

Comparison of Quantum Statistical Models: Equivalent Conditions for Sufficiency

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Received: 29 November 2010 / Accepted: 8 September 2011
Published online: 21 January 2012 – © Springer-Verlag 2012

Abstract: A family of probability distributions (i.e. a statistical model) is said to be sufficient for another, if there exists a transition matrix transforming the probability distributions in the former to the probability distributions in the latter. The Blackwell-Sherman-Stein (BSS) Theorem provides necessary and sufficient conditions for one statistical model to be sufficient for another, by comparing their information values in statistical decision problems. In this paper we extend the BSS Theorem to quantum statistical decision theory, where statistical models are replaced by families of density matrices defined on finite-dimensional Hilbert spaces, and transition matrices are replaced by completely positive, trace-preserving maps (i.e. coarse-grainings). The framework we propose is suitable for unifying results that previously were independent, like the BSS theorem for classical statistical models and its analogue for pairs of bipartite quantum states, recently proved by Shmaya. An important role in this paper is played by *statistical morphisms*, namely, affine maps whose definition generalizes that of coarse-grainings given by Petz and induces a corresponding criterion for statistical sufficiency that is weaker, and hence easier to be characterized, than Petz's.

1. Introduction

The task in which an experimenter tries to learn about the true value of an unknown parameter by observing a random variable whose distribution depends on such a value, is generally called a statistical estimation task or a statistical decision problem. The mathematical structure used to describe such a situation is called *statistical model*, i.e. a family of probability distributions (or, more generally, measures) indexed by a parameter set, which represents the unknown parameter one wants to learn about in the estimation process.

An important subject in classical statistics is the comparison of statistical models in terms of their “information value” in statistical decision problems [1–10]. Within this area, one of the most important results has been proved by Blackwell, Sherman, and

Stein [3–6]: the theorem states equivalent conditions for one statistical model being *more informative* than another. More explicitly, the Blackwell-Sherman-Stein (from now on, BSS) Theorem proves that one statistical model carries more information than another if and only if the former is *sufficient* for the latter, namely, if and only if there exists a transition matrix (a Markov kernel) mapping the probability distributions (measures) in the former to the probability distributions (measures) in the latter.

In quantum statistical decision theory [11, 12], where statistical models are replaced by families of non-commuting density operators (i.e. quantum statistical models), the notion of sufficiency has been introduced and developed by Petz [13, 14], by replacing Markov kernels with completely positive (or, at least, two-positive) trace-preserving maps, i.e. coarse-grainings. However, the idea of applying to the quantum case concepts from the theory of comparison of statistical models *à la* BSS, like e.g. the concept of information value, has not been explicitly pursued until recently, in a work by Shmaya [15]: there, partial ordering relations between pairs of bipartite quantum states (analogous, in a way, to the partial ordering relations used in the BSS Theorem) are introduced, and an equivalence relation between such partial orderings is established. Subsequently, in [16], Chefles reformulated Shmaya’s result for the comparison of pairs of quantum channels. However, the equivalence relations proved in [15] and [16] neither imply any criterion for the comparison of quantum statistical models, nor are they more general than the BSS Theorem, with which they are, in fact, logically *unrelated*. This is due to the fact that both Shmaya and Chefles need, in their proofs, quantum entanglement: as such, their results are *purely quantum* and cannot be compared with the case of classical statistics, where quantum entanglement is not available.

The aim of this paper is to bridge the gap mentioned above, by developing a general theory for the comparison of statistical models, which can be applied both to the classical and the quantum (i.e. non-commutative) setting. In order to do this, it is mathematically convenient to relax the definition of sufficiency introduced by Petz [13, 14] and define a weaker notion of sufficiency, which we call *m-sufficiency*, based on the concept of *statistical morphisms*.¹ Statistical morphisms are affine maps satisfying the minimum requirements necessary to make them meaningful in a statistical sense: in fact, as we will carefully argue in what follows, even the requirement of positivity can be lifted, without compromising the formalism. In spite of their generality, statistical morphisms are sufficiently well-behaved, so that, in some cases, they can be extended to completely positive coarse-grainings. This fact is proved in two extension theorems, of crucial importance in this paper, analogous to those proved for positive maps by Choi (Thm. 6 in [18]) and Arveson (Prop. 1.2.2 in [19]).

The generality of the definition of statistical morphisms makes the main result proved here applicable to both commutative and non-commutative scenarios. When specialized to the classical setting, our result provides an alternative proof of the BSS Theorem, while, in the quantum setting, an equivalent characterization of Petz’s sufficiency criterion is obtained. An intermediate, ‘hybrid’ quantum-classical case is also considered and completely characterized. We are also able to recover Shmaya’s result as a special case, although here, in contrast with Refs. [15] and [16], we *never* need to resort to any additional entangled resource.

