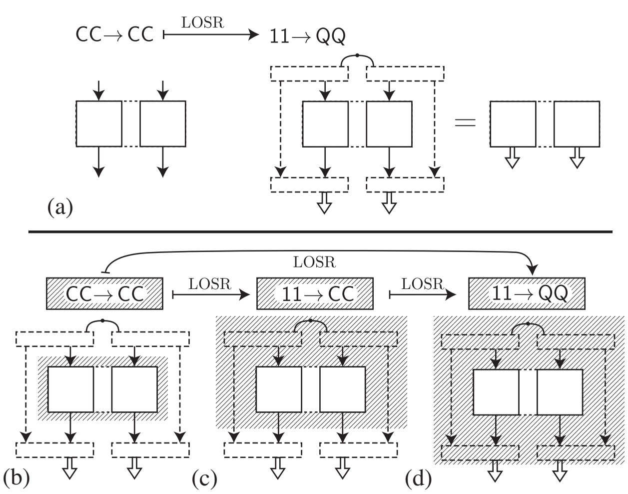


FIG. 3. Graphical proof of case 1. of Proposition 2. (a) The most general transformation from type $\mathsf{CC} \to \mathsf{CC}$ to type $\mathsf{II} \to \mathsf{QQ}$; without loss of generality, the side channels can be taken to be classical. One can imagine the transformation in (a) as a two-step procedure: the initial resource (b) of type $\mathsf{CC} \to \mathsf{CC}$ is transformed to a resource (c) of type $\mathsf{II} \to \mathsf{CC}$, and then to a resource (d) of type $\mathsf{II} \to \mathsf{QQ}$. In (b)–(d), the shading indicates the portion of the figure whose type is labeled.

loss of generality. As shown in Figs. 3(b)–3(d), any such LOSR transformation can be seen as a composition of two LOSR transformations, where the first takes type $\mathsf{CC} \to \mathsf{CC}$ to type $\mathsf{II} \to \mathsf{CC}$, and the second takes type $\mathsf{II} \to \mathsf{CC}$ to type $\mathsf{II} \to \mathsf{QQ}$. Now, Proposition 1 guarantees that the result of the first transformation is free, since its type is T-trivial. Since LOSR operations preserve the free set, the final resource resulting after the second transformation is also free.

The other two cases listed in the proposition have analogous proofs, but where the T-trivial intermediate types are $\mathsf{CI} \to \mathsf{CC}$ and $\mathsf{II} \to \mathsf{CQ}$, respectively.

To establish the nonconvertibility of $\{\mu_{a|x}^{XZ}\}$ to $\{\mu_{a|x}^{XZY}\}$, it suffices to exhibit a monotone whose value would increase under the conversion. Such a monotone is the type-independent absolute robustness, which we introduce below and depict in Fig. 1.

Type-independent monotones.—One can quantitatively measure the nonclassicality of a resource using any function that does not increase under LOSR transformations, which is a monotone of our resource theory. We focus on the type-independent absolute robustness here, but consider others in Sec. I of [23].

Definition 4 The type-independent absolute robustness $M_{\mathrm{abs}}(R)$ of a resource $R \in \mathbf{R}$ of arbitrary type and dimension is

$$M_{\mathrm{abs}}(R) = \min s \text{ such that } \quad (R + sS)/(1+s) \in \mathbf{R}^{\mathrm{free}},$$
$$s \geq 0, \qquad S \in \mathbf{R}^{\mathrm{free}}, \qquad \mathsf{d}[S] = \mathsf{d}[R], \mathsf{T}[S] = \mathsf{T}[R].$$

Our innovation here is to consider functions which behave monotonically even under operations that change the resource type, which allows us to compare the nonclassicality of resources across different types, as in Fig. 1. We prove in Sec. I of [23] that the absolute robustness (as defined here) has this property—despite the fact that the special cases of it which have been previously studied [17,54–58] have been type specific, and despite the fact that the computation of $M_{\mathrm{abs}}(R)$ involves a specification of the type and dimension of R.

The values of this monotone on the examples in Fig. 1 are exact, and the manner by which they are computed is described in Sec. VI of [23]. (We also compute these values for a family of parametrized versions of four of these resources.) Note that monotones can be used to prove that some conversions under LOSR are impossible, namely, those which would increase the value of M_{abs}. One cannot conclude anything about which conversions are possible. This can be seen in Fig. 1, where M_{abs} assigns the same value to the Tsirelson box and to the assemblage $\{\mu_{a|x}^{XZ}\}$, despite the fact that the two are not equivalent, as the former cannot be freely converted to the latter by Proposition 2.

