# Phase-covariant virtual broadcasting of qubits and its optimal physical approximation

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Virtual quantum maps simulate nonphysical transformations using physical operations and classical postprocessing, but known constructions of virtual broadcasting are operationally inefficient. Motivated by the expectation that relaxing symmetry requirements may improve operational performance, we study virtual quantum broadcasting for qubits under phase covariance. We show that phase covariance, flip covariance, permutation invariance, and classical consistency already fix the structure of the virtual map and imply broadcasting on equatorial states, without assuming broadcasting explicitly. Within this class, we identify a unique map of minimal simulation cost. Remarkably, and in direct analogy with the fully unitarily covariant case, the positive part of this optimal map coincides with the *optimal symmetric phase-covariant cloning* channel, showing that optimal cloning again emerges as the physically realizable map closest to virtual broadcasting, despite the reduced symmetry. Although phase covariance lowers both the simulation cost and this distance compared with the unitarily covariant case, the resulting virtual map remains sample inefficient.

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## I. INTRODUCTION

It is a trivial fact that, in classical information theory, information can be copied. In quantum information theory, by contrast, fundamental constraints prevent such copying. The no-cloning theorem [1,2] states that no completely positive trace-preserving (CPTP) map can create a perfect copy of an unknown quantum state. The no-broadcasting theorem [3] strengthens this insight: if the set of states to be broadcast contains at least one pair of noncommuting states, then no CPTP map can output systems whose reduced states all coincide with the input. Broadcasting is less demanding than cloning because it allows the outputs to be entangled, yet even this weaker task cannot be achieved physically.

To work around this limitation, recent studies have explored *virtual maps* [4–9]. A virtual map is a Hermitian-preserving trace-preserving (HPTP) map that need not correspond to any physical process. Instead, it reproduces the expectation values that would result from broadcasting by combining physical operations with classical postprocessing [10–12]. Parzygnat *et al.* [4] showed that unitary covariance, permutation invariance, and classical consistency single out a unique virtual broadcasting map, known as the *canonical broadcasting map*. Although not physically implementable, this map has found applications in the evaluation of two-point correlation functions [10,11], error mitigation [13–15], quantum states over time [16–18], and the simulation of non-Markovian dynamics [12]. Indeed, the $1 \to 2$ virtual broadcasting map was first developed to reproduce two-point correlation functions [10,11], and this remains one of its most natural applications. However, if one insists on using it merely to mimic access to two independent copies of the input state, the question arises whether its simulation cost is justified when compared with the trivial redistribution of the available copies.

This issue is central to the operational meaning of virtual maps. Because they are implemented through sampling and classical postprocessing, they typically require multiple copies of the input state. This leads to a natural question: when estimating expectation values on two outputs, is it more efficient to use a virtual broadcasting map or simply distribute the available input copies to two parties? Xiao *et al.* [7,8] showed that direct distribution is strictly more efficient, revealing the large simulation cost of the canonical map. This negative result motivates the search for alternative virtual broadcasting constructions with reduced cost, obtained by relaxing some of the assumptions underlying the unitarily covariant framework.

A closely related phenomenon appears in quantum cloning, where restricting symmetry often leads to improved performance. Cloning channels are frequently designed to satisfy group covariance, meaning that they act uniformly on all states invariant under a chosen symmetry group [19–22]. Two prominent examples are *unitary* (or universal) covariance, which enforces uniform fidelity for all pure states, and *phase covariance*, which requires uniform fidelity only for states belonging to a restricted subset, such as the equator of the Bloch sphere in the qubit case. These restricted settings typically allow higher fidelities and more economical implementations. By analogy, one may expect that restricting the class of input states in virtual broadcasting could similarly reduce simulation cost and improve operational performance.

In this work, we investigate virtual broadcasting for qubits under phase covariance. Our approach replaces the unitary covariance assumed in Refs. [4,7,8] with phase covariance and supplements it with bit-flip covariance, permutation invariance, and classical consistency. A key conceptual point

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is that we do *not* assume the broadcasting condition. Instead, we show that these symmetry and consistency requirements already fix the structure of the virtual map on equatorial states and *force* the broadcasting property and trace preservation as derived consequences. Broadcasting thus emerges here as a structural implication of the assumptions, rather than as an independent postulate.

Within the resulting family of virtual maps, we then identify a unique map that minimizes the simulation cost. Remarkably, and in direct analogy with the fully unitarily covariant case, the positive part of the optimal map coincides exactly with the *optimal symmetric phase-covariant cloning* channel. As a consequence, optimal phase-covariant cloning *emerges naturally* as the completely positive trace-preserving map that provides the closest physically realizable approximation to virtual broadcasting in diamond norm, despite the reduced symmetry. While phase covariance lowers both the simulation cost and this distance compared with the unitarily covariant case, the resulting virtual broadcasting map remains sample inefficient. Even under these relaxed assumptions, the trivial strategy of directly distributing input copies remains strictly more efficient.

The paper is organized as follows. In Sec. II, we review quantum broadcasting and introduce the symmetry and consistency conditions used in our analysis. In Sec. III, we derive and characterize the virtual phase-covariant broadcasting map, evaluate its simulation cost, and compare it with the optimal physical approximation. In Sec. IV, we summarize the results and discuss open problems.

