Open Systems & Information Dynamics
Vol. 27, No. 4 (2020) 2050017 (23 pages)
DOI:S1230161220500171

World Scientific
www.worldscientific.com

Explicit Construction of Optimal Witnesses for Input-Output Correlations Attainable by Quantum Channels

Michele Dall'Arno

Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwakecho, Sakyoku, Kyoto 606-8502, Japan

and

Faculty of Education and Integrated Arts and Sciences, Waseda University
1-6-1 Nishiwaseda, Shinjuku-ku, Tokyo 169-8050, Japan

e-mail: dallarno.michele@yukawa.kyoto-u.ac.jp

Sarah Brandsen

Department of Physics, Duke University, Durham, North Carolina 27708, USA

sarah.brandsen@duke.edu

Francesco Buscemi

Graduate School of Informatics, Nagoya University, Chikusa-ku, Nagoya, 464-8601, Japan

e-mail: buscemi@i.nagoya-u.ac.jp

(Received: September 2, 2020; Accepted: December 2, 2020; Published: April 29, 2021)

Abstract. Given a quantum channel — that is, a completely positive trace-preserving linear map — as the only communication resource available between two parties, we consider the problem of characterizing the set of classical noisy channels that can be obtained from it by means of suitable classical-quantum encodings and quantum-classical decodings, respectively, on the sender's and the receiver's side. We consider various classes of linear witnesses and compute their optimum values in closed form for several classes of quantum channels. The witnesses that we consider here are formulated as *communication games*, in which Alice's aim is to exploit a single use of a given quantum channel to help Bob guess some information she has received from an external referee.

Keywords: Quantum channels, quantum correlations, communication games.

1. Introduction

Suppose Alice and Bob play a two-party, quantum-enhanced version of the popular game charades. In each run, Alice's aim is to help Bob guess some piece of information that she has received from a referee. As in the traditional charades game, there is a bottleneck in the communication channel, in this case created by a given noisy quantum channel. After each round, the referee provides a payoff that depends on both the information Alice was provided and Bob's guess. The parties' aim is to maximize the average payoff, which depends only on the channel and the game, by optimizing Alice's encoding and Bob's decoding.

Here, we introduce the communication utility of any given quantum channel for any given communication game as the average payoff after asymptotically many runs. Communication games are linear functionals (i.e., *witnesses*) on the set of classical noisy channels (i.e., *quantum signalling correlations*) that can be obtained from the given quantum channel, and the corresponding communication utility constitutes the optimal value for any such a witness.

For any given channel and game, the problem of computing the communication utility, as well as the encoding-decoding achieving it, can be generally framed as a semi-definite programming problem. However, here we are interested in those cases where a closed-form solution is possible. To this aim, we restrict to the following classes of games:

- *Unbiased games*, where any of Bob's possible outcomes generates the same average payoff;
- *Discrimination games* where the payoff is a diagonal matrix;
- *Binary-output games* where Bob has two possible outcomes;
- *Binary-input-output games* where Alice has two possible inputs and Bob has two possible outcomes.

For any arbitrary game, we derive the communication utility of any unitary, trace-class, erasure, dephasing, and quantum-classical channel, generalizing a result by Frenkel and Weiner [1] that applies to the identity channel. For any unbiased game, we derive the communication utility of any depolarizing channel. We apply our result for unitary channels to the case of discrimination games, providing a simplified proof of a result by Elron and Eldar [2]. We show that any binary-output game is either trivial, or equivalent to a binary-input-output discrimination game. For any such game, we show that the optimal encoding consists of two orthogonal pure states, regardless of whether the channel is commutativity preserving. Using these facts, we extend previous results [3, 4] to derive the communication utility of any Pauli, amplitude-damping, optimal 1-to-2 universal cloning, and shifted-depolarizing channel for any binary-output game.

The paper is structured as follows. In Sect. 2 we formalize the problem of evaluating the communication utility of quantum channels. We address such a problem for arbitrary games in Sect. 3. We specialize our results to the cases of unbiased, discrimination, and binary-output games in Sects. 4,

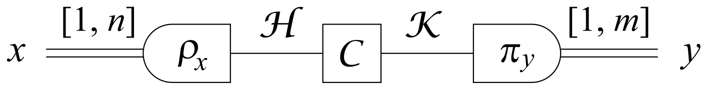

Fig. 1: The communication setup.

5, and 6, respectively. Finally, we summarize our results and discuss some outlooks in Sect. 7.

2. Utility of Quantum Channels

We recall some standard facts from quantum information theory [5]. Any quantum system is associated with a Hilbert space $\mathcal{H}$, and we denote with $\mathcal{L}(\mathcal{H})$ the space of linear operators on $\mathcal{H}$. Any quantum state in $\mathcal{H}$ is represented by a density matrix $\rho \in \mathcal{L}(\mathcal{H})$, namely a positive semidefinite unit-trace operator. Any quantum transformation from $\mathcal{H}$ to $\mathcal{K}$ is represented by a quantum channel $\mathcal{C} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$, namely a completely positive trace-preserving linear map. Any quantum measurement on $\mathcal{H}$ is represented by a positive operator valued measure (POVM) $\{\pi_y \in \mathcal{L}(\mathcal{H})\}$, namely a family of positive semidefinite operators such that $\sum_y \pi_y = \mathbb{1}$, where $\mathbb{1}$ denotes the identity operator.

The communication setup under consideration can be framed as a quantum game played by two parties, Alice and Bob, and an external referee. Prior to starting the game, the parties are allowed to establish a common strategy. In each run, the following occurs:

1. The referee gives as an input to Alice the value $x \in [1, n]$ of a random variable X with prior probability p_x, known in advance to the parties;
2. Alice and Bob are allowed to perform one-way communication over a single use of a quantum channel $\mathcal{C}$ from Alice to Bob;
3. Bob returns to the referee the value $y \in [1, m]$ of a random variable Y;
4. The referee provides a payoff according to the function $u_{x,y} \in \mathbb{R}$, known in advance to the parties.

In the absence of any previously shared resource, the most general strategy allowed by quantum theory is for Alice to encode her input x into a quantum state ρ_x, and for Bob to decode his input $\mathcal{C}(\rho_x)$ by means of a POVM $\{\pi_y\}$. The resulting setup is shown in Fig. 1.

Then the expected average payoff is given by

$$\langle u \rangle_{\{\rho_x\},\{\pi_y\}} = \sum_{x=1}^{n} \sum_{y=1}^{m} p_x \operatorname{Tr}[\mathcal{C}(\rho_x)\pi_y] u_{x,y} \,. \tag{1}$$

From (1) it immediately follows that two games with the same quantity

$$g_{x,y} := p_x u_{x,y}$$

have the same average payoff, hence g identifies a class of equivalence among games. We can now introduce a measure of how useful a channel $\mathcal{C}$ is in maximizing the average payoff of a game g.

DEFINITION 1 (Communication utility) The *communication utility* $U(\mathcal{C}, g)$ of a quantum channel $\mathcal{C}$ for game g is the maximum over any encoding $\{\rho_x\}$ and decoding $\{\pi_y\}$ of the average payoff $\langle u \rangle_{\{\rho_x\},\{\pi_y\}}$, namely

$$U(\mathcal{C}, g) = \max_{\{\rho_x\},\{\pi_y\}} \langle u \rangle_{\{\rho_x\},\{\pi_y\}} \,.$$

In the remaining of this work we derive the utility of several classes of quantum channels. For clarity, we provide a glossary of the channels considered in this work in Tab. 1. We also provide a summary of our main results in Tab. 2.

3. Utility for Arbitrary Games

In this section we consider arbitrary games. Let us begin with some general results. The following Lemma provides a simple, channel-independent upper bound to the utility for any given game.