A hierarchy to characterize nonclassicality.—We now describe how one can in practice determine whether or not a given resource $R^\star$ is LOSR free. This can be done using a hierarchy of SDPs which ultimately checks if $R^\star$ is a member of $\mathbf{R}^{\mathrm{free}}$. The reasoning is as follows.

If $R^\star_{\mathcal{AB}|\mathcal{XY}} \in \mathbf{R}^{\mathrm{free}}$, then $R^\star$ has a convex decomposition as in (14). By copying the shared randomness to more parties (who can then locally emulate any other party) it follows that $R^\star$ has an n-symmetric extension [59,60] for any n, obtained by copying the second party n times in the product. That is, for any n there exists

$$R^{(n)} = \sum_i p_i (R^i_{\mathcal{A}|\mathcal{X}} \otimes R^i_{\mathcal{B}_1|\mathcal{Y}_1} \otimes, \ldots, \otimes R^i_{\mathcal{B}_n|\mathcal{Y}_n}), \tag{16}$$

such that (i) $R^{(n)}$ is no signaling from any party to any other, (ii) $R^{(n)}$ is symmetric under all $n!$ permutations of the copies, and (iii) the reduced resource $\mathrm{tr}_{\mathcal{B}_2,\ldots,\mathcal{B}_k} R^{(n)} = R^\star$.

Whether or not an n-symmetric extension $R^{(n)}$ exists can be tested by an SDP. Hence:

Proposition 3 The set of classical resources $\mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}}$ of any given type and dimensionalities has a sequence of outer approximations

$$\mathbf{R}_{\mathcal{AB}|\mathcal{XY}} \supseteq \mathbf{F}^{(1)}_{\mathcal{AB}|\mathcal{XY}} \supseteq, \ldots, \mathbf{F}^{(n)}_{\mathcal{AB}|\mathcal{XY}}, \ldots, \supseteq \mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}}, \tag{17}$$

where each $\mathbf{F}^{(n)}_{\mathcal{AB}|\mathcal{XY}}$ is representable by a SDP, such that $\lim_{n\to\infty} \mathbf{F}^{(n)}_{\mathcal{AB}|\mathcal{XY}} = \mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}}$.

Proof.—Each $\mathbf{F}^{(n)}_{\mathcal{AB}|\mathcal{XY}}$ is defined by the set of resources that admit of an n-symmetric extension, and each such set is SDP-representable (see Definition 1). Leveraging the Choi matrix representation of resources, convergence follows immediately from Ref. [[60], Theorem 3.4], since the constraints of Eqs. (2)–(7) and (when applicable) (8), (9) have the form required by the theorem; see Sec. IV of [23] for explicit details.

Application: witnessing nonclassicality.—Because this sequence of approximations converges on $\mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}}$, for any nonclassical resource $R \notin \mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}}$ of arbitrary type and dimension, there exists some level n of the hierarchy such that $R \notin \mathbf{F}^{(n)}_{\mathcal{AB}|\mathcal{XY}}$, which witnesses the fact that R is not free. We stress again that our technique applies to all types of resources, since all that distinguishes different types is the inclusion (or not) of constraints of the form in (8) and (9) in the relevant SDPs.

Application: computation of monotones.—This hierarchy enables the practical computation of monotones. For example, consider again the absolute robustness M_{abs}. By replacing $\mathbf{R}^{\mathrm{free}}_{\mathcal{AB}|\mathcal{XY}}$ with the outer approximation $\mathbf{F}^{(n)}_{\mathcal{AB}|\mathcal{XY}}$ in the definition of M_{abs}, one relaxes the constraints in the minimization, thus obtaining a lower bound on $M_{\mathrm{abs}}(R)$. This gives a sequence of approximations

$$\tilde{M}^{(1)}_{\mathrm{abs}}(R) \leq, \ldots, \leq \tilde{M}^{(n)}_{\mathrm{abs}}(R) \leq, \ldots, \leq M_{\mathrm{abs}}(R), \tag{18}$$

each of which is explicitly computable using a SDP (see Sec. V of [23]), such that $\lim_{n\to\infty} \tilde{M}^{(n)}_{\mathrm{abs}}(R) = M_{\mathrm{abs}}(R)$.

Application: LOSR convertibility of states.—Since convertibility relations fundamentally determine the resourcefulness of resources, it is of central importance to be able to determine when one resource can be freely converted to another. By definition, free transformations that convert states (type $\mathsf{II} \to \mathsf{QQ}$) into states are themselves free resources of type $\mathsf{QQ} \to \mathsf{QQ}$. Given states $\rho \in \mathsf{D}(\mathcal{AB})$ and $\rho' \in \mathsf{D}(\mathcal{A}'\mathcal{B}')$, one can test for the existence of an LOSR-free transformation (viewed as a resource R) such that $\rho' = R[\rho]$. Since this is a linear constraint, one can modify the hierarchy of Proposition 3 to include it, generating a new hierarchy which tests for the possibility of LOSR convertibility between ρ and ρ'. A natural extension of this hierarchy to allow generic local supermaps [53] would allow one to test for the possibility of LOSR conversions between arbitrary types of resources, but we leave this for future work.