## II. PRELIMINARIES

A two-dimensional quantum system is associated with a Hilbert space $\mathcal{H} \cong \mathbb{C}^2$. We denote the set of linear operators on $\mathcal{H}$ as $\mathsf{L}(\mathcal{H})$, the set of density operators as $\mathsf{S}(\mathcal{H})$, and the set of unitary operators as $\mathsf{U}(\mathcal{H})$. We denote the identity operator on $\mathcal{H}$ as $\mathbb{1}$ and the identity channel on $\mathsf{L}(\mathcal{H})$ as $\mathcal{I}$. Throughout the paper, $\mathcal{B}$ denotes a linear map from $\mathsf{L}(\mathcal{H})$ to $\mathsf{L}(\mathcal{H}\otimes\mathcal{H})$. We define the Choi operator of $\mathcal{B}$ as $C_{\mathcal{B}} := \sum_{i,j} \mathcal{B}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$, where $\{|i\rangle\}_i$ is a fixed orthonormal basis.

Quantum broadcasting refers to an operation that generates an output state whose reduced states on each subsystem coincide with the input state. A map $\mathcal{B} : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H}\otimes\mathcal{H})$ is called a $1 \to 2$ $\mathscr{S}$*-broadcasting map* if

$$\mathrm{Tr}_1\mathcal{B}(\rho) = \mathrm{Tr}_2\mathcal{B}(\rho) = \rho \tag{1}$$

holds for every state $\rho$ in the subset $\mathscr{S} \subseteq \mathsf{S}(\mathcal{H})$. We refer to condition (1) as the $\mathscr{S}$*-broadcasting condition* ($\mathscr{S}$-BROAD).

The no-broadcasting theorem [3] states that there is no CPTP map that satisfies $\mathscr{S}$-BROAD whenever $\mathscr{S}$ contains at least one pair of noncommuting states. However, it has been shown that a unique HPTP map that satisfies $\mathsf{S}(\mathcal{H})$-BROAD, namely, broadcasting *everywhere* on $\mathsf{S}(\mathcal{H})$, is singled out by the following three assumptions [4,7,8]:

(1) $\mathcal{B}$ satisfies *unitary covariance* (UNITARY), expressed as

$$(\mathcal{U}\otimes\mathcal{U})\circ\mathcal{B} = \mathcal{B}\circ\mathcal{U}, \tag{2}$$

where $\mathcal{U} : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H})$ is defined by $\mathcal{U}(\cdot) := U\cdot U^\dagger$ for any arbitrary but fixed $U \in \mathsf{U}(\mathcal{H})$;

(2) *permutation invariance* (PERM), expressed as

$$\mathcal{S}\circ\mathcal{B} = \mathcal{B}, \tag{3}$$

where $\mathcal{S} : \mathsf{L}(\mathcal{H}\otimes\mathcal{H}) \to \mathsf{L}(\mathcal{H}\otimes\mathcal{H})$ is defined by $\mathcal{S}(\cdot) := S\cdot S$ for the swap operator $S : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ defined by $S(|\phi\rangle\otimes|\psi\rangle) = |\psi\rangle\otimes|\phi\rangle$, for any $|\phi\rangle, |\psi\rangle \in \mathcal{H}$;

(3) when both the input and output systems of $\mathcal{B}$ undergo decoherence, its action coincides with the classical broadcasting map. The classical broadcasting map $\mathcal{B}_{\mathrm{cl}} : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H}\otimes\mathcal{H})$ is the unique map defined by

$$\mathcal{B}_{\mathrm{cl}}(|i\rangle\langle j|) := \delta_{ij}|i\rangle\langle i| \otimes |i\rangle\langle i| \quad \forall i, j, \tag{4}$$

for a fixed orthonormal basis $\{|i\rangle\}_i$. We say that $\mathcal{B}$ satisfies *classical consistency* (CLASSIC) if

$$(\mathcal{D}\otimes\mathcal{D})\circ\mathcal{B}\circ\mathcal{D} = \mathcal{B}_{\mathrm{cl}}, \tag{5}$$

where $\mathcal{D} : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H})$ is the decoherence map defined by $\mathcal{D}(\rho) := \sum_i \langle i|\rho|i\rangle\, |i\rangle\langle i|$. If a broadcasting $\mathcal{B}$ satisfies UNITARY and CLASSIC for a certain basis, then it automatically satisfies CLASSIC also for any other orthonormal basis [4].

The assumption of *phase covariance* (PHASE) used in this work is a relaxation of UNITARY covariance, obtained by deliberately restricting attention to a smaller set of input states. The motivation for this restriction is operational: by asking a less stringent broadcasting requirement, we expect the corresponding virtual broadcasting map to exhibit improved performance, in particular a reduced simulation cost.