LEMMA 1 *For any channel $\mathcal{C}$ and any game g, the utility $U(\mathcal{C}, g)$ is upper bounded by*

$$U(\mathcal{C}, g) \le \sum_x \max_y g_{x,y} \,.$$

Proof. By replacing the conditional probability $\operatorname{Tr}[\mathcal{C}(\rho_x)\pi_y]$ with an arbitrary conditional probability $p_{y|x}$ and taking the supremum over such conditional probabilities, one clearly has

$$U(\mathcal{C}, g) \le \sup_{p_{y|x}} \sum_{x=1}^{n} \sum_{y=1}^{m} p_{y|x} g_{x,y} \,.$$

Table 1: Glossary of quantum channels considered in this work. All channels are formally defined in the text. Not otherwise specified channels are denoted with $\mathcal{C}$ in the text.

Definition	**Name**
$\mathcal{U}(\rho) := U\rho U^\dagger$	Unitary
$\mathcal{F}_\lambda(\rho) := \lambda\rho + (1-\lambda)\sum_k \langle k\vert\rho\vert k\rangle\vert k\rangle\!\langle k\vert$	Dephasing
$\mathcal{T}(\rho) := \mathrm{Tr}[\rho]\sigma$	Trace class
$\mathcal{E}_\lambda(\rho) := \lambda\rho \oplus (1-\lambda)\,\mathrm{Tr}[\rho]\vert\phi\rangle\!\langle\phi\vert$	Erasure
$\mathcal{M}(\rho) := \sum_y \mathrm{Tr}[\rho\pi_y]\vert y\rangle\!\langle y\vert$	QC
$\mathcal{D}_\lambda(\rho) := \lambda\rho + (1-\lambda)\,\mathrm{Tr}[\rho]\mathbb{1}/d$	Depolarizing
$\mathcal{P}_{\vec{\lambda}}(\rho) := \lambda_0\rho + \sum_{k=1}^3 \lambda_k\sigma_k\rho\sigma_k$	Pauli
$\mathcal{A}_\eta\begin{pmatrix} 1-\beta & \gamma \\ \gamma^* & \beta \end{pmatrix} = \begin{pmatrix} 1-\eta\beta & \sqrt{\eta}\gamma \\ \sqrt{\eta}\gamma^* & \eta\beta \end{pmatrix}$	Amplitude damping
$\mathcal{S}_\lambda(\rho) := \lambda\rho + (1-\lambda)\,\mathrm{Tr}[\rho]\sigma$	Shifted depolarizing
$\mathcal{N}(\rho) := \frac{2}{d+1}P_s(\rho\otimes\mathbb{1})P_s$	1 → 2 Cloning

For any fixed x, the optimal probability distribution $p_{y|x}$ is given by $p_{y|x} = \delta_{y,y^*}$, where $y^* := \sup_y g_{x,y}$, therefore one has

$$\sup_{p_{y|x}} \sum_{x=1}^{n}\sum_{y=1}^{m} p_{y|x} g_{x,y} \;=\; \sum_x \sup_y g_{x,y}\,,$$

from which the statement immediately follows. □

The following Lemma characterizes a subclass of the linear transformations of game g under which the utility $U(\mathcal{C}, g)$ transforms linearly, for any channel $\mathcal{C}$.

LEMMA 2 *For any game g consider the game g' such that $g'_{x,y} := \alpha(g_{x,y} + \beta_x)$, for any $\alpha \geq 0$ and some $\{\beta_x\}$. Then for any channel $\mathcal{C}$ we have*

$$U(\mathcal{C}, g') \;=\; \alpha\Big[U(\mathcal{C}, g) + \sum_x \beta_x\Big]\,.$$

Moreover, $U(\mathcal{C}, g)$ and $U(\mathcal{C}, g')$ are attained by the same encoding $\{\rho_x\}$ and decoding $\{\pi_y\}$.

Table 2: Summary of our main results, namely the utilities $U(\mathcal{C}, g)$ of several channels $\mathcal{C}$ for several classes of games.

$\mathcal{C}$	**Game** g	**Utility** $U(\mathcal{C}, g)$
$\mathcal{F}_\lambda$	Any	$U(\mathcal{U}, g)$
$\mathcal{T}$	Any	$\max_y \sum_x g_{x,y}$
$\mathcal{E}_\lambda$	Any	$\lambda U(\mathcal{U}, g) + (1-\lambda) \max_y \sum_x g_{x,y}$
$\mathcal{M}$	Any	$\sum_x \lambda_x$
$\mathcal{D}_\lambda$	Unbiased	$\lambda U(\mathcal{U}, g) + (1-\lambda) \sum_x g_{x,0}$
$\mathcal{U}$	Discrimination	$\sum_{x=0}^{d-1} g_x \Theta(g_x)$
$\mathcal{P}_{\vec{\lambda}}$	Binary	$\max \left(g_0, \frac{1+\max_{k\geq 1} \lvert 2(\lambda_0+\lambda_k)-1 \rvert}{2} \right)$
$\mathcal{A}_\eta$	Binary	$\frac{1+\sqrt{1-4p(1-\eta)+4p^2(1-\eta)}}{2}$
$\mathcal{S}_\lambda$	Binary	$\max \left[g_0, \frac{1+\lambda+(1-\lambda)(1-2s_{d-1})(2g_0-1)}{2} \right]$
$\mathcal{N}$	Binary	$\frac{d+g_0}{d+1}$

Proof. By Definition 1 one immediately has

$$U(\mathcal{C}, g') \ := \ \alpha \sup_{\{\rho_x\}, \{\pi_y\}} \sum_{x,y} \mathrm{Tr}[\mathcal{C}(\rho_x)\pi_y](g_{x,y} + \beta_x) \, .$$

Since POVMs are decompositions of the identity, namely $\sum_y \pi_y = \mathbb{1}$, and channels are trace-preserving, namely $\mathrm{Tr}[\mathcal{C}(\rho_x)] = \mathrm{Tr}[\rho_x] = 1$, one has

$$U(\mathcal{C}, g') \ = \ \alpha \Big[\sup_{\rho_x, \pi_y} \sum_{x,y} \mathrm{Tr}[\mathcal{C}(\rho_x)\pi_y] g_{x,y} + \sum_x \beta_x \Big] \, ,$$

for any encoding $\{\rho_x\}$ and POVM $\{\pi_y\}$. Since only the first term depends on the encoding $\{\rho_x\}$ and decoding $\{\pi_y\}$, one has that $U(\mathcal{C}, g)$ and $U(\mathcal{C}, g')$ are attained by the same encoding and decoding. □

As an immediate consequence of a recent breakthrough by Frenkel and Weiner [1], the utility of any identity quantum channel, that we denote by id, is equal to the utility of any identity classical channel in the same dimension. Accordingly, the utility of any unitary and dephasing channel is also equal to the utility of the identity classical channel in the same dimension, as follows.

DEFINITION 2 (Unitary channel) The action of any unitary channel $\mathcal{U} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ on any state $\rho \in \mathcal{L}(\mathcal{H})$ is given by $\mathcal{U}(\rho) = U\rho U^\dagger$, for some unitary U.

PROPOSITION 1 (Frenkel and Weiner [1]) *The utility $U(\mathcal{U}, g)$ of any unitary channel $\mathcal{U}$ for any game g is attained by an orthonormal encoding $\{\rho_x\}$ and a projective decoding $\{\pi_y\}$ that are simultaneously diagonalizable.*

DEFINITION 3 The action of the dephasing channel $\mathcal{F}_\lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ on any state $\rho \in \mathcal{L}(\mathcal{H})$ is given by

$$\mathcal{F}_\lambda(\rho) \;:=\; \lambda\rho + (1-\lambda)\sum_{k=1}^{d} \langle k|\rho|k\rangle |k\rangle\langle k| \,,$$

where $d := \dim \mathcal{H}$ is the dimension of Hilbert space $\mathcal{H}$ and $\{|k\rangle\}$ is some o.n.b.

PROPOSITION 2 (Frenkel and Weiner [1]) *The utility $U(\mathcal{F}_\lambda, g)$ of any dephasing channel $\mathcal{F}_\lambda$ for any game g is attained by an orthonormal encoding $\{\rho_x\}$ and a projective decoding $\{\pi_y\}$ along basis $\{|k\rangle\}$.*

At the other side of the spectrum of channels there lie the trace class channels, that is those channel that cannot convey any information. Hence, their utility corresponds to a trivial guessing on Bob's side, as follows.