The paper is organized as follows: in Sect. 2 we briefly review the notions of statistical models, statistical decision problems, and comparison of statistical models in classical statistics. In Sect. 3 we introduce some basic definitions, extending the idea of

¹ The term “statistical morphism” has been introduced in the classical setting by Morse and Sacksteder [17]. In this paper we use the same term, but in a non-commutative setting.

comparison of statistical models to finite dimensional quantum systems. In Sect. 4, we introduce the notions of statistical morphisms and m-sufficiency. In Sect. 5 we prove two extension theorems for statistical morphisms. Section 6 contains the main result, which is then applied, in Sect. 7, in order to recover the BSS Theorem, characterize a semi-classical scenario, and obtain an equivalent characterization of Petz’s sufficiency relation. Section 8 deals with the scenario originally considered in Ref. [15] and the result proved by Shmaya is recovered without the need of any entangled auxiliary resource. Finally, Sect. 9 concludes the paper with the summary of its contents and one remark about generalized probabilistic theories.

2. Classical Formulation

A (finite) *statistical model* \mathcal{E} is defined by a triple (Θ, Δ, α) , where Θ is a (finite) parameter set $\{\theta\}_{\theta \in \Theta}$, Δ is a (finite) sample set $\{\delta\}_{\delta \in \Delta}$, and α is a family $(p_\theta; \theta \in \Theta)$ of probability distributions p_θ on Δ , i.e., $p_\theta(\delta) \geq 0$ and $\sum_{\delta \in \Delta} p_\theta(\delta) = 1$. In the following, it will sometimes be convenient to think of each p_θ as a $|\Delta|$ -dimensional probability vector $\vec{p}_\theta = (p_\theta^1, \dots, p_\theta^\delta, \dots, p_\theta^{|\Delta|})$, whose components are defined as $p_\theta^\delta := p_\theta(\delta)$.

Remark 1. In many relevant situations, Δ can be considered as the set of possible states of a physical system, so that the probability distribution p_θ becomes the statistical description of the state of the system. This point of view, which is the guiding one in Ref. [11], will be implicitly adopted here as well.

A *statistical decision problem* is defined by a triple $(\mathcal{E}, \mathcal{X}, \ell)$, where $\mathcal{E} = (\Theta, \Delta, \alpha)$ is a statistical model, \mathcal{X} is a (finite) decision set $\{i\}_{i \in \mathcal{X}}$, and $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$ is a payoff function. The decision problem works as follows: upon the observation (or state) $\delta \in \Delta$, occurring with probability $p_\theta(\delta)$, the statistician performs a decision, namely, he applies a \mathcal{X} -decision function $u : \Delta \rightarrow \mathcal{X}$, gaining a payoff (or suffering a loss, if negative) of $\ell(\theta, i)$, depending on the “true” law of nature θ that determined the observed state δ . The choice of the function $u : \Delta \rightarrow \mathcal{X}$ corresponds to the experimenter’s choice of a *strategy*.

The deterministic \mathcal{X} -decision function $u : \Delta \rightarrow \mathcal{X}$ is often generalized to a *randomized \mathcal{X} -decision function* (or \mathcal{X} -r.d.f.) ϕ , which is a convex combination of \mathcal{X} -decision functions, i.e., a function mapping each $\delta \in \Delta$ to a probability distribution t_δ on \mathcal{X} . A convenient way to represent a \mathcal{X} -r.d.f. ϕ is by giving conditional probabilities $t_\phi(i|\delta) \geq 0$, i.e. non-negative real numbers such that $\sum_{i \in \mathcal{X}} t_\phi(i|\delta) = 1$, for all $\delta \in \Delta$.

Given a decision problem $(\mathcal{E}, \mathcal{X}, \ell)$, for each \mathcal{X} -r.d.f. ϕ , we introduce the payoff vector $\vec{v}(\phi; \mathcal{E}, \mathcal{X}, \ell) \in \mathbb{R}^{|\Theta|}$, whose θ^{th} component, representing the payoff gained if the true law of nature is θ , is defined as

$$v^\theta(\phi; \mathcal{E}, \mathcal{X}, \ell) := \sum_{i \in \mathcal{X}} \ell(\theta, i) \sum_{\delta \in \Delta} t_\phi(i|\delta) p_\theta(\delta). \tag{1}$$

Then, the following set

$$\mathcal{C}(\mathcal{E}, \mathcal{X}, \ell) := \{\vec{v}(\phi; \mathcal{E}, \mathcal{X}, \ell) \mid \phi \text{ is a } \mathcal{X}\text{-r.d.f. on } \Delta\} \tag{2}$$

forms a (closed and bounded) convex subset of $\mathbb{R}^{|\Theta|}$, since it inherits the convex structure from the set of randomized decision functions.