We focus on the subset

$$\mathsf{S}'(\mathcal{H}) := \{\rho \mid \rho \in \mathsf{S}(\mathcal{H}),\ \mathrm{Tr}[\rho\sigma_z] = 0\} \tag{6}$$

$$= \left\{\tfrac{1}{2}(\mathbb{1} + r\cos\varphi\,\sigma_x + r\sin\varphi\,\sigma_y) \mid 0 \leqslant r \leqslant 1,\ 0 \leqslant \varphi < 2\pi\right\}, \tag{7}$$

namely, the equatorial states of the Bloch sphere. The relevant symmetry group is the set of diagonal unitaries

$$\mathsf{U}'(\mathcal{H}) := \{U_\varphi \mid 0 \leqslant \varphi < 2\pi\}, \tag{8}$$

where $U_\varphi := |0\rangle\langle 0| + e^{i\varphi}|1\rangle\langle 1|$; global phases are immaterial. Condition PHASE demands covariance only with respect to $\mathsf{U}'(\mathcal{H})$:

$$(\mathcal{U}'\otimes\mathcal{U}')\circ\mathcal{B} = \mathcal{B}\circ\mathcal{U}', \tag{9}$$

for any $\mathcal{U}'(\cdot) := U_\varphi \cdot U_\varphi^\dagger$ with $U_\varphi \in \mathsf{U}'(\mathcal{H})$.

Because $\mathsf{S}'(\mathcal{H})$ is also closed under conjugation by $\sigma_x$, it is natural to supplement PHASE with covariance under a bit flip about the $x$ axis. Together, these symmetries capture the full dihedral symmetry of the equatorial plane, namely, phase rotations and reflections. A map $\mathcal{B}$ satisfies *flip covariance* (FLIP) if

$$(X \otimes X) \circ \mathcal{B} = \mathcal{B} \circ X, \tag{10}$$

where $X(\cdot) := \sigma_x \cdot \sigma_x$.

In this setting, both BROAD and CLASSIC are required only on states in $\mathsf{S}'(\mathcal{H})$. In particular, in the phase-covariant setting it is natural to impose CLASSIC with respect to equatorial bases; indeed, in Sec. III we formulate CLASSIC starting from the $\{|+\rangle, |-\rangle\}$ basis, where $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

Note that any orthonormal basis of equatorial states is obtained from the basis $\{|+\rangle, |-\rangle\}$ by conjugation with some $U_\varphi \in \mathsf{U}'(\mathcal{H})$. Hence, if a map $\mathcal{B}$ satisfies PHASE covariance and the classical consistency condition (5) for the basis $\{|+\rangle, |-\rangle\}$, then, by covariance and linearity, it also satisfies CLASSIC for every equatorial basis.

### A. Sample efficiency

Although broadcasting cannot be physically realized, its expectation values can be simulated through virtual operations, namely, HPTP maps. To assess whether such simulations are operationally meaningful, we use the notion of *sample efficiency* (SAMPLE) [7,8]. Operationally, SAMPLE compares the number of input copies required by a virtual operation with that required by the trivial strategy in which copies of the unknown input state $\rho$ are directly distributed to each party.

Consider the following task. Alice holds many copies of an unknown state $\rho$. She sends systems to Bob and Claire, who locally measure observables $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively. To estimate the corresponding expectation values with given accuracy, Bob requires $n_1$ copies of $\rho$, and Claire requires $n_2$ copies. The trivial strategy therefore consumes $n_1 + n_2$ input copies.

A virtual broadcasting map offers an alternative. While it reproduces the correct expectation values, the number of input copies required depends on how the map is simulated via physical operations and classical postprocessing. A virtual broadcasting map is regarded as operationally useful only if it allows Alice to use fewer than $n_1 + n_2$ copies. This requirement defines SAMPLE.

We now formalize this comparison. Bob and Claire must each pass an $\epsilon_i - \delta_i$ test, meaning that the probability that their estimation error exceeds $\epsilon_i$ is at most $\delta_i$. By Hoeffding's inequality, the required number of samples satisfies

$$n_i \geqslant \frac{c^2}{2\epsilon_i^2} \ln \frac{2}{\delta_i} \quad (i \in \{1, 2\}), \tag{11}$$

where $c$ is the range of possible measurement outcomes.

Next, consider a HPTP map $\mathcal{B}$ decomposed as $\mathcal{B} = a\mathcal{E} - b\mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are CPTP maps and $a, b \geqslant 0$. Then,

$$\mathrm{Tr}[\mathcal{O}_1 \mathrm{Tr}_2 \mathcal{B}(\rho)] = (a+b)\bigg(\frac{a}{a+b}\mathrm{Tr}[\mathcal{O}_1 \mathrm{Tr}_2 \mathcal{E}(\rho)] - \frac{b}{a+b}\mathrm{Tr}[\mathcal{O}_1 \mathrm{Tr}_2 \mathcal{F}(\rho)]\bigg). \tag{12}$$

Thus, estimating the expectation value associated with $\mathcal{B}$ can be achieved by sampling either $\mathcal{E}$ or $\mathcal{F}$ and rescaling the outcome by a factor $a + b$. This rescaling amplifies statistical fluctuations by the same factor. Combining this with (11), simulating $\mathcal{B}$ requires $(a+b)^2 n_Q$ input copies, where $n_Q := \max\{n_1, n_2\}$.