DEFINITION 4 (Trace-type channel) A channel $\mathcal{T}$ is trace-type if and only if there exists a state σ such that $\mathcal{T}(\rho) = \mathrm{Tr}[\rho]\sigma$ for any state ρ.

PROPOSITION 3 *For any game g, the utility $U(\mathcal{T}, g)$ of any trace-type channel $\mathcal{T}$ is given by*

$$U(\mathcal{T}, g) \;=\; \max_y \sum_x g_{x,y} \,.$$

Any encoding is optimal, and the optimal decoding is $\pi_y = \delta_{y,y^*}\mathbb{1}$, where $y^* := \arg\max_y \sum_x g_{x,y}$.

Proof. The statement directly follows from Definition 4. □

Between unitary and trace class channels are erasure channels, that is, channels that probabilistically declare an error while otherwise achieving noiseless communication. Accordingly, their utility is the convex combination of the utility of a noiseless channel and a trace class channel, as follows.

DEFINITION 5 (Erasure channel) The action of the erasure channel $\mathcal{E}_\lambda(\rho) : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ on any state $\rho \in \mathcal{L}(\mathcal{H})$ is given by

$$\mathcal{E}_\lambda(\rho) \ := \ \lambda\rho \oplus (1-\lambda)\operatorname{Tr}[\rho]|\phi\rangle\langle\phi|\,,$$

where $|\phi\rangle \in \mathcal{K}$.

PROPOSITION 4 *For any game g, the utility $U(\mathcal{E}_\lambda, g)$ of erasure channel $\mathcal{E}_\lambda$ is given by*

$$U(\mathcal{E}_\lambda, g) \ = \ \lambda U(\mathrm{id}, g) + (1-\lambda)U(\mathcal{T}, g)\,.$$

By denoting with $\{\rho_x^*\}$ and $\{\pi_y^*\}$ any encoding and decoding attaining $U(\mathrm{id}, g)$, the encoding $\{\rho_x^*\}$ and the decoding $\{\pi_y^* \oplus \delta_{y,y^*}\mathbb{1}_\mathcal{K}\}$ attain $U(\mathcal{E}_\lambda, g)$, where $y^* := \arg\max_y \sum_x g_{x,y}$.

Proof. By direct computation one has

$$\begin{aligned} U(\mathcal{E}_\lambda, g) \ &= \ \max_{\{\rho_x\},\{\pi_y\}} \sum_{x,y} \Big[\lambda \operatorname{Tr}[\rho_x \pi_y] + (1-\lambda)\langle\phi|\pi_y|\phi\rangle\Big] g_{x,y} \\ &\leq \ \lambda \max_{\{\rho_x\},\{\pi_y\}} \sum_{x,y} \operatorname{Tr}[\rho_x\pi_y] g_{x,y} + (1-\lambda) \max_{\{\pi_y\}} \sum_{x,y} \langle\phi|\pi_y|\phi\rangle g_{x,y}\,, \end{aligned}$$

where the maxima are over encodings $\{\rho_x \in \mathcal{L}(\mathcal{H})\}$ and decodings $\{\pi_y \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})\}$. Since $\operatorname{Tr}[\rho_x\pi_y] = \operatorname{Tr}[\rho_x P_\mathcal{H} \pi_y P_\mathcal{H}]$ and $\langle\phi|\pi_y|\phi\rangle = \langle\phi|P_\mathcal{K}\pi_y P_\mathcal{K}|\phi\rangle$, where $P_\mathcal{H}$ and $P_\mathcal{K}$ are the projectors on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, w.l.o.g. the first and second maxima in the last step can be taken over decodings $\{\pi_y \in \mathcal{L}(\mathcal{H})\}$ and $\{\pi_y \in \mathcal{L}(\mathcal{K})\}$ respectively. Then by Definition 1 for the first maximum one has

$$\max_{\rho_x,\pi_y} \sum_{x,y} \operatorname{Tr}[\rho_x\pi_y] g_{x,y} \ =: \ U(\mathrm{id}, g)\,,$$

and the second maximum is trivially achieved when $\pi_y = \delta_{y,y^*}\mathbb{1}_\mathcal{K}$, where $y^* := \arg\max_y \sum_x g_{x,y}$, namely

$$\max_{\pi_y} \sum_{x,y} \langle\phi|\pi_y|\phi\rangle g_{x,y} \ = \ \max_y \sum_x g_{x,y}\,.$$

By explicit computation the encodings $\{\rho_x^*\}$ and decodings $\{\pi_y^* \oplus \delta_{y,y^*}\mathbb{1}_\mathcal{K}\}$ saturate this upper bound. □

We conclude our study of the utility of quantum channels for arbitrary games by considering quantum–classical channels, which are used to represent the most general demolishing quantum measurement.

DEFINITION 6 (Quantum–classical channel) The action of the quantum–classical (QC) channel $\mathcal{M} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ over any state $\rho \in \mathcal{L}(\mathcal{H})$ is given by

$$\mathcal{M}(\rho) \; := \; \sum_y \mathrm{Tr}[\rho \pi_y] |y\rangle\langle y| \, ,$$

for some POVM $\{\pi_y \in \mathcal{L}(\mathcal{H})\}$ and some o.n.b. $\{|y\rangle \in \mathcal{K}\}$.

PROPOSITION 5 *For any game g, the utility $U(\mathcal{M}, g)$ of QC channel $\mathcal{M}$ is given by*

$$U(\mathcal{M}, g) \; = \; \max_S \sum_x \lambda_x \, ,$$

where S is any deterministic stochastic matrix, that is, any matrix with entries 0 or 1 such that $\sum_z S_{z,y} = 1$ for any y, and λ_x is the largest eigenvalue of $\sum_{y,z} g_{x,z} S_{z,y} \pi_y$. If $|\lambda_x\rangle$ is the corresponding eigenvector, the optimal encoding is given by $\rho_x = |\lambda_x\rangle\langle\lambda_x|$.

Proof. Since classical decodings are represented by stochastic matrices, due to the linearity of the figure of merit the optimal decoding is a deterministic stochastic matrix. Hence, by Definition 1 one has

$$U(\mathcal{M}, g) \; := \; \max_{\{\rho_x\}, S} \sum_{x,y,z} g_{x,z} S_{z,y} \, \mathrm{Tr}\left[\rho_x \pi_y\right] \; \leq \; \max_S \sum_x \lambda_x \, ,$$

where the inequality is saturated if and only if encoding $\{\rho_x\}$ is as given in Proposition 5. □

4. Utility for Unbiased Games

In this section we consider unbiased games, which are games where any of Bob's possible outcomes generate the same average payoff. Due to Lemma 2, without loss of generality (w.l.o.g. for short) we can take such an average to be zero.

DEFINITION 7 (Unbiased game) We call unbiased game any game g such that $\sum_x g_{x,y} = 0$.

DEFINITION 8 (Depolarizing channel) The action of the depolarizing channel $\mathcal{D}_\lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ on any state $\rho \in \mathcal{L}(\mathcal{H})$ is given by

$$\mathcal{D}_\lambda(\rho) \; := \; \lambda\rho + (1 - \lambda) \, \mathrm{Tr}[\rho] \frac{\mathbb{1}}{d} \, ,$$

where $d := \dim \mathcal{H}$ is the dimension of Hilbert space $\mathcal{H}$.

PROPOSITION 6 *For any unbiased game g, the utility $U(\mathcal{D}_\lambda, g)$ of the depolarizing channel $\mathcal{D}_\lambda$ is given by*

$$U(\mathcal{D}_\lambda, g) \;=\; \lambda U(\mathrm{id}, g)\,,$$

The encoding $\{\rho_x\}$ and decoding $\{\pi_y\}$ attaining $U(\mathrm{id}, g)$ also attain $U(\mathcal{D}_\lambda, g)$.