Let now $\mathcal{F} = (\Theta, \Delta', \boldsymbol{\beta})$ be another statistical model, with the same parameter set Θ as for \mathcal{E} , but with a different sample set Δ' and a different family of probability distributions on Δ' . $\boldsymbol{\beta} = (q_\theta; \theta \in \Theta)$. Also for \mathcal{F} , we can define, for each decision set \mathcal{X} and each payoff function $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, the convex set of achievable payoff vectors as

$$\mathcal{C}(\mathcal{F}, \mathcal{X}, \ell) := \{ \bar{v}(\phi'; \mathcal{F}, \mathcal{X}, \ell) \mid \phi' \text{ is a } \mathcal{X}\text{-r.d.f. on } \Delta' \}. \quad (3)$$

In classical statistics, the following partial ordering between statistical models with the same parameter set Θ is introduced (see, e. g., Ref. [6]):

Definition 1 (Information Ordering). *The statistical model $\mathcal{E} = (\Theta, \Delta, \boldsymbol{\alpha})$ is said to be **always more informative than** $\mathcal{F} = (\Theta, \Delta', \boldsymbol{\beta})$, in formula $\mathcal{E} \supset \mathcal{F}$, if and only if, for every finite set of decisions \mathcal{X} and every payoff function $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, $\mathcal{C}(\mathcal{E}, \mathcal{X}, \ell) \supseteq \mathcal{C}(\mathcal{F}, \mathcal{X}, \ell)$.*

In other words, \mathcal{E} is said to be more informative than \mathcal{F} if every payoff vector attainable in the problem $(\mathcal{F}, \mathcal{X}, \ell)$ is also attainable in the problem $(\mathcal{E}, \mathcal{X}, \ell)$. The definition of information ordering between statistical models can be simplified as follows. Given a statistical model $\mathcal{E} = (\Theta, \Delta, \boldsymbol{\alpha})$, for every decision problem $(\mathcal{E}, \mathcal{X}, \ell)$ and every \mathcal{X} -r.d.f. ϕ , we define

$$\mathbf{s}(\mathcal{E}, \mathcal{X}, \ell, \phi) := \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \sum_{i \in \mathcal{X}} \ell(\theta, i) \sum_{\delta \in \Delta} t_\phi(i|\delta) p_\theta(\delta). \quad (4)$$

The maximum of $\mathbf{s}(\mathcal{E}, \mathcal{X}, \ell, \phi)$ over all \mathcal{X} -r.d.f. ϕ is correspondingly defined as

$$\begin{aligned} \$(\mathcal{E}, \mathcal{X}, \ell) &:= \max_{\phi: \mathcal{X}\text{-r.d.f.}} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \sum_{i \in \mathcal{X}} \ell(\theta, i) \sum_{\delta \in \Delta} t_\phi(i|\delta) p_\theta(\delta) \\ &= \max_{\phi: \mathcal{X}\text{-r.d.f.}} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} v^\theta(\phi; \mathcal{E}, \mathcal{X}, \ell). \end{aligned} \quad (5)$$

In the Bayesian approach, when there is no compelling reason to treat the sample set differently from the parameter set, it is reasonable to interpret the factor $1/|\Theta|$ as an *a priori* probability distribution over the unknown parameter $\theta \in \Theta$. In this framework, the function $\$(\mathcal{E}, \mathcal{X}, \ell)$ is understood as the optimal expected payoff, and the following partial ordering between statistical models governed by the same parameter set Θ is introduced (see, e. g., Ref. [10]):

Definition 2 (Bayesian Information Ordering). *The statistical model $\mathcal{E} = (\Theta, \Delta, \boldsymbol{\alpha})$ is said to be **bayesianly always more informative than** $\mathcal{F} = (\Theta, \Delta', \boldsymbol{\beta})$, in formula $\mathcal{E} \supset_{\text{Bayes}} \mathcal{F}$, if and only if, for every finite decision set \mathcal{X} and every payoff function $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, $\$(\mathcal{E}, \mathcal{X}, \ell) \geq \$(\mathcal{F}, \mathcal{X}, \ell)$.*

In other words, \mathcal{E} is said to be bayesianly more informative than \mathcal{F} if every expected payoff attainable in the problem $(\mathcal{F}, \mathcal{X}, \ell)$ is also attainable in the problem $(\mathcal{E}, \mathcal{X}, \ell)$. In Appendix A we report the proof of the following basic fact:

Proposition 1. $\mathcal{E} \supset \mathcal{F}$ if and only if $\mathcal{E} \supset_{\text{Bayes}} \mathcal{F}$.

Since the two orderings $\mathcal{E} \supset \mathcal{F}$ and $\mathcal{E} \supset_{\text{Bayes}} \mathcal{F}$ are equivalent, from now on we will keep only the notation $\mathcal{E} \supset \mathcal{F}$, stressing the fact that the “Bayesian information ordering” relation given in Definition 2 does not really depend on any strong Bayesian assumption.