We say that $\mathcal{B}$ satisfies SAMPLE if

$$n_1 + n_2 > \min(a+b)^2 n_Q,$$

where the minimum is taken over all decompositions of $\mathcal{B}$ into CPTP maps.

The problem of minimizing the overhead factor $a + b$ was first identified in Refs. [10,11] as the natural figure of merit for the classical simulation of linear quantum maps. Building on this insight and on the general framework developed in Ref. [23], the minimal achievable value of $a + b$ is given by the base norm associated with completely positive trace-nonincreasing (CPTNI) maps:

$$\|\mathcal{B}\|_\blacklozenge := \min\{a + b \mid \mathcal{B} = a\mathcal{E} - b\mathcal{F},\ \mathcal{E}, \mathcal{F} : \mathrm{CPTNI}\}. \tag{13}$$

When $\mathcal{B}$ is trace preserving, the minimization can be restricted to CPTP maps [23]:

$$\|\mathcal{B}\|_\blacklozenge = \min\{a + b \mid \mathcal{B} = a\mathcal{E} - b\mathcal{F},\ a, b \geqslant 0,\ \mathcal{E}, \mathcal{F} : \mathrm{CPTP}\}. \tag{14}$$

Since we consider only trace-preserving maps, we refer to $\|\mathcal{B}\|_\blacklozenge$ as the *simulation cost* of $\mathcal{B}$.

Finally, the base norm can be bounded in terms of the trace norm of the Choi operator $C_\mathcal{B}$ [23]:

$$\frac{1}{d}\|C_\mathcal{B}\|_1 \leqslant \|\mathcal{B}\|_\blacklozenge \leqslant \|C_\mathcal{B}\|_1, \tag{15}$$

where $d = \dim(\mathcal{H})$. These bounds provide practical estimates for the simulation cost and hence for sample efficiency.

## III. VIRTUAL PHASE-COVARIANT QUANTUM BROADCASTING FOR QUBITS

In this section we characterize all linear maps $\mathcal{B} : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H} \otimes \mathcal{H})$ satisfying the symmetry and consistency assumptions introduced in Sec. II, and we identify within this family the map of minimal simulation cost.

TABLE I. Decomposition of $\mathcal{H}^{\otimes 3}$ into equivalence classes and invariant subspaces, under the combined action of $U_\varphi$, the swap $S$, and the bit flip $\sigma_x$. The first column labels the inequivalent representations, and the second column lists irreducible representations (all one-dimensional) within the same equivalence class. The integer $n$ in the third column specifies the character of the action of $U_\varphi \otimes U_\varphi \otimes U_\varphi^*$ on that subspace, which acts as multiplication by $e^{in\varphi}$. The sign in the fourth column records the eigenvalue of the operator $S_{12} \otimes \mathbb{1}_3$. The fifth column indicates the partner class obtained from the action of $\sigma_x \otimes \sigma_x \otimes \sigma_x^*$. Note that the superscript $*$ denotes complex conjugation: since $\sigma_x$ is real, here it plays no role, but we keep it for clarity.

| Subspace | Un-normalized basis | $U_\varphi$ | $S$ | $\sigma_x$ |
|---|---|---|---|---|
| $\mathcal{H}_1$ | $\|000\rangle,\ \|011\rangle + \|101\rangle$ | 0 | 1 | $\mathcal{H}_2$ |
| $\mathcal{H}_2$ | $\|111\rangle,\ \|100\rangle + \|010\rangle$ | 1 | 1 | $\mathcal{H}_1$ |
| $\mathcal{H}_3$ | $\|011\rangle - \|101\rangle$ | 0 | $-1$ | $\mathcal{H}_4$ |
| $\mathcal{H}_4$ | $\|100\rangle - \|010\rangle$ | 1 | $-1$ | $\mathcal{H}_3$ |
| $\mathcal{H}_5$ | $\|001\rangle$ | $-1$ | 1 | $\mathcal{H}_6$ |
| $\mathcal{H}_6$ | $\|110\rangle$ | 2 | 1 | $\mathcal{H}_5$ |

The group covariance of a map is expressed as the invariance of its Choi operator under the corresponding group action [20]. When $\mathcal{B}$ satisfies PHASE, the Choi operator $C_{\mathcal{B}}$ obeys

$$(U_\varphi \otimes U_\varphi \otimes U_\varphi^*)\, C_{\mathcal{B}}\, (U_\varphi \otimes U_\varphi \otimes U_\varphi^*)^\dagger = C_{\mathcal{B}} \quad \forall\, U_\varphi \in \mathsf{U}'(\mathcal{H}), \tag{16}$$

where $U_\varphi^*$ denotes the complex conjugate of $U_\varphi$. Since

$$(U_\varphi \otimes U_\varphi \otimes U_\varphi^*)\,|jkl\rangle = e^{i(j+k-l)\varphi}\,|jkl\rangle \quad (j,k,l \in \{0,1\}), \tag{17}$$

the joint action of $U_\varphi$, the swap $S$, and the bit flip $\sigma_x$ decomposes the space $\mathcal{H}^{\otimes 3}$ into the invariant subspaces listed in Table I. By Schur's Lemma, the Choi operator takes the block form

$$C_{\mathcal{B}} = \bigoplus_{j=1}^{6} C_j,$$

where each block $C_j$ acts on the corresponding subspace $\mathcal{H}_j$. In particular, the symmetries reduce the description of $C_{\mathcal{B}}$ to finitely many scalar parameters, which we introduce explicitly below.