Proof. By Definition 1 one immediately has

$$U(\mathcal{D}_\lambda, g) \;:=\; \max_{\{\rho_x\},\{\pi_y\}} \sum_{x,y} \left[\lambda \operatorname{Tr}[\rho_x \pi_y] g_{x,y} + \frac{1-\lambda}{d} \operatorname{Tr}[\pi_y] g_{x,y}\right].$$

Since any POVM is a decomposition of the identity, namely $\sum_y \pi_y = \mathbb{1}$, and $\sum_x g_{x,y} = 0$ for any y, one has

$$\sum_{x,y} \left[\lambda \operatorname{Tr}[\rho_x \pi_y] g_{x,y} + \frac{1-\lambda}{d} \operatorname{Tr}[\pi_y] g_{x,y}\right] \;=\; \lambda \sum_{x,y} \operatorname{Tr}[\rho_x \pi_y] g_{x,y}\,,$$

for any encoding $\{\rho_x\}$ and decoding $\{\pi_y\}$. Then one has

$$\max_{\rho_x, \pi_y} \lambda \sum_{x,y} \operatorname{Tr}\left[\rho_x \pi_y\right] g_{x,y} \;=\; \lambda U\left(\mathrm{id}, g\right)$$

for any unitary channel $\mathcal{U}$. □

5. Utility for Discrimination Games

In this section we consider discrimination games, that is games where the payoff is a diagonal matrix and thus the numbers of inputs and outputs are equal, that is $m = n$.

DEFINITION 9 (Discrimination game) We call discrimination game any game g such that $g_{x,y} = \delta_{x,y} g_x$, for some g_x.

According to Def. 9, the utility of any channel $\mathcal{C}$ for discrimination game g is given by

$$U(\mathcal{C}, g) \;=\; \sup_{\{\rho_x\},\{\pi_x\}} \sum_x \operatorname{Tr}[\mathcal{C}(\rho_x)\pi_x] g_x\,.$$

We have shown in Proposition 1 that w.l.o.g. the optimization of the utility for any unitary channel can be restricted to a finite set of encodings and decodings. In the following we specify such a result by deriving a closed form for the utility $U(\mathcal{U}, g)$ of any unitary channel $\mathcal{U}$ for any discrimination game g. This extends and simplifies the proof of a result by Elron and Eldar [2].

PROPOSITION 7 *For any discrimination game g, the utility $U(\mathcal{U}, g)$ of any unitary channel $\mathcal{U}$ is given by*

$$U(\mathcal{U}, g) \;=\; \sum_{x=0}^{d-1} g_x \Theta(g_x)\,,$$

where w.l.o.g. we take $g_0 \geq g_1 \geq \ldots \geq g_{n-1}$ and $\Theta(x)$ is the Heaviside step function.

Proof. For a unitary channel $\mathcal{U}$, the unitary operator can be absorbed into the encoding and the decoding such that, w.l.o.g., we can consider only the case of the identity channel $\mathcal{U} = \mathrm{id}$.

If $g_x \leq 0$ for any x then by Lemma 1 one has that $U(\mathcal{U}, g) \leq 0$. This bound is attained by encoding $\{\rho_x\}$ and decoding $\{\pi_y\}$ given by

$$\rho_x \;=\; \begin{cases} |1\rangle\langle 1| & \text{if} \quad x = 0, \\ |0\rangle\langle 0| & \text{if} \quad x > 0, \end{cases}$$

and

$$\pi_x \;=\; \begin{cases} |0\rangle\langle 0| & \text{if} \quad x = 0, \\ \dfrac{1}{n-1}(\mathbb{1} - |0\rangle\langle 0|) & \text{if} \quad x > 0\,. \end{cases}$$

Then let $g_0 > 0$. For any $\{\rho_x\}$ and $\{\pi_x\}$ one has

$$\sum_x \mathrm{Tr}[\rho_x \pi_x] g_x \;\leq\; \sum_x \mathrm{Tr}[\rho_x \pi'_x] g_x\,,$$

where for any $x \geq 1$ one has

$$\pi'_x \;:=\; \begin{cases} \pi_x & \text{if} \quad g_x \geq 0, \\ 0 & \text{if} \quad g_x < 0 \end{cases}$$

and $\pi'_0 = \mathbb{1} - \sum_{x \neq 0} \pi'_x$, therefore $\pi'_0 \geq \pi_0$.

Therefore, the discrimination game g'_x given by

$$g'_x \;:=\; g_x \Theta(g(x))\,,$$

is such that $U(\mathcal{U}, g') = U(\mathcal{U}, g)$. Moreover $U(\mathcal{U}, g)$ and $U(\mathcal{U}, g')$ are attained by the same encoding and decoding.

By denoting with $||\cdot||_\infty$ the largest singular value, By Def. 1 one has

$$U(\mathcal{U}, g') \;:=\; \max_{\{\rho_x\},\{\pi_x\}} \sum_x \mathrm{Tr}[\rho_x \pi_x] g'_x \;=\; \max_{\{\pi_x\}} \sum_x g'_x ||\pi_x||_\infty\,,$$

where the second equality represents an upper bound saturated when ρ_x is the projector on the largest eigenvalue of π_x, for any x.

We relax the condition $\sum_x \pi_x = \mathbb{1}$ to the weaker condition $\sum_x \mathrm{Tr}[\pi_x] = d$, where $d := \dim \mathcal{H}$. Thus we have $\sum_x ||\pi_x||_\infty \leq d$. Moreover, since $\pi_x \leq \mathbb{1}$ for any x, one has $||\pi_x||_\infty \leq 1$ for any x. Then one clearly has

$$\max_{\{\pi_x\}} \sum_x g'_x ||\pi_x||_\infty = \sum_{x=0}^{d-1} g'_x \,,$$

where the equality represents an upper bound saturated when $\pi_x = |x\rangle\langle x|$ for any $x = 0, \ldots, d-1$, so the statement remains proved. □

6. Utility for Binary-Output Games

In this section we consider binary-output games, which are games with $n = 2$ outputs (but an arbitrary number m of inputs). It is perhaps surprising that any binary-output game can be recast as a binary-input-output (binary for short) discrimination game, that is a diagonal game with $m = n = 2$ inputs and outputs, parametrized by a single real parameter, as shown by the following Lemma.

LEMMA 3 *For any binary-output game g, there exists a binary-input-output discrimination game g' such that*

$$U(\mathcal{C}, g) = a \cdot U(\mathcal{C}, g') + b \,,$$

for any channel $\mathcal{C}$. In the above formula one has $g' = \mathrm{diag}(g_0, 1 - g_0)$, where

$$\begin{aligned} a &:= \sum_x |g_{x,0} - g_{x,1}| \,, \\ b &:= \sum_x \min_y g_{x,y} \,, \\ g_0 &:= \frac{1}{a} \sum_x (g_{x,0} - g_{x,1})\, \Theta\, (g_{x,0} - g_{x,1}) \,, \end{aligned}$$

where $\Theta(x)$ is the Heaviside step function. Moreover, the same encoding and decoding achieving $U(\mathcal{C}, g')$ also achieve $U(\mathcal{C}, g)$.

Proof. For any binary-output game g, consider another binary-output game $\tilde{g}$ defined as

$$\tilde{g}_{x,y} := \frac{1}{a}(g_{x,y} - \min_z g_{x,z}) \,,$$

for any x and y. Hence, for any x one has that $\tilde{g}_{x,y} \geq 0$ for any y, with equality for at least one value of y. Due to Lemma 2 one immediately has $U(\mathcal{C}, g) = a \cdot U(\mathcal{C}, \tilde{g}) + b$ and the encodings and decodings attaining $U(\mathcal{C}, g)$ and $U(\mathcal{C}, \tilde{g})$ are the same.