Outlook.—We have given a type independent description of nonclassical resources when local operations and shared randomness are free, generalizing a series of existing results, including those in Refs. [1,3–10,61]. This Letter opens many new research avenues. First, although generalizing our definitions and arguments to n-party scenarios is straightforward, multipartite nonclassicality in general scenarios is likely to have a rich structure, as happens for multipartite LOCC entanglement [62–64]. Still more interesting would be to expand our framework to encompass supermaps; for example, this would allow for a description of filtering [65], and for the construction of a hierarchy of SDPs that can witness the possibility or impossibility of any LOSR conversion between any two resources of any type. Much work remains to be done in defining monotones which are computable and which have operational significance across multiple resource types. These questions can also be asked for the resource theory of postquantumness, wherein the free operations are local operations and shared randomness [66]. For all these, the methods of Ref. [67], which presents a unified framework to study monotones in resource theories, are likely to be useful. Finally, we hope that our framework will be used to unify the expanding set of scenarios under study, and to generalize the tools developed to characterize them.

We acknowledge useful discussions with S.-L. Chen, R. W. Spekkens, and E. Wolfe. D. S. is supported by a Vanier Canada Graduate Scholarship. F. B. is grateful for the hospitality of Perimeter Institute where part of this work was carried out and acknowledges partial support from the Japan Society for the Promotion of Science (JSPS) KAKENHI, Grants No. 19H04066 and No. 20K03746, and from MEXT Quantum Leap Flagship Program (MEXT Q-LEAP), Grant No. JPMXS0120319794. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade. This publication was made possible through the support of a grant from the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[2] A. Fine, Phys. Rev. Lett. **48**, 291 (1982).
[3] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[4] D. Cavalcanti and P. Skrzypczyk, Rep. Prog. Phys. **80**, 024001 (2017).
[5] D. Cavalcanti, P. Skrzypczyk, and I. Šupić, Phys. Rev. Lett. **119**, 110501 (2017).
[6] M. J. Hoban and A. B. Sainz, New J. Phys. **20**, 053048 (2018).
[7] F. Buscemi, Phys. Rev. Lett. **108**, 200401 (2012).
[8] E. G. Cavalcanti, M. J. W. Hall, and H. M. Wiseman, Phys. Rev. A **87**, 032306 (2013).
[9] M. Piani, J. Opt. Soc. Am. B **32**, A1 (2015).
[10] A. B. Sainz, M. J. Hoban, P. Skrzypczyk, and L. Aolita, Phys. Rev. Lett. **125**, 050404 (2020).
[11] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).
[12] D. Cavalcanti, P. Skrzypczyk, G. H. Aguilar, R. V. Nery, P. H. S. Ribeiro, and S. P. Walborn, Nat. Commun. **6**, 7941 (2015).
[13] F. Xu, X. M. Q. Zhang, H.-K. Lo, and J.-W. Pan, Rev. Mod. Phys. **92**, 025002 (2020).
[14] Reference [6] also recognizes the value of treating various types of resources as channels, but does not study the full range of possibilities, as we have done here. The chief differences between our work and Ref. [6] are that our work focuses on nonclassicality (while Ref. [6] focuses on postquantumness), and that we leverage the resource theoretic framework to unify the distinct types of nonclassicality, as discussed in the main text.
[15] D. Schmid, D. Rosset, and F. Buscemi, Quantum **4**, 262 (2020).
[16] B. Coecke, T. Fritz, and R. W. Spekkens, Inf. Comput. Quantum Phys. Logic, **250**, 59 (2016).
[17] E. Wolfe, D. Schmid, A. B. Sainz, R. Kunjwal, and R. W. Spekkens, Quantum **4**, 280 (2020).
[18] D. Schmid, T. C. Fraser, R. Kunjwal, A. B. Sainz, E. Wolfe, and R. W. Spekkens, arXiv:2004.09194.
[19] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).
[20] D. Schmid, K. Ried, and R. W. Spekkens, Phys. Rev. A **100**, 022112 (2019).
[21] M.-D. Choi, Linear Algebra Appl. **10**, 285 (1975).
[22] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).

[23] See the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.210402 for technical details regarding our techniques, which includes Refs. [24–49].