The condition FLIP imposes an additional restriction. The operator $\sigma_x \otimes \sigma_x \otimes \sigma_x^*$ maps $\mathcal{H}_j$ onto $\mathcal{H}_{j+1}$ for $j = 1, 3, 5$. Because $C_{\mathcal{B}}$ is invariant under this action, the two blocks on each such pair of subspaces must coincide. In other words, $C_j = C_{j+1}$ as operators for $j = 1, 3, 5$. We therefore use the same scalar parameters to describe each pair. We denote the parameters for $C_1$, $C_3$, and $C_5$ by $\{c_1, c_2, c_3, c_4\}$, $c_5$, and $c_6$, respectively, with $c_1, \dots, c_6 \in \mathbb{C}$.

In the computational basis, the Choi operator of a map $\mathcal{B}$ satisfying PHASE, FLIP, and PERM can be written as

$$C_{\mathcal{B}} = \begin{bmatrix} c_1 & 0 & 0 & c_2 & 0 & c_2 & 0 & 0 \\ 0 & c_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_4+c_5 & 0 & c_4-c_5 & 0 & 0 & c_3 \\ c_3 & 0 & 0 & c_4+c_5 & 0 & c_4-c_5 & 0 & 0 \\ 0 & 0 & c_4-c_5 & 0 & c_4+c_5 & 0 & 0 & c_3 \\ c_3 & 0 & 0 & c_4-c_5 & 0 & c_4+c_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_6 & 0 \\ 0 & 0 & c_2 & 0 & c_2 & 0 & 0 & c_1 \end{bmatrix}. \tag{18}$$

We now impose the classical consistency requirement CLASSIC for equatorial bases. Since we consider qubit states lying on the $xy$ plane (equatorial states), the decoherence map $\mathcal{D}$ and the classical broadcasting map $\mathcal{B}_{\mathrm{cl}}$ with respect to which CLASSIC is defined are taken with respect to the basis $\{|+\rangle, |-\rangle\}$. Using the reverse-Choi relation $\mathcal{B}(\rho) = \mathrm{Tr}_3[\mathbb{1}_{12} \otimes \rho_3^T\, C_{\mathcal{B}}]$, we obtain

$$\begin{aligned}(\mathcal{D}\otimes\mathcal{D})\circ\mathcal{B}(|+\rangle\langle+|) = {}& \left(\frac{c_1+c_6}{4} + \frac{c_2+c_3}{2} + c_4\right)|{++}\rangle\langle{++}| + \left(\frac{c_1+c_6}{4} + c_5\right)(|{+-}\rangle\langle{+-}| + |{-+}\rangle\langle{-+}|) \\ & + \left(\frac{c_1+c_6}{4} - \frac{c_2+c_3}{2} - c_4\right)|{--}\rangle\langle{--}|. \end{aligned} \tag{19}$$

Then, as a consequence of linearity and PHASE, a necessary and sufficient condition for $\mathcal{B}$ to satisfy CLASSIC is that (19) equals $|{++}\rangle\langle{++}|$, which is in turn equivalent to the following simultaneous linear conditions:

$$\begin{aligned} c_3 &= 1 - c_2, \\ c_5 &= c_4 - \tfrac{1}{2}, \\ c_6 &= 2 - c_1 - 4c_4. \end{aligned} \tag{20}$$

Note that, since we work with qubits and phase covariance, knowing the action of $\mathcal{B}$ on $|+\rangle\langle+|$ already determines its action on $|-\rangle\langle-|$, because $|-\rangle = \sigma_z\,|+\rangle$ and phase covariance fixes the behavior of $\mathcal{B}$ on all equatorial states once it is fixed on $|+\rangle$.

Substituting (20) into (18), we obtain

$$\begin{aligned}\mathcal{B}(|+\rangle\langle+|) = {}& (1-2c_4)(|00\rangle\langle00| + |11\rangle\langle11|) + \left(2c_4 - \frac{1}{2}\right)(|01\rangle\langle01| + |10\rangle\langle10|) \\ & + \frac{c_2}{2}(|00\rangle\langle11|)(\langle01| + \langle10|) + \frac{1-c_2}{2}(|01\rangle + |10\rangle)(\langle00| + \langle11|) \\ & + \frac{1}{2}(|01\rangle\langle10| + |10\rangle\langle01|). \end{aligned} \tag{21}$$

Taking partial traces yields

$$\mathrm{Tr}_1[\mathcal{B}(|+\rangle\langle+|)] = \mathrm{Tr}_2[\mathcal{B}(|+\rangle\langle+|)] = |+\rangle\langle+|. \tag{22}$$

Therefore, by linearity and PHASE, $\mathcal{B}$ satisfies $\mathsf{S}'(\mathcal{H})$-BROAD. Moreover, since $\mathrm{Tr}_{12}[C_{\mathcal{B}}] = \mathbb{1}$, $\mathcal{B}$ also satisfies TP everywhere.