For any encodings $\{\rho_x\}$, any decodings $\{\pi_y\}$, and any x_0 and x_1 such that $\tilde{g}_{x_0,1} = \tilde{g}_{x_1,1} = 0$, let w.l.o.g. $\mathrm{Tr}[\mathcal{C}(\rho_{x_1})\pi_0] \geq \mathrm{Tr}[\mathcal{C}(\rho_{x_0})\pi_0]$. Therefore, replacing state ρ_{x_0} with state ρ_{x_1} increases the average payoff. Hence, the utility is attained when states $\{\rho_x\}$ coincide for any x such that $\tilde{g}_{x,1} = 0$, and the same for any x such that $\tilde{g}_{x,0} = 0$. Hence, the utility $U(\mathcal{C}, \tilde{g}) = U(\mathcal{C}, g')$, and the statement remains proved. □

Therefore, w.l.o.g. in the following we consider binary discrimination games g, which are games with $m = 2$ inputs and $n = 2$ outputs, such that the payoff g is diagonal and constitutes a probability distribution.

For any such a game, as an immediate consequence of Helstrom's theorem [6] one has that

$$U(\mathcal{C}, g) \ = \ \max_{\{\rho_x\}} \frac{1}{2} \left(1 + \|H\|_1\right).$$

where $|| \cdot ||_1 := \mathrm{Tr}[| \cdot |]$ denotes the 1-Schatten norm, and

$$H \ := \ g_0 \mathcal{C}(\rho_0) - (1 - g_0) \mathcal{C}(\rho_1),$$

is the Helstrom matrix.

For any commutativity-preserving channel, that is any channel $\mathcal{C}$ such that $[\mathcal{C}(\rho), \mathcal{C}(\sigma)] = 0$ whenever $[\rho, \sigma] = 0$, and any binary discrimination game g, it is straightforward that the optimal encoding is also orthogonal, that is $\langle \phi_0 | \phi_1 \rangle = 0$. However, it is perhaps surprising that this fact holds true also for noncommutativity preserving channels, as stated by the following Lemma.

LEMMA 4 *For any channel $\mathcal{C}$ and any binary discrimination game g, the utility $U(\mathcal{C}, g)$ is attained by an orthonormal encoding $\{\phi_x^*\}$, that is $\langle \phi_{x_0}^* | \phi_{x_1}^* \rangle = \delta_{x_0,x_1}$.*

Proof. For any encoding $\{|\phi_x\rangle\}$, let us define the matrix K and a spectral decomposition as follows:

$$K \ := \ g_0 |\phi_0\rangle\langle\phi_0| - (1 - g_0) |\phi_1\rangle\langle\phi_1| \ =: \ \sum_k \lambda_k |k\rangle\langle k|,$$

and consider the dephasing channel $\mathcal{P}_0(\cdot) := \sum_k \langle k| \cdot |k\rangle |k\rangle\langle k|$ on a basis of eigenvectors of K. One has that

$$K \ = \ \mathcal{P}_0(K) \ = \ g_0 \sigma_0 - (1 - g_0) \sigma_1,$$

where $\sigma_x := \mathcal{P}_0(|\phi_x\rangle\langle\phi_x|)$ and therefore $\sigma_x \geq 0$, $\mathrm{Tr}[\sigma_x] = 1$, and $[\sigma_0, \sigma_1] = 0$, namely σ_x are commuting states. Since $U(\mathcal{C}, g)$ only depends on the encoding $\{\phi_x\}$ through the Helstrom matrix $H := \mathcal{C}(K)$, encoding $\{\sigma_x\}$ performs as well as encoding $\{\phi_x\}$, and therefore w.l.o.g. one can maximize over commuting encodings only. By the convexity of $\mathrm{Tr}[\mathrm{Pos}(X - Y)]$ in X and Y, a pure orthonormal encoding $\{\phi_x\}$ suffices. □

Notice that Lemma 4 cannot be generalized to discrimination games with more than two alternatives. For example, let $\mathcal{M}$ be the quantum–classical channel corresponding to the trine POVM of a qubit, that is

$$\mathcal{M}(\rho) \;=\; \sum_y \langle \pi_y | \rho | \pi_y \rangle |y\rangle\langle y|$$

with $|\pi_y\rangle := \frac{2}{3} U^y |0\rangle$ and $U := e^{-i\frac{2\pi}{3}\sigma_Y}$, and let g be the discrimination game $g_{x,y} := \delta_{x,y}$. Then one has $U(\mathcal{M}, g) = 2$ and the optimal encoding is the trine encoding, that is $|\phi_x\rangle = U^x|0\rangle$, which is of course not pairwise commuting. For comparison, the best pairwise commuting encoding is $|\phi_0\rangle = |0\rangle$ and $|\phi_1\rangle = |\phi_2\rangle = |1\rangle$, and in this case the average payoff is given by $\sum_{x,y} |\langle \phi_x | \pi_y \rangle|^2 = 5/3 < 2$.

The Pauli channel is an example of a qubit channel which is commutativity preserving.

DEFINITION 10 (Pauli channel) The action of the Pauli channel

$$\mathcal{P}_{\vec{\lambda}} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$$

with $\dim(\mathcal{H}) = 2$ on any state ρ is given by

$$\mathcal{P}_{\vec{\lambda}}(\rho) \;=\; \lambda_0 \rho + \sum_{k=1}^{3} \lambda_k \sigma_k \rho \sigma_k \,,$$

where $\sigma_1 = \sigma_X$, $\sigma_2 = \sigma_Y$, and $\sigma_3 = \sigma_Z$ are the Pauli matrices.

PROPOSITION 8 *For any binary discrimination game g, the utility $U(\mathcal{P}_{\vec{\lambda}}, g)$ of the Pauli channel $\mathcal{P}_{\vec{\lambda}}$ is given by*

$$U(\mathcal{P}_{\vec{\lambda}}, g) \;=\; \max\left\{ g_0, \frac{1 + \max_{k\geq 1} |2(\lambda_0 + \lambda_k) - 1|}{2} \right\}.$$

Proof. By Lemma 4 it suffices to consider a pure orthonormal encoding $\{\phi_\pm\}$. Upon decomposition over Pauli matrices one has

$$\phi_\pm \;=\; \frac{\mathbb{1}}{2} \pm \sum_{i=1}^{3} \alpha_i \frac{\sigma_i}{2} \,, \qquad \sum_{i=1}^{3} \alpha_i^2 = 1 \,.$$

By direct computation one has

$$\mathcal{P}_{\vec{\lambda}}(\phi_\pm) = \frac{\mathbb{1}}{2} \pm \sum_{i=}^{3} \alpha_i (2(\lambda_0 + \lambda_i) - 1) \frac{\sigma_i}{2} .$$

Since the eigenvalues of the state $\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z$ are $1 \pm \sqrt{x^2 + y^2 + z^2}$, one has that the eigenvalues of the Helstrom matrix

$$H = g_0 \mathcal{P}_{\vec{\lambda}}(\phi_+) - (1 - g_0)\mathcal{P}_{\vec{\lambda}}(\phi_-)$$

are

$$\frac{2g_0 - 1 \pm \sqrt{\sum_{i=1}^{3} \alpha_i^2 [2(\lambda_0 + \lambda_i) - 1]^2}}{2} ,$$

and thus

$$U(\mathcal{P}_{\vec{\lambda}}, g) = \max \left\{ g_0 , \max_{\substack{\vec{\alpha} \\ \sum_i \alpha_i^2 = 1}} \frac{1 + \sqrt{\sum_{i=1}^{3} \alpha_i^2 [2(\lambda_0 + \lambda_i) - 1]^2}}{2} \right\} .$$

By making the substitution $\beta_i := \alpha_i^2$, one has that $\vec{\beta}$ is a probability distribution and therefore the maximum over $\vec{\alpha}$ can be explicitly computed as

$$\max_{\substack{\vec{\beta},\, \vec{\beta} \geq 0 \\ \sum_i \beta_i = 1}} \sqrt{\sum_{i=1}^{3} \beta_i [2(\lambda_0 + \lambda_i) - 1]^2} = \max_{i \geq 1} |2(\lambda_0 + \lambda_i) - 1| ,$$

and the statement immediately follows. □

The amplitude damping channel is an example of a qubit channel which is not commutativity preserving.