Another partial ordering between statistical models with the same parameter set Θ is relevant, and it is defined as follows, according to [6]:

Definition 3 (Sufficiency). *The statistical model $\mathcal{E} = (\Theta, \Delta, \alpha)$ is said to be **sufficient** for $\mathcal{F} = (\Theta, \Delta', \beta)$, in formula $\mathcal{E} \succ \mathcal{F}$, if and only if there exists a $|\Delta'| \times |\Delta|$ transition matrix \mathbf{M} , i.e. a matrix of non-negative numbers $M_{\delta', \delta}$ with $\sum_{\delta' \in \Delta'} M_{\delta', \delta} = 1$ for all $\delta \in \Delta$, for which $\bar{q}_\theta = \mathbf{M} \bar{p}_\theta$, for all $\theta \in \Theta$.*

The Blackwell-Sherman-Stein (BSS) Theorem states the following important equivalence relation:

Theorem 1 (BSS Theorem [3–5]). *Given two statistical models $\mathcal{E} = (\Theta, \Delta, \alpha)$ and $\mathcal{F} = (\Theta, \Delta', \beta)$, governed by the same parameter set Θ , $\mathcal{E} \succ \mathcal{F}$ if and only if $\mathcal{E} \supset \mathcal{F}$.*

3. The Formulation in Quantum Theory

In what follows, we only consider quantum systems defined on finite dimensional Hilbert spaces \mathcal{H} . We denote by $\mathfrak{L}(\mathcal{H})$ the set of all linear operators (identified with their representing matrices) acting on \mathcal{H} , and by $\mathfrak{S}(\mathcal{H})$ the set of all density matrices (or states) $\rho \in \mathfrak{L}(\mathcal{H})$, with $\rho \geq 0$ and $\text{Tr}[\rho] = 1$. The identity matrix will be denoted by the symbol $\mathbb{1}$, whereas the identity map will be denoted by id .

Most of the concepts used here are introduced and rigorously formalized in Refs. [11] and [12]. For reader’s clarity, however, we will report the definitions we need, in a simplified fashion. According with [11] (see also Remark 1), we adopt the following definition:

Definition 4 (Quantum Statistical Models). *A **quantum statistical model** is defined by a triple $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$, where Θ is a (finite) parameter set, \mathcal{H} is a (finite dimensional) Hilbert space, and $\rho = (\rho_\theta; \theta \in \Theta)$ is a family of density matrices in $\mathfrak{S}(\mathcal{H})$. A quantum statistical model \mathbf{R} is said to be **abelian** when $[\rho_\theta, \rho_{\theta'}] = 0$, for all $\theta, \theta' \in \Theta$.*

Definition 5 (POVM’s). *For any (finite) decision set $\mathcal{X} = \{i\}$, a **positive-operator-valued \mathcal{X} -measure** (\mathcal{X} -POVM) $\mathbf{P}^\mathcal{X}$ on the Hilbert space \mathcal{H} is a family $(P^i; i \in \mathcal{X})$ of operators $P^i \in \mathfrak{L}(\mathcal{H})$, such that $P^i \geq 0$ for all $i \in \mathcal{X}$ and $\sum_{i \in \mathcal{X}} P^i = \mathbb{1}$. From now on, the superscript \mathcal{X} will be dropped when clear from the context.*

Definition 6 (Quantum Statistical Decision Problems). *A **quantum statistical decision problem** is defined by a triple $(\mathbf{R}, \mathcal{X}, \ell)$, where $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ is a quantum statistical model, \mathcal{X} is a (finite) decision set $\{i\}_{i \in \mathcal{X}}$, and $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$ is a payoff function. The choice of a **strategy** for the problem $(\mathbf{R}, \mathcal{X}, \ell)$ corresponds to the choice of a \mathcal{X} -POVM $\mathbf{P} = (P^i; i \in \mathcal{X})$ on \mathcal{H} . The corresponding expected payoff is computed as*

$$s_q(\mathbf{R}, \mathcal{X}, \ell, \mathbf{P}) := \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \sum_{i \in \mathcal{X}} \ell(\theta, i) \text{Tr}[\rho_\theta P^i]. \tag{6}$$

The maximum expected payoff for the decision problem $(\mathbf{R}, \mathcal{X}, \ell)$ is defined as

$$\$_{\mathbf{q}}(\mathbf{R}, \mathcal{X}, \ell) := \max_{\mathbf{P}} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} \sum_{i \in \mathcal{X}} \ell(\theta, i) \text{Tr}[\rho_{\theta} P^i]. \tag{7}$$

Notice the use of the subscript “q”, for “quantum”, to distinguish the expressions above from their classical analogues appearing in (4) and (5).

Given two quantum statistical models $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ and $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$, governed by the same parameter set Θ , but with different Hilbert spaces \mathcal{H} and \mathcal{H}' and different families of quantum states $\rho = (\rho_{\theta} \in \mathfrak{S}(\mathcal{H}); \theta \in \Theta)$ and $\sigma = (\sigma_{\theta} \in \mathfrak{S}(\mathcal{H}'); \theta \in \Theta)$, the following partial ordering is introduced:

Definition 7 (Information Ordering). *A quantum statistical model $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ is said to be **always more informative than** $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$, in formula $\mathbf{R} \supset \mathbf{S}$, if and only if, for every finite decision set \mathcal{X} and every payoff function $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, $\$_{\mathbf{q}}(\mathbf{R}, \mathcal{X}, \ell) \geq \$_{\mathbf{q}}(\mathbf{S}, \mathcal{X}, \ell)$.*

In other words, \mathbf{R} is said to be more informative than \mathbf{S} if every expected payoff attainable in the problem $(\mathbf{S}, \mathcal{X}, \ell)$ is also attainable in the problem $(\mathbf{R}, \mathcal{X}, \ell)$.