Crucially, the assumptions imposed so far do more than merely constrain the form of the Choi operator: they already *force* the broadcasting behavior on the whole equatorial set, and they do so without ever postulating the broadcasting condition itself. In particular, `CLASSIC` and the symmetries imply both $\mathsf{S}'(\mathcal{H})$-`BROAD` and trace preservation, so that broadcasting emerges here as a *derived* property rather than an additional requirement. Hence, we obtain the following proposition.

*Proposition 1.* A linear map $\mathcal{B} : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H} \otimes \mathcal{H})$ satisfying `PHASE`, `FLIP`, `PERM`, and `CLASSIC` also satisfies $\mathsf{S}'(\mathcal{H})$-`BROAD` and `TP`.

Note that, if $\mathcal{B}$ satisfies $\mathsf{S}'(\mathcal{H})$-`BROAD`, it automatically follows that $\mathcal{B}$ is `TP` on states of $\mathsf{S}'(\mathcal{H})$. However, $\mathrm{Tr}_{12}[C_\mathcal{B}] = \mathbb{1}$ implies that $\mathcal{B}$ is `TP` on the whole of $\mathsf{S}(\mathcal{H})$, i.e., on arbitrary input states, including those outside the equatorial plane. This does not follow, in principle, from $\mathsf{S}'(\mathcal{H})$-`BROAD` alone. The fact that trace preservation holds *everywhere* is therefore a nontrivial consequence of the symmetry and consistency assumptions. Note also that, at this point, `HP` need not be satisfied.

### A. Singling out the optimal map

Unlike the unitarily covariant case, the operational properties introduced so far do not single out a unique map. We therefore introduce an optimization criterion to select, among all maps satisfying the required symmetries and consistency conditions, the one with minimal simulation cost, namely, to minimize $\|\mathcal{B}\|_\blacklozenge$. Since the base norm is bounded in terms of the trace norm by (15), it is natural to first search for parameters that minimize $\|C_\mathcal{B}\|_1$. At this stage we also impose Hermiticity preservation (`HP`), which is not needed for Proposition 1 but is required for $\mathcal{B}$ to define a consistent real-valued simulation of expectation values.

Using the Wolfram Cloud platform [24], by numerically minimizing the trace norm of the Choi operator (18) under the `CLASSIC` constraints (20) and the `HP` condition, which for $C_\mathcal{B}$ is equivalent to $c_1, c_4 \in \mathbb{R}$ and $c_2 = c_3^*$, we obtain

$$\min \|C_\mathcal{B}\|_1 = \tfrac{10}{3},$$

with a unique minimizer given by

$$c_1 = \tfrac{1}{3}, \quad c_2 = c_3 = \tfrac{1}{2}, \quad c_4 = \tfrac{5}{12}, \quad c_5 = -\tfrac{1}{12}, \quad c_6 = 0.$$

Using (15), every map $\mathcal{B}$ satisfying `PHASE`, `PERM`, `FLIP`, `CLASSIC`, and `HP` must therefore obey

$$\|\mathcal{B}\|_\blacklozenge \geqslant \tfrac{5}{3}. \tag{23}$$

The Choi operator achieving this minimum is

$$\hat{C}_\mathcal{B} := \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{3} \end{bmatrix}. \tag{24}$$

This operator admits a decomposition into two positive semidefinite operators with orthogonal support:

$$\hat{C}_\mathcal{B} = \tfrac{4}{3}\hat{C}_\mathcal{B}^+ - \tfrac{1}{3}\hat{C}_\mathcal{B}^-, \tag{25}$$

where the positive and negative parts are

$$\hat{C}_\mathcal{B}^+ := \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} \tag{26}$$

and

$$\hat{C}_\mathcal{B}^- := \begin{bmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} \\ -\frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{6} \\ -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & \frac{1}{3} \end{bmatrix}. \tag{27}$$

A direct calculation shows that both operators correspond to `CPTP` maps:

$$\mathrm{Tr}_{1,2}[\hat{C}_\mathcal{B}^+] = \mathrm{Tr}_{1,2}[\hat{C}_\mathcal{B}^-] = \mathbb{1}. \tag{28}$$

### B. Emergence of optimal symmetric phase-covariant cloning

As in the unitarily covariant case [4,10], the optimal Choi operator $\hat{C}_\mathcal{B}$ admits a decomposition into positive and negative parts with orthogonal support. Remarkably, the positive part $\hat{C}_\mathcal{B}^+$ coincides exactly with the Choi operator of the optimal symmetric phase-covariant cloning channel derived in Ref. [21]. This observation identifies the phase-covariant cloner as the distinguished physical map naturally associated with the virtual broadcasting transformation.

This fact has a direct operational consequence: the optimal symmetric phase-covariant cloning map is the `CPTP` map that best approximates the virtual broadcasting map in the diamond norm.