DEFINITION 11 (Amplitude damping channel) The action of the amplitude damping channel $\mathcal{A}_\eta : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ with $\dim(\mathcal{H}) = 2$ on any state $\rho := \begin{pmatrix} 1 - \beta & \gamma \\ \gamma^* & \beta \end{pmatrix}$ is given by

$$\mathcal{A}_\eta(\rho) = \begin{pmatrix} 1 - \eta\beta & \sqrt{\eta}\gamma \\ \sqrt{\eta}\gamma^* & \eta\beta \end{pmatrix} .$$

PROPOSITION 9 *The utility $U(\mathcal{A}_\eta, g)$ of any amplitude damping channel $\mathcal{A}_\eta$ for any binary discrimination game g is given by*

$$U(\mathcal{A}_\eta, g) = \frac{1 + \sqrt{1 - 4g_0(1 - \eta) + 4g_0^2(1 - \eta)}}{2} ,$$

and an optimal encoding is given by

$$\begin{aligned} |\psi_0^*\rangle &= \sqrt{g_0}|0\rangle + \sqrt{1-g_0}|1\rangle\,, \\ |\psi_1^*\rangle &= \sqrt{1-g_0}|0\rangle - \sqrt{g_0}|1\rangle\,. \end{aligned}$$

Proof. Due to Lemma 4 the optimal encoding is pure and orthonormal. W.l.o.g. we consider

$$\begin{aligned} |\psi_0\rangle &= \sqrt{1-\gamma^2}|0\rangle + \gamma|1\rangle\,, \\ |\psi_1\rangle &= \gamma|0\rangle - \sqrt{1-\gamma^2}|1\rangle\,. \end{aligned}$$

The eigenvalues $\lambda_\pm$ of the Helstrom matrix

$$H := g_0\mathcal{A}_\eta(|\psi_0\rangle\langle\psi_0|) - (1-g_0)\mathcal{A}_\eta(|\psi_1\rangle\langle\psi_1|)$$

are given by

$$\lambda_\pm = \frac{2g_0 - 1 \pm \sqrt{a\gamma^4 + b\gamma^2 + c}}{2}\,,$$

where

$$\begin{aligned} a &= 4\eta(\eta-1)\,, \\ b &= 8\eta\left[(\eta-1)g_0 - \eta + 1\right], \\ c &= 4(\eta-1)^2g_0^2 - 4(2\eta^2 - 3\eta + 1)g_0 + 4\eta(\eta-1) + 1\,. \end{aligned}$$

W.l.o.g. we take $g_0 \geq \frac{1}{2}$. Hence, the discrimination utility $U(\mathcal{A}_\eta, g)$ is given by

$$U(\mathcal{A}_\eta, g) = \max_\gamma \frac{1 + \lambda_+ + |\lambda_-|}{2}\,.$$

If the utility is achieved for γ such that $\lambda_- \geq 0$, by direct inspection one has

$$\frac{1 + \lambda_+ + |\lambda_-|}{2} = g_0\,,$$

for any η, namely $U(\mathcal{A}_\eta, g) = g_0$. However, this is absurd, because upon setting $\gamma = 0$ (i.e., $|\psi_x\rangle = |x\rangle$) it immediately follows that

$$\frac{1 + \lambda_+ + |\lambda_-|}{2} = g_0 + (1-g_0)\eta\,,$$

namely $U(\mathcal{A}_\eta, g) \geq g_0 + (1-g_0)\eta$. Therefore, $U(\mathcal{A}_\eta, g) > g_0$ for any $0 < \eta \leq 1$ and γ and $g_0 < 1$, which is absurd. Then one can conclude $\lambda_- < 0$ and

$$U(\mathcal{A}_\eta, g) = \max_\gamma \frac{1 + \lambda_+ - \lambda_-}{2}\,.$$

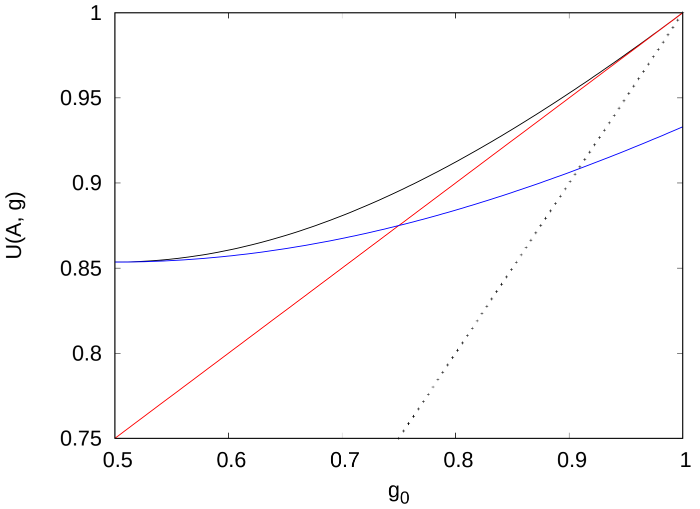

Fig. 2: (Colour online) Binary utility $U(\mathcal{A}_\eta, g)$ (upper black line) of the amplitude damping channel $\mathcal{A}_\eta$ for a binary discrimination game $g = \mathrm{diag}(g_0, 1 - g_0)$ as a function of g_0 for fixed $\eta = \frac{1}{2}$. The average payoff with encodings $|\psi_i\rangle = |\pm\rangle$ (blue curved line) and $|\psi_i\rangle = |i\rangle$ (red straight line) are optimal for the balanced case $g_0 = \frac{1}{2}$ and for the maximally unbalanced case $g_0 = 1$, respectively. The trivial guess (dotted black line) is optimal only in the maximally unbalanced case.

By direct inspection, the zeros of the first derivative of $(1 + \lambda_+ - \lambda_-)/2$ are attained when $\gamma = \pm\sqrt{1 - g_0}$ and $\gamma = 0$. The zeros of the second derivative are attained when $\gamma = \pm\sqrt{(1 - g_0)/3}$ and the second derivative is positive when $\gamma = 0$. Therefore the maximum is attained when $\gamma = \pm\sqrt{1 - g_0}$, and the statement follows by direct computation. □

Let us consider the two extremal cases $g_0 = 1/2$ (balanced) and $g_0 = 1$ (maximally unbalanced). For $g_0 = 1/2$ the optimal encoding given by Proposition 9 becomes $|\psi_\pm\rangle = |\pm\rangle$, and the corresponding binary discrimination utility becomes $U(\mathcal{A}_\eta, g) = (1 + \sqrt{\eta})/2$. For $g_0 = 1$, the optimal encoding given by Proposition 9 becomes $|\psi_i\rangle = |i\rangle$, and the corresponding binary discrimination utility becomes $U(\mathcal{A}_\eta, g) = 1$. This situation is depicted in Fig. 2.

An example of arbitrary dimensional, noncommutativity-preserving channel is the shifted-depolarizing channel. Following [7, 8, 9, 10], we define the shifted-depolarizing channel as follows.

DEFINITION 12 (Shifted-depolarizing channel) The action of the shifted-depolarizing channel $\mathcal{S}_\lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ on any state $\rho \in \mathcal{L}(\mathcal{H})$ is given by

$$\mathcal{S}_\lambda(\rho) \ := \ \lambda\rho + (1-\lambda)\operatorname{Tr}[\rho]\sigma \,,$$

where $\sigma \in \mathcal{L}(\mathcal{H})$ is a state.

PROPOSITION 10 *For any binary game g, the utility $U(\mathcal{S}_\lambda, g)$ of the shifted-depolarizing channel $\mathcal{S}_\lambda$ is given by*

$$U(\mathcal{S}_\lambda, g) \ = \ \max\left\{g_0, \frac{1+\lambda+(1-\lambda)(1-2s_{d-1})(2g_0-1)}{2}\right\}, \qquad (2)$$

where s_{d-1} is the smallest eigenvalue of σ.