Remark 2. We stress once more that the information ordering relation between quantum statistical models introduced above does not depend on the *a priori* distribution on Θ used to compute the expected payoff (7). One could in fact adopt an information ordering for quantum statistical models analogous to that introduced in Definition 1, and prove that the two ordering relations are equivalent. This is due to the fact that Proposition 1 is valid also for quantum statistical models.

Remark 3 (Quantum-Classical Correspondence). Given an abelian quantum statistical model $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$, it is always possible to construct, from \mathbf{R} , a (classical) statistical model $\mathcal{E}_{\mathbf{R}} = (\Theta, \Delta_{\mathcal{H}}, \alpha_{\rho})$ that is completely equivalent to \mathbf{R} , in the sense that, for every finite decision set \mathcal{X} , every payoff function $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, and every \mathcal{X} -POVM \mathbf{P} on \mathcal{H} , there exists a \mathcal{X} -r.d.f. ϕ on $\Delta_{\mathcal{H}}$ such that $\mathbf{s}(\mathcal{E}_{\mathbf{R}}, \mathcal{X}, \ell, \phi) = \mathbf{s}_{\mathbf{q}}(\mathbf{R}, \mathcal{X}, \ell, \mathbf{P})$. Such a correspondence is obtained by first introducing a sample set $\Delta_{\mathcal{H}} = \{\delta\}$ with $|\Delta_{\mathcal{H}}| = \dim \mathcal{H}$, so that any orthonormal basis for \mathcal{H} can be indexed by $\Delta_{\mathcal{H}}$. Then, since all density matrices ρ_{θ} are pairwise commuting, an orthonormal basis $\{|\varphi_{\delta}\rangle \in \mathcal{H}\}_{\delta \in \Delta_{\mathcal{H}}}$ for \mathcal{H} exists, with respect to which all ρ_{θ} are simultaneously diagonal. Finally, the family of probability distributions $\alpha_{\rho} = (p_{\theta}; \theta \in \Theta)$ on $\Delta_{\mathcal{H}}$ is defined according to the relation $p_{\theta}(\delta) := \langle \varphi_{\delta} | \rho_{\theta} | \varphi_{\delta} \rangle$, for all $\delta \in \Delta_{\mathcal{H}}$ and $\theta \in \Theta$. Then, it is easy to check that the statistical model $\mathcal{E}_{\mathbf{R}} = (\Theta, \Delta_{\mathcal{H}}, \alpha_{\rho})$, obtained in this way from $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$, is completely equivalent to the initial quantum statistical model \mathbf{R} , in the sense explained above. This in particular implies that, for every finite decision set \mathcal{X} and every payoff function $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, $\$(\mathcal{E}_{\mathbf{R}}, \mathcal{X}, \ell) = \$_{\mathbf{q}}(\mathbf{R}, \mathcal{X}, \ell)$.

Conversely, given a (classical) statistical model $\mathcal{E} = (\Theta, \Delta, \alpha)$, it is always possible to construct an equivalent abelian quantum statistical model $\mathbf{R}_{\mathcal{E}} = (\Theta, \mathcal{H}_{\Delta}, \rho_{\alpha})$, by introducing a Hilbert space \mathcal{H}_{Δ} , with $\dim \mathcal{H}_{\Delta} = |\Delta|$, and a family $\rho_{\alpha} = (\rho_{\theta}; \theta \in \Theta)$ of diagonal density matrices on \mathcal{H}_{Δ} , defined by the relation $\rho_{\theta} = \sum_{\delta \in \Delta} p_{\theta}(\delta) |\varphi_{\delta}\rangle \langle \varphi_{\delta}|$, where $\{|\varphi_{\delta}\rangle \in \mathcal{H}_{\Delta}\}_{\delta \in \Delta}$ is any orthonormal basis for \mathcal{H}_{Δ} . Also in this case, it is easy to check that the quantum statistical model $\mathbf{R}_{\mathcal{E}} = (\Theta, \mathcal{H}_{\Delta}, \rho_{\alpha})$, obtained in this way from $\mathcal{E} = (\Theta, \Delta, \alpha)$, is completely equivalent to the initially given statistical model \mathcal{E} , in the sense that, for every finite decision set \mathcal{X} , every payoff function $\ell : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, and every \mathcal{X} -r.d.f. ϕ on Δ , there exists a \mathcal{X} -POVM \mathbf{P} on \mathcal{H}_{Δ} such that $\mathbf{s}_{\mathbf{q}}(\mathbf{R}_{\mathcal{E}}, \mathcal{X}, \ell, \mathbf{P}) = \mathbf{s}(\mathcal{E}, \mathcal{X}, \ell, \phi)$. In particular, $\$_{\mathbf{q}}(\mathbf{R}_{\mathcal{E}}, \mathcal{X}, \ell) = \$(\mathcal{E}, \mathcal{X}, \ell)$.