*Proposition 2.* Let $\hat{\mathcal{B}}$ be the `HPTP` map corresponding to $\hat{C}_\mathcal{B}$ in Eq. (24). Then,

$$\min_{\mathcal{E}:CPTP} \|\hat{\mathcal{B}} - \mathcal{E}\|_\diamond = \tfrac{2}{3}, \tag{29}$$

where $\|\mathcal{E}\|_\diamond := \max_{\rho \in \mathsf{S}(\mathcal{H}\otimes\mathcal{H})} \|(\mathcal{E} \otimes \mathcal{I})(\rho)\|_1$ is the diamond norm. The minimum is achieved when $\mathcal{E}$ is the optimal symmetric $1 \to 2$ phase-covariant qubit cloning map [21].

The proof follows the same strategy as in Ref. [4]. This result shows that optimal cloning remains the closest physically implementable approximation to virtual broadcasting even under the weaker assumption of phase covariance, extending the correspondence identified in the unitarily covariant case [4]. Quantitatively, whereas the canonical broadcasting map has minimal diamond norm distance $d - 1$ from the set of `CPTP` maps, in the qubit phase-covariant setting this distance decreases to $2/3$.

The same decomposition also provides an explicit realization of $\hat{\mathcal{B}}$ as a signed mixture of CPTP maps, which allows the simulation cost to be computed immediately.

*Proposition 3.* For any linear map $\mathcal{B} : \mathsf{L}(\mathcal{H}) \to \mathsf{L}(\mathcal{H} \otimes \mathcal{H})$ satisfying PHASE, FLIP, PERM, CLASSIC, and HP, it holds that

$$\|\mathcal{B}\|_{\blacklozenge} = \tfrac{5}{3}.$$

In the unitarily covariant case [7,8], the simulation cost of the virtual broadcasting map is $d = \dim(\mathcal{H})$. Thus, for qubits, relaxing unitary covariance to phase covariance lowers the simulation cost. Nevertheless,

$$\left(\tfrac{5}{3}\right)^2 n_Q > 2n_Q \geqslant n_1 + n_2, \tag{30}$$

so SAMPLE is still not satisfied. We therefore obtain:

*Corollary 1.* No linear map exists that simultaneously satisfies PHASE, FLIP, PERM, CLASSIC, HP, and SAMPLE.

Hence, relaxing unitary covariance to phase covariance not only brings the virtual broadcasting map strictly closer to the set of physically implementable channels but also lowers the simulation cost.

## IV. CONCLUSION

In this work, we investigated virtual phase-covariant quantum broadcasting for qubits, with the aim of understanding whether relaxing symmetry requirements can improve the operational performance of virtual broadcasting maps. By imposing phase covariance, flip covariance, permutation invariance, and classical consistency, we showed that the structure of a virtual broadcasting map is already highly constrained and can be fully characterized. Remarkably, these conditions are sufficient to imply the broadcasting property on equatorial states, as well as trace preservation on the entire state space, without assuming either requirement *a priori*.

Beyond this structural characterization, our most notable finding is the emergence of optimal phase-covariant cloning as the distinguished physical approximation to virtual broadcasting. Within the class of maps satisfying the above symmetry and consistency conditions, we identified a unique virtual broadcasting map that minimizes the simulation cost. The positive part of its optimal decomposition coincides exactly with the Choi operator of the optimal phase-covariant cloning channel. As a consequence, the latter is also the completely positive trace-preserving map closest to the virtual broadcasting map in diamond norm. This result is conceptually striking: even after relaxing full unitary covariance to phase covariance, the intimate connection between virtual broadcasting and optimal quantum cloning persists, revealing a robust structural link between nonphysical broadcasting maps and optimal physical cloning transformations.

From an operational perspective, relaxing unitary covariance does lead to quantitative improvements. In the phase-covariant setting, both the simulation cost and the diamond norm distance to the closest physical map are strictly smaller than in the unitarily covariant case. Nevertheless, these improvements are not sufficient to overcome the fundamental limitation captured by the sample-efficiency criterion. We showed that the optimal phase-covariant virtual broadcasting map still fails to satisfy SAMPLE, implying that the trivial strategy of directly distributing input copies remains strictly more efficient. Rather than a shortcoming, this no-go result strengthens the conclusion that sample inefficiency is a robust obstruction for virtual broadcasting, persisting even under substantial symmetry relaxation.

Our analysis focused on qubits. A natural direction for future work is to extend this framework to qudits. Existing results on virtual broadcasting and simulation cost [4,5,7–9] rely heavily on tools from representation theory and semidefinite optimization that exploit full unitary covariance. Since the present work operates under weaker symmetry assumptions, such techniques do not directly apply, and new structural or algebraic methods may be required. In this broader setting, it would also be interesting to investigate whether the structural features observed here, in particular the emergence of optimal cloning as the positive component of virtual broadcasting, persist beyond the qubit case.

Another promising avenue is to study virtual broadcasting beyond the single-shot regime by increasing the number of input copies and analyzing the resulting redistribution rate. In the physical setting, perfect broadcasting at infinite rate becomes possible already with a finite number of input copies—six in the unitarily covariant case and four in the phase-covariant case [25]. It is plausible that virtual operations may lower these thresholds further. In this context, a deeper understanding of the optimal signed decomposition of virtual broadcasting maps may prove crucial. While the positive part admits a clear operational interpretation as the optimal phase-covariant cloning channel, the negative part defines a completely positive trace-preserving map with the same symmetry constraints but with support orthogonal to the cloning sector. Clarifying whether this negative component admits an interpretation in terms of a known physical transformation, such as a phase-covariant analog of anticloning or optimal state inversion [26], or whether it represents a genuinely new primitive arising from virtual constructions, remains an open problem.