Proof. Due to Lemma 4 w.l.o.g. we take the encoding $\{\rho_x\}$ to be pure and orthogonal. Denote the eigenvalues of σ with $\{s_0, s_1, \dots, s_{d-1}\}$, where w.l.o.g. we take $s_0 \geq s_1 \geq \cdots \geq s_{d-1}$. The Helstrom matrix becomes

$$\begin{aligned} H &= g_0\mathcal{S}_\lambda(\rho_0) - (1-g_0)\mathcal{S}_\lambda(\rho_1) \\ &= -\lambda(1-g_0)\rho_1 + \lambda g_0\rho_0 + (1-\lambda)(2g_0-1)\sigma \,. \end{aligned}$$

Let us set $R := -\lambda(1-g_0)\rho_1$ and $C := \lambda g_0\rho_0 + (1-\lambda)(2g_0-1)\sigma$, so that $H = R + C$, where the only non-null eigenvalue of R is $r_{d-1} := -\lambda(1-g_0) \leq 0$ and the eigenvalues of C are $\{c_0, \dots, c_{d-1}\}$, where $c_j \geq c_{j+1} \geq 0$.

Then we can label the eigenvalues of H as $\{h_0, \dots, h_{d-1}\}$, where $h_j \geq h_{j+1}$, and applying Weyl's inequality immediately results in the system:

$$\begin{cases} r_i + c_{d-1} \ \leq \ h_i \leq r_i + c_0 \,, \\ r_{d-1} + c_i \ \leq \ h_i \leq r_0 + c_i \,. \end{cases}$$

Since $r_i = 0$ for any $i < d-1$, this becomes

$$\begin{cases} c_{d-1} \ \leq \ h_i \ \leq \ c_i \quad \forall\, i < d-1 \,, \\ r_{d-1} + c_{d-1} \ \leq \ h_{d-1} \ \leq \ r_{d-1} + c_0 \,. \end{cases}$$

If $h_{d-1} \geq 0$, then $\|H\|_1 = 2g_0 - 1$, which corresponds to the strategy of trivial guessing. However, if $h_{d-1} \leq 0$, then $\|H\|_1 \leq \sum_{i=0}^{d-2} c_i - (r_{d-1} + c_{d-1})$. This upper bound can be obtained by selecting R to have its single non-zero eigenvalue r_{d-1} corresponding to the eigenvector for which C has the eigenvalue c_{d-1}. Making use of the identity

$$\sum_i^{d-2} c_i \ = \ \operatorname{Tr}[C] - c_{d-1} \ = \ [\lambda g_0 + (1-\lambda)(2g_0-1)] - c_{d-1} \,,$$

one has $\|H\|_1 = \mathrm{Tr}[C] - r_{d-1} - 2c_{d-1}$, and thus $\|H\|_1$ is maximized when c_{d-1} is minimized.

Let us set $A = \lambda g_0 \rho_0$ with only non-null eigenvalue $a_0 \geq 0$ and $B = (1-\lambda)(2g_0 - 1)\sigma$ with eigenvalues $\{b_0, \dots, b_{d-1}\}$ where $b_j \geq b_{j+1}$, so that $C = A+B$. Then another application of Weyl's inequality yields $b_{d-1} \leq c_{d-1}$, with equality saturated if and only if ρ_0 is orthogonal to the eigenvector of σ corresponding to eigenvalue s_{d-1} (w.l.o.g. we take here $\sigma \neq \mathbb{1}/d$, as the case of the depolarizing channel has already been discussed in Prop. 6), and thus $c_{d-1} = (1-\lambda)(2g_0-1)s_{d-1}$. Finally, $\|H\|_1 = \lambda + (1-\lambda)(1-2s_{d-1})(2g_0-1)$, from which the statement immediately follows. □

Further results can be derived for the utility of group-covariant quantum channels.

DEFINITION 13 (Covariant channel) A channel $\mathcal{C} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is G-covariant if group G admits unitary representations $U_k \in \mathcal{L}(\mathcal{H})$ and $V_k \in \mathcal{L}(\mathcal{K})$ such that

$$\mathcal{C}(U_k \rho U_k^\dagger) \; = \; V_k \mathcal{C}(\rho) V_k^\dagger \tag{3}$$

for any $\rho \in \mathcal{L}(\mathcal{H})$ and any $k \in G$.

LEMMA 5 *For any G-covariant channel $\mathcal{C}$ and any binary game g, if an encoding $\{\rho_x\}$ attains the utility $U(\mathcal{C}, g)$, then also any encodings $\{\sigma_x := U_k \rho_x U_k^\dagger\}$, where k is any element of G, attains the same utility $U(\mathcal{C}, g)$.*

Proof. The statement follows by direct inspection, namely

$$\begin{aligned} ||g_0\mathcal{C}(\sigma_0) - (1-g_0)\mathcal{C}(\sigma_1)||_1 &= ||g_0\mathcal{C}(U_k\rho_0U_k^\dagger) - (1-g_0)\mathcal{C}(U_k\rho_1U_k^\dagger)||_1 \\ &= ||g_0V_k\mathcal{C}(\rho_0)V_k^\dagger - (1-g_0)V_k\mathcal{C}(\rho_1)V_k^\dagger||_1 \\ &= ||V_k\left(g_0\mathcal{C}(\rho_0) - (1-g_0)\mathcal{C}(\rho_1)\right)V_k^\dagger||_1 \\ &= ||g_0\mathcal{C}(\rho_0) - (1-g_0)\mathcal{C}(\rho_1)||_1 \,, \end{aligned}$$

where the second equality follows from (3), and the fourth from the invariance of trace distance under unitary transformations. □

Then, the utility of universally covariant channels follows. As an example, let us consider the 1 to 2 optimal universally covariant quantum cloning channel [11].

DEFINITION 14 (Universal optimal cloning) The action of the universal optimal cloning channel $\mathcal{N} : \mathcal{L}(\mathcal{H}^{\otimes N}) \to \mathcal{L}(\mathcal{H}^{\otimes M})$ on any state $\rho \in \mathcal{L}(\mathcal{H})$ is given by

$$\mathcal{N}(\rho) \; := \; \frac{f(N)}{f(M)} P_s(\rho^{\otimes N} \otimes \mathbb{1}^{\otimes(M-N)})P_s$$

where $d = \dim \mathcal{H}$, P_s is the projector on the symmetric subspace of $\mathcal{H}^{\otimes M}$, and $f(x) := \binom{d+x-1}{x}$.

PROPOSITION 11 *For any binary discrimination game g, the utility $U(\mathcal{N}, g)$ of the $1 \to 2$ universal optimal cloning channel $\mathcal{N} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}^{\otimes 2})$ is given by*

$$U(\mathcal{N}, g) \;=\; \frac{d + g_0}{d+1}\,,$$

and any orthonormal pure encoding is optimal.

Proof. By Lemmas 4 and 5 any orthonormal pure encoding is optimal. W.l.o.g. let us fix a computational basis and set $\rho_x = |x\rangle\langle x|$ for $x = 0, 1$. Then we have

$$P_s \;=\; \sum_{n,m} \frac{(|n,m\rangle + |m,n\rangle)(\langle n,m| + \langle m,n|)}{4}\,,$$

and thus

$$\mathcal{N}(|0\rangle\langle 0|) \;=\; \frac{2}{d+1} \sum_i \frac{(|0,i\rangle + |i,0\rangle)(\langle 0,i| + \langle i,0|)}{4}\,,$$

and analogously for $\mathcal{N}(|1\rangle\langle 1|)$ upon replacing $|0\rangle$ with $|1\rangle$. Therefore we have

$$\begin{aligned} \mathcal{N}(|0\rangle\langle 0|)\mathcal{N}(|1\rangle\langle 1|) \;&=\; \frac{1}{(d+1)^2} \frac{(|0,1\rangle + |1,0\rangle)(\langle 0,1| + \langle 1,0|)}{2} \\ &=\; \mathcal{N}\langle 1|)\mathcal{N}(|0\rangle\langle 0|)\,, \end{aligned}$$

and thus $[\mathcal{N}(|0\rangle\langle 0|), \mathcal{N}(|1\rangle\langle 1|)] = 0$. Therefore, $\mathcal{N}(|0\rangle\langle 0|)$ and $\mathcal{N}(|1\rangle\langle 1|)$ admit a basis of common eigenvectors, namely

$$\mathcal{N}(|x\rangle\langle x|) \;=\; \sum_k \lambda_{k|x} |k\rangle\langle k|\,.$$