These ideas can be compactly re-expressed as follows:

Postulate 1 (Correspondence principle). *Classical statistical models are identified with abelian quantum statistical models, and viceversa.*

A quantum statistical model \mathbf{R} involves a parameter set Θ and a Hilbert space \mathcal{H} . In a sense, then, a quantum statistical model constitutes an asymmetric structure, where a quantum system carries information about a classical parameter. It is useful hence to provide a notion for a “fully quantum” information structure. This can be done as follows: given a finite parameter set Θ , let \mathcal{H}_Θ be a Hilbert space such that $\dim \mathcal{H}_\Theta = |\Theta|$, i.e., such that there exists a complete set of orthonormal vectors $\{|\varphi_\theta\rangle \in \mathcal{H}_\Theta\}_{\theta \in \Theta}$, labeled by θ , which form a basis for \mathcal{H}_Θ . For the sake of notation, let us denote $|\varphi_\theta\rangle$ simply by $|\theta\rangle$. Then, each quantum model $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ defines a corresponding bipartite quantum state

$$\rho_{AB}^{\mathbf{R}} := \frac{1}{|\Theta|} \sum_{\theta \in \Theta} |\theta\rangle\langle\theta|_A \otimes \rho_B^\theta, \tag{8}$$

where $\mathcal{H}_A \cong \mathcal{H}_\Theta$, $\mathcal{H}_B \cong \mathcal{H}$, and $\rho_B^\theta \equiv \rho_\theta$. The particular “classical-quantum” structure of the state given in (8) reflects the above mentioned “hybrid” structure of a quantum statistical model. Instead, by allowing ρ_{AB} to be an arbitrary bipartite state, we arrive at the following definition:

Definition 8 (Quantum Information Structures [15]). A **quantum information structure** \mathfrak{q}_{AB} is defined as a triple $(\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$, where \mathcal{H}_A and \mathcal{H}_B are finite dimensional Hilbert spaces, and $\rho_{AB} \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

The notion of quantum information structure is hence the “fully quantized” analogue of a quantum statistical model. In the same way in which a quantum statistical model can be used to define a quantum statistical decision problem, a quantum information structure can be used to define a quantum game² as follows:

Definition 9 (Quantum Statistical Decision Games [15]). A **quantum statistical decision game** is defined as a triple $(\mathfrak{q}_{AB}, \mathcal{X}, \mathcal{O}_A^\mathcal{X})$, where $\mathfrak{q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ is a quantum information structure, \mathcal{X} is a (finite) decision set $\{i\}_{i \in \mathcal{X}}$, and $\mathcal{O}_A^\mathcal{X}$ is a family $(O_A^i; i \in \mathcal{X})$ of self-adjoint **payoff operators** $O_A^i \in \mathfrak{L}(\mathcal{H}_A)$. (From now on, the superscript \mathcal{X} in $\mathcal{O}_A^\mathcal{X}$ will be dropped when clear from the context.) The choice of a **strategy** for player B corresponds to the choice of a POVM $\mathbf{P}_B = (P_B^i; i \in \mathcal{X})$ on \mathcal{H}_B . The corresponding expected payoff is computed as

$$\mathfrak{s}_q(\mathfrak{q}_{AB}, \mathcal{X}, \mathcal{O}_A, \mathbf{P}_B) := \sum_{i \in \mathcal{X}} \text{Tr} \left[\left(O_A^i \otimes P_B^i \right) \rho_{AB} \right]. \tag{9}$$

The maximum expected payoff is given by

$$\mathfrak{s}_q(\mathfrak{q}_{AB}, \mathcal{X}, \mathcal{O}_A) := \max_{\mathbf{P}_B} \sum_{i \in \mathcal{X}} \text{Tr} \left[\left(O_A^i \otimes P_B^i \right) \rho_{AB} \right]. \tag{10}$$

The following definition was introduced in [15] as a very natural analogue of Definition 2:

² In the very specific sense given in Ref. [15].

Definition 10 (Information Ordering). *Given two quantum information structures $\mathcal{Q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ and $\mathcal{S}_{AB'} = (\mathcal{H}_A, \mathcal{H}_{B'}, \sigma_{AB'})$, \mathcal{Q}_{AB} is said to be **always more informative than** $\mathcal{S}_{AB'}$, in formula,*

$$\mathcal{Q}_{AB} \supset_A \mathcal{S}_{AB'}, \tag{11}$$

if and only if, for every finite decision set \mathcal{X} and every family of self-adjoint payoff operators $O_A = (O_A^i; i \in \mathcal{X})$ on \mathcal{H}_A ,

$$\$_{\mathcal{Q}}(\mathcal{Q}_{AB}, \mathcal{X}, O_A) \geq \$_{\mathcal{S}}(\mathcal{S}_{AB'}, \mathcal{X}, O_A). \tag{12}$$