More broadly, our results clarify the structural constraints imposed on virtual quantum operations by operational requirements. The persistence of the correspondence between virtual broadcasting and optimal cloning, together with the failure of sample efficiency even under reduced symmetry assumptions, indicates that these features are not artifacts of full unitary covariance but reflect more intrinsic limitations. From this perspective, virtual broadcasting provides a controlled setting in which to analyze how nonphysical linear maps approximate physical transformations, and to delineate the limits of what can be achieved through classical simulation of quantum processes.

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## DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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[1] W. K. Wootters and W. H. Zurek, A single quantum cannot be cloned, Nature (London) **299**, 802 (1982).

[2] D. Dieks, Communication by EPR devices, Phys. Lett. A **92**, 271 (1982).

[3] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Noncommuting mixed states cannot be broadcast, Phys. Rev. Lett. **76**, 2818 (1996).

[4] A. J. Parzygnat, J. Fullwood, F. Buscemi, and G. Chiribella, Virtual quantum broadcasting, Phys. Rev. Lett. **132**, 110203 (2024).

[5] H. Yao, X. Liu, C. Zhu, and X. Wang, Optimal unilocal virtual quantum broadcasting, Phys. Rev. A **110**, 012458 (2024).

[6] Y. Zheng, X. Nie, H. Liu, Y. Luo, D. Lu, and X. Liu, Experimental virtual quantum broadcasting, Phys. Rev. A **111**, L060402 (2025).

[7] Y. Xiao, X. Liu, and Z. Liu, No practical quantum broadcasting: Even virtually, Phys. Rev. Lett. **135**, 090202 (2025).

[8] Y. Xiao, X. Liu, and Z. Liu, No practical quantum broadcasting: General framework, Phys. Rev. Res. **7**, 033194 (2025).

[9] H. Yao, J. Xie, X. Zhao, C. Zhu, R. Chen, and X. Wang, Quantifying unxtendibility via virtual state extension, arXiv:2510.24895.

[10] F. Buscemi, M. Dall'Arno, M. Ozawa, and V. Vedral, Direct observation of any two-point quantum correlation function, arXiv:1312.4240.

[11] F. Buscemi, M. Dall'Arno, M. Ozawa, and V. Vedral, Universal optimal quantum correlator, Int. J. Quantum Inf. **12**, 1560002 (2014).

[12] M. Rossini, D. Maile, J. Ankerhold, and B. I. C. Donvil, Single-qubit error mitigation by simulating non-Markovian dynamics, Phys. Rev. Lett. **131**, 110603 (2023).

[13] K. Temme, S. Bravyi, and J. M. Gambetta, Error mitigation for short-depth quantum circuits, Phys. Rev. Lett. **119**, 180509 (2017).

[14] J. Jiang, K. Wang, and X. Wang, Physical implementability of linear maps and its application in error mitigation, Quantum, **5**, 600 (2021).

[15] R. Takagi, S. Endo, S. Minagawa, and M. Gu, Fundamental limits of quantum error mitigation, npj Quantum Information **8**, 114 (2022).

[16] J. Fullwood and A. J. Parzygnat, On quantum states over time, Proc. R. Soc. A **478**, 20220104 (2022).

[17] A. J. Parzygnat and J. Fullwood, From time-reversal symmetry to quantum Bayes' rules, PRX Quantum **4**, 020334 (2023).

[18] S. H. Lie and N. H. Y. Ng, Uniqueness of quantum state over time function, Phys. Rev. Res. **6**, 033144 (2024).

[19] R. F. Werner, Optimal cloning of pure states, Phys. Rev. A **58**, 1827 (1998).

[20] G. M. D'Ariano and P. L. Presti, Optimal nonuniversally covariant cloning, Phys. Rev. A **64**, 042308 (2001).

[21] G. M. D'Ariano and C. Macchiavello, Optimal phase-covariant cloning for qubits and qutrits, Phys. Rev. A **67**, 042306 (2003).

[22] F. Buscemi, G. M. D'Ariano, and C. Macchiavello, Economical phase-covariant cloning of qudits, Phys. Rev. A **71**, 042327 (2005).

[23] B. Regula, R. Takagi, and M. Gu, Operational applications of the diamond norm and related measures in quantifying the non-physicality of quantum maps, Quantum, **5**, 522 (2021).

[24] Wolfram Research Inc., Wolfram cloud (2025), Available at https://www.wolframcloud.com

[25] F. Buscemi, G. M. D'Ariano, C. Macchiavello, and P. Perinotti, Universal and phase-covariant superbroadcasting for mixed qubit states, Phys. Rev. A **74**, 042309 (2006).

[26] F. Buscemi, G. M. D'Ariano, P. Perinotti, and M. F. Sacchi, Optimal realization of the transposition maps, Phys. Lett. A **314**, 374 (2003).