The only common eigenvector such that $\lambda_{k|x} \neq 0$ for any x is $(|0,1\rangle + |1,0\rangle)(\langle 0,1| + \langle 1,0|)/2$ and its corresponding eigenvalue is $(d+1)^{-1}$. Then the trace norm of the Helstrom matrix is given by

$$\| g_0 \mathcal{N}(|0\rangle\langle 0|) - (1 - g_0)\mathcal{N}(|1\rangle\langle 1|) \|_1 \;=\; \frac{d + 2g_0 - 1}{d+1}\,,$$

from which the statement immediately follows. □

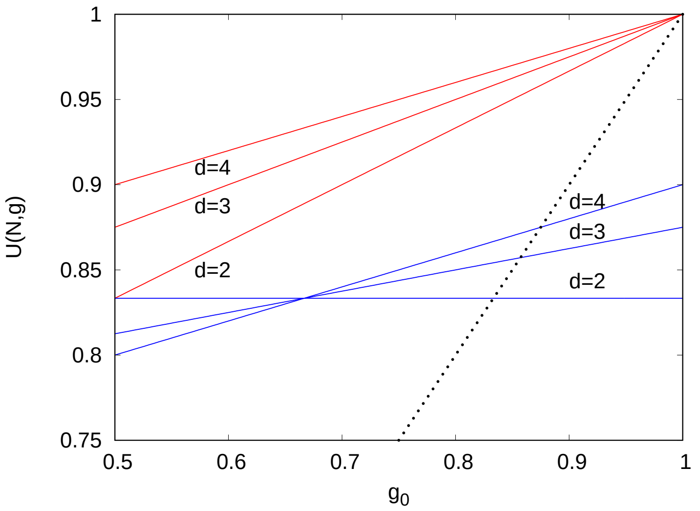

Fig. 3: (Color online) Binary utility $U(\mathcal{N}, g)$ (upper red lines) and binary utility $U(\mathcal{D}_\lambda, g)$ (lower blue lines) of the covariant cloning channel $\mathcal{N}$ and depolarizing channel $\mathcal{D}_\lambda(\rho) := \mathrm{Tr}_2[\mathcal{N}(\rho)]$ with $\lambda = (d+2)/[2(d+1)]$, respectively, as a function of g_0, for fixed dimension $d = 2$, 3, and 4. The trivial guess (dotted line) is also depicted. Notice that, perhaps surprisingly, for $d = 2$ and $g_0 = \frac{1}{2}$ the utilities $U(\mathcal{N}, g)$ and $U(\mathcal{D}_\lambda, g)$ coincide. This can be regarded as an entangled analogy of the readily verifiable fact that, for any λ, the success probabilities in the discrimination of equiprobable qubit states $\mathcal{D}_\lambda(|0\rangle\langle 0|)$ and $\mathcal{D}_\lambda(|1\rangle\langle 1|)$ are the same when one or two copies of the unknown state are available.

Notice that the partial trace of the 1 → 2 cloning channel $\mathcal{N}$ is a depolarizing channel given by

$$\mathrm{Tr}_2[\mathcal{N}(\rho)] \;=\; \frac{1}{2(d+1)}\left((d+2)\rho + \mathrm{Tr}[\rho]\frac{\mathbb{1}}{d}\right) \;=\; \mathcal{D}_\lambda(\rho)\,,$$

with $\lambda = (d+2)/[2(d+1)]$. Its utility is given by (2), i.e.,

$$U(\mathrm{Tr}_2[\mathcal{N}(\rho)], g) \;=\; \max\left\{g_0, \frac{(d-2)g_0 + d + 3}{2(d+1)}\right\}.$$

This situation is depicted in Fig. 3.

7. Conclusion and Outlook

In this work we considered quantum communication games, where Alice's task is to communicate some information received by a referee to Bob through a quantum channel, in order to maximize a payoff that depends on both the received information and Bob's output. The maximum average payoff defines the communication utility of the channel for that particular game. Hence, communication games act as witnesses on the set of classical noisy channels that can be obtained from the given quantum channel, and the corresponding communication utility constitutes the optimal value for any such a witness. We derived general results and closed-form, analytic solutions for the utility of several classes of quantum channels and several classes of games.

A natural extension of our setup consists of allowing Alice and Bob to share some entangled state, thus generalizing superdense coding [12] to cases involving noisy communication channels. In such a case, the object being considered is not the channel $\mathcal{C}$ alone, but an extension $\mathcal{C} \otimes \mathrm{id}$. For the noiseless channel, clearly the entangled assisted utility of the identity channel in dimension d is equivalent to the utility of the identity channel in dimension d^2. Based on this example, it is clear that in at least some cases entanglement can increase the utility of quantum channels.

Our results shed new light on the problem of creating quantum correlations. It is a known result [13, 14] that classically correlated bipartite states can be transformed by a local channel in a state exhibiting quantum correlations, if and only if the channel is not commutativity preserving. However, not much is known about the problem of characterizing commutativity preserving channels in the general case where the dimensions of the input and output Hilbert spaces differ. Notice that from the proof of Theorem 11, it immediately follows that universal optimal 1 to 2 cloning is a commutativity preserving channel.

Our results have important applications in the quantification of non-markovianity of quantum channels. In Theorem 3 of [15], it was shown that the supremum of non-Markovianity over the encoding and the binary discrimination game is attained by an orthogonal encoding. From the proof of our Lemma 4, it immediately follows that this is actually the case for *any* binary discrimination game.

Finally, our results generalize to the quantum case the partial ordering among classical channels derived by Shannon [16]. Therein, it is shown that if one classical channel $\mathcal{C}_0$ can be reproduced by another $\mathcal{C}_1$ upon classical (possibly correlated) pre- and post-processes, then $\mathcal{C}_1$ has larger Shannon capacity than $\mathcal{C}_0$. However, the validity of the converse remains an open problem.

Acknowledgements

M. D. acknowledges support from MEXT Quantum Leap Flagship Program (MEXT Q-LEAP) Grant No. JPMXS0118067285, JSPS KAKENHI Grant Number JP20K03774, and the International Research Unit of Quantum Information, Kyoto University. F. B. acknowledges support from MEXT Quantum Leap Flagship Program (MEXT Q-LEAP), Grant Number JPMXS0120319794, and from the Japan Society for the Promotion of Science (JSPS) KAKENHI, Grants Number 19H04066 and Number 20K03746.

Bibliography

[1] P. E. Frenkel and M. Weiner, Commun. Math. Phys. **340**, 563 (2015).

[2] N. Elron and Y. C. Eldar, IEEE Trans. Inf. Theory **53**, 1900 (2007).

[3] M. Dall'Arno, S. Brandsen, and F. Buscemi, Proc. R. Soc. A **473**, 20160721 (2017).

[4] F. Buscemi and M. Dall'Arno, New J. Phys. **21**, 113029 (2019).

[5] M. M. Wilde, *Quantum Information Theory*, Cambridge Univ. Press, 2013.

[6] C. W. Helstrom, *Quantum Detection and Estimation Theory*, Academic Press, 1976.

[7] V. Giovannetti, S. Lloyd, and M. B. Ruskai, J. Math. Phys. **46**, 042105 (2005).

[8] C. King, M. Nathanson, and M. B. Ruskai, Lin. Alg. Appl. **404**, 367 (2005).

[9] M. Fukuda, J. Phys. A: Math. Gen. **38**, 753 (2005).

[10] Y. Ouyang, Quant. Inf. Comp. **14**, 0917 (2014).

[11] R. F. Werner, Phys. Rev. A **58**, 1827 (1998).

[12] C. Bennett and S. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[13] X. Hu, H. Fan, D. L. Zhou, and W.-M. Liu, Phys. Rev. A **85**, 032102 (2012).

[14] Z. Guo and H. Cao, J. Phys. A **46**, 065303 (2013).

[15] S. Wißmann, B. Vacchini, and H.-P. Breuer, Phys. Rev. A **92**, 042108 (2015).

[16] C. E. Shannon, Information and Control **1**, 390 (1958).