Remark 4. In analogy with Remark 3, here we note that any quantum information structure $\mathcal{Q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$, for which a decomposition like that in Eq. (8) exists, naturally induces a corresponding quantum statistical model $\mathbf{R}_{\mathcal{Q}} = (\Theta, \mathcal{H}_B, (\rho_B^\theta; \theta \in \Theta))$, where the states ρ_B^θ are those appearing in (8). Moreover, any quantum statistical decision game $(\mathcal{Q}_{AB}, \mathcal{X}, O_A)$ built upon such a classical-quantum structure \mathcal{Q}_{AB} is completely equivalent to a quantum statistical decision problem $(\mathbf{R}_{\mathcal{Q}}, \mathcal{X}, \ell_O)$, in the sense that $\$_{\mathcal{Q}}(\mathcal{Q}_{AB}, \mathcal{X}, O_A) = \$_{\mathcal{S}}(\mathbf{R}_{\mathcal{Q}}, \mathcal{X}, \ell_O)$, where the payoff function ℓ_O is defined by $\ell(\theta, i) := \langle \theta_A | O_A^i | \theta_A \rangle$, with the vectors $|\theta_A\rangle$ being the same as in (8).

For the reader’s convenience, we end this section by summarizing the contents of Remarks 3 and 4 as follows:

1. the most general notion is that of quantum statistical decision games over quantum information structures;
2. quantum statistical decision problems over quantum statistical models are equivalent to quantum statistical decision games over hybrid classical-quantum information structures;
3. classical statistical decision problems over statistical models are equivalent to quantum decision problems over abelian quantum statistical models.

In other words, quantum information structures contain quantum statistical models (as hybrid structures), which, in turn, contain classical statistical models (as abelian models). For this reason we will first formulate our results for quantum information structures: quantum statistical models and classical statistical models will be considered afterwards, as particular cases.

4. Sufficiency Conditions for Quantum Information Structures and Statistical Morphisms

In the previous section we extended the notion of information ordering to quantum statistical models and quantum information structures, depending on their “information value” in quantum statistical decision problems and quantum statistical decision games, respectively. In the following we will carefully define what it means that a quantum information structure is sufficient for another. In order to do this, we will need to consider a formalism which is slightly more general than the one we used before.

We begin with the following definition:

Definition 11 (State Spaces). *The **state space** \mathfrak{S} of a quantum system defined on a Hilbert space \mathcal{H} is a non-empty subset of $\mathfrak{S}(\mathcal{H})$, containing all possible physical states of the system.*

Remark 5. Usually, the state space \mathfrak{S} coincides with the set $\mathfrak{S}(\mathcal{H})$ of all possible density matrices acting on \mathcal{H} . However, there are cases in which the states accessible to the system form a proper subset of $\mathfrak{S}(\mathcal{H})$, for example, when the system is known to obey a conservation law. For later convenience, we keep our definition of state space as general as possible. This is also the reason why, in the above definition, there is no assumption about the convexity of the state space, as we do not need such assumption in general (even though, in many physically relevant situations, that would seem rather natural).

Definition 12 (Effects and Tests). *An operator $X \in \mathfrak{L}(\mathcal{H})$ is called an **effect** on a state space \mathfrak{S} (defined on \mathcal{H}) if and only if there exists an operator $P \in \mathfrak{L}(\mathcal{H})$, with $0 \leq P \leq \mathbb{1}$, such that $\text{Tr}[X\rho] = \text{Tr}[P\rho]$, for all $\rho \in \mathfrak{S}$. For any (finite) decision set $\mathcal{X} = \{i\}$, a family $(M^i; i \in \mathcal{X})$ of operators $M^i \in \mathfrak{L}(\mathcal{H})$ is called a **\mathcal{X} -test** $\mathfrak{M}^{\mathcal{X}}$ on a state space \mathfrak{S} (defined on \mathcal{H}) if and only if there exists a \mathcal{X} -POVM $P^{\mathcal{X}} = (P^i; i \in \mathcal{X})$ on \mathcal{H} with $\text{Tr}[M^i \rho] = \text{Tr}[P^i \rho]$, for all $i \in \mathcal{X}$ and for all $\rho \in \mathfrak{S}$. Any such POVM $P^{\mathcal{X}}$ is said to realize the test $\mathfrak{M}^{\mathcal{X}}$ on \mathfrak{S} . From now on, the superscript \mathcal{X} will be dropped when clear from the context.*

Remark 6. For a given state space \mathfrak{S} and a given decision set \mathcal{X} , two families $\mathfrak{M} = (M^i; i \in \mathcal{X})$ and $\mathfrak{N} = (N^i; i \in \mathcal{X})$ of operators in $\mathfrak{L}(\mathcal{H})$ are statistically equivalent on \mathfrak{S} , in formula $\mathfrak{M} \sim_{\mathfrak{S}} \mathfrak{N}$, if and only if $\text{Tr}[M^i \rho] = \text{Tr}[N^i \rho]$, for all $i \in \mathcal{X}$ and all $\rho \in \mathfrak{S}$. For any family $\mathfrak{M} = (M^i; i \in \mathcal{X})$, let $[\mathfrak{M}]_{\mathfrak{S}}$ be the corresponding equivalence class induced by $\sim_{\mathfrak{S}}$. Any \mathcal{X} -test on \mathfrak{S} can hence be thought of as the equivalence class of some \mathcal{X} -POVM on \mathcal{H} .