[24] M. ApS, The MOSEK optimization toolbox for MATLAB manual. Version 7.1 (Revision 28). (2019), http://docs.mosek.com/9.0/toolbox/index.html.

[25] D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill, Phys. Rev. A **64**, 052309 (2001).

[26] S. Boyd and L. Vandenberghe, *Convex Optimization*, new. ed. (Cambridge University Press, Cambridge, England, 2004).

[27] G. M. D'Ariano, S. Facchini, and P. Perinotti, Phys. Rev. Lett. **106**, 010501 (2011).

[28] W. van Dam, R. D. Gill, and P. D. Grunwald, IEEE Trans. Inf. Theory **51**, 2812 (2005).

[29] R. Gallego and L. Aolita, Phys. Rev. X **5**, 041008 (2015).

[30] A. Garg and N. D. Mermin, Found. Phys. **14**, 1 (1984).

[31] J. Helton and J. Nie, SIAM J. Optim. **20**, 759 (2009).

[32] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[33] N. Johnston (2016) https://doi.org/10.5281/zenodo.44637.

[34] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. **80**, 2261 (1998).

[35] J. Lofberg, in *IEEE International Conference on Robotics and Automation, 2004 (IEEE Cat. No.04CH37508), New Orleans, LA* (IEEE, 2004), pp. 284–289.

[36] MATLAB (The MathWorks Inc., Natick, Massachusetts), https://www.mathworks.com/products/matlab.html.

[37] M. Navascués, M. Owari, and M. B. Plenio, Phys. Rev. Lett. **103**, 160404 (2009).

[38] J. W. Eaton, D. Bateman, S. Hauberg, and R. Wehbring, GNU Octave version 5.2.0 manual: A high-level interactive language for numerical computations (2020), https://www.gnu.org/software/octave/doc/v5.2.0/.

[39] A. Ostermann and G. Wanner, in *Geometry by Its History*, Undergraduate Texts in Mathematics, edited by A. Ostermann and G. Wanner (Springer, Berlin, Heidelberg, 2012), pp. 3–26.

[40] M. Piani and J. Watrous, Phys. Rev. Lett. **114**, 060404 (2015).

[41] I. Pitowsky, Math. Program. **50**, 395 (1991).

[42] S. Portmann, C. Branciard, and N. Gisin, Phys. Rev. A **86**, 012104 (2012).

[43] P. Skrzypczyk, M. Navascués, and D. Cavalcanti, Phys. Rev. Lett. **112**, 180404 (2014).

[44] W. Slofstra, arXiv:1703.08618.

[45] M. Steiner, Phys. Rev. A **67**, 054305 (2003).

[46] J. F. Sturm, Optim. Methods Software **11**, 625 (1999).

[47] R. Takagi and B. Regula, Phys. Rev. X **9**, 031053 (2019).

[48] B. M. Terhal, A. C. Doherty, and D. Schwab, Phys. Rev. Lett. **90**, 157903 (2003).

[49] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).

[50] E. Verbanis, A. Martin, D. Rosset, C. C. W. Lim, R. T. Thew, and H. Zbinden, Phys. Rev. Lett. **116**, 190501 (2016).

[51] D. Rosset, A. Martin, E. Verbanis, C. C. W. Lim, and R. Thew, Phys. Rev. A **98**, 052332 (2018).

[52] B. S. Tsirel'son, J. Sov. Math. **36**, 557 (1987).

[53] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Europhys. Lett. **83**, 30004 (2008).

[54] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).

[55] J. Geller and M. Piani, J. Phys. A **47**, 424030 (2014).

[56] D. Cavalcanti and P. Skrzypczyk, Phys. Rev. A **93**, 052112 (2016).

[57] A. B. Sainz, L. Aolita, N. Brunner, R. Gallego, and P. Skrzypczyk, Phys. Rev. A **94**, 012308 (2016).

[58] J. I. de Vicente, J. Phys. A **47**, 424017 (2014).

[59] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A **69**, 022308 (2004).

[60] M. Berta, F. Borderi, O. Fawzi, and V. Scholz, arXiv:1810.12197.

[61] J. Bowles, F. Hirsch, M. T. Quintino, and N. Brunner, Phys. Rev. Lett. **114**, 120401 (2015).

[62] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, Phys. Rev. Lett. **106**, 250404 (2011).

[63] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Phys. Rev. Lett. **109**, 070401 (2012).

[64] J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Phys. Rev. A **88**, 014102 (2013).

[65] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[66] D. Schmid, H. Du, M. Mudassar, G. Coulter-de Wit, D. Rosset, and M. J. Hoban, arXiv:2004.06133.

[67] T. Gonda and R. W. Spekkens, arXiv:1912.07085.