Remark 7. A second, more physically motivated way to think of tests is the following: \mathcal{X} -tests on a state space \mathfrak{S} are those affine mappings, from \mathfrak{S} to probability distributions on \mathcal{X} , which can be *physically realized* as quantum measurements. This is guaranteed by requiring, in the definition of test, the existence of at least one POVM that is statistically equivalent to it: in fact, all physically realizable quantum measurements give rise to a POVM, and any POVM can be physically measured [21]. Such a restriction in the definition of tests is meaningful only if there exist cases of affine mappings from a state space \mathfrak{S} to probability distributions on a decision set \mathcal{X} , which cannot be realized by any POVM. If the state space is the totality of states $\mathfrak{S}(\mathcal{H})$, then, all such affine mappings are in one-to-one correspondence with POVM's, and there is no need to introduce further definitions. However, in the general case in which $\mathfrak{S} \subset \mathfrak{S}(\mathcal{H})$, the distinction between tests and “unphysical” affine mappings become relevant, and Definition 12 is necessary.

We are now in the position to rigorously introduce the idea which will be the basis of our analysis:

Definition 13 (Statistical Morphisms). *Given two state spaces \mathfrak{S}_{in} (defined on a Hilbert space \mathcal{H}_{in}) and $\mathfrak{S}_{\text{out}}$ (defined on a Hilbert space \mathcal{H}_{out}), we say that a linear map $\mathcal{L} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{out}})$ induces a statistical morphism $\mathcal{L} : \mathfrak{S}_{\text{in}} \rightarrow \mathfrak{S}_{\text{out}}$ if and only if the following conditions are both satisfied:*

1. *for every $\rho \in \mathfrak{S}_{\text{in}}$, $\mathcal{L}(\rho) \in \mathfrak{S}_{\text{out}}$;*
2. *the dual transformation $\mathcal{L}^* : \mathfrak{L}(\mathcal{H}_{\text{out}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{in}})$, defined by trace duality,³ maps tests on $\mathfrak{S}_{\text{out}}$ into tests on \mathfrak{S}_{in} .*

³ For any operator $X \in \mathfrak{L}(\mathcal{H}_{\text{out}})$, $\mathcal{L}^*(X) \in \mathfrak{L}(\mathcal{H}_{\text{in}})$ is defined by the relation $\text{Tr}[\mathcal{L}^*(X) Y] = \text{Tr}[X \mathcal{L}(Y)]$, for every $Y \in \mathfrak{L}(\mathcal{H}_{\text{in}})$.

Remark 8. Notice that the notion of statistical morphism, introduced in Definition 13, is in principle strictly weaker than the notion of positive map, which is a linear map that transforms positive operators into positive operators. In fact, given a positive operator $P \leq \mathbb{1}$ on \mathcal{H}_{out} , the operator $\mathcal{L}^*(P)$ might have negative eigenvalues, and yet be an effect on \mathfrak{S}_{in} , according to Definition 12. On the contrary, a linear, trace-preserving, positive map from $\mathfrak{L}(\mathcal{H}_{\text{in}})$ to $\mathfrak{L}(\mathcal{H}_{\text{out}})$ always constitutes a statistical morphism. An open question is whether any statistical morphism can always be extended to a positive map. Indications that this might not be true in general are provided in Ref. [22], Cor. 10. However, at the moment of writing, an explicit counterexample is not available.

Definition 14. Given a quantum information structure $\mathfrak{Q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$, the associated state space $\mathfrak{S}_B(\mathfrak{Q}_{AB}) \subseteq \mathfrak{S}(\mathcal{H}_B)$ of physical states of the subsystem B is defined as

$$\mathfrak{S}_B(\mathfrak{Q}_{AB}) := \left\{ \frac{\text{Tr}_A[(P_A \otimes \mathbb{1}_B)\rho_{AB}]}{\text{Tr}[(P_A \otimes \mathbb{1}_B)\rho_{AB}]} \mid P_A \in \mathfrak{L}(\mathcal{H}_A) : 0 \leq P_A \leq \mathbb{1}_A \right\}. \quad (13)$$

Remark 9. The state spaces associated with given information structures turn out to be convex state spaces. This can be easily verified by direct inspection.

Remark 10. From now on, it is convenient to think that, in Eq. (10), the maximum over POVM's P_B on \mathcal{H}_B is replaced by a maximum over tests \mathfrak{M}_B on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$. Such a replacement, which is formally convenient, is quantitatively irrelevant, since it does not affect the value of the maximum expected payoff, nor does it modify the information ordering relation introduced in Definition 10.

We are now able to rigorously define the notion of sufficiency (in a sense analogous to the one used by Blackwell in [6]) for quantum information structures, in its two variants: sufficiency and m-sufficiency.

Definition 15 (Sufficiency and m-sufficiency). Given two quantum information structures $\mathfrak{Q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ and $\mathfrak{S}_{AB'} = (\mathcal{H}_A, \mathcal{H}_{B'}, \sigma_{AB'})$, we say that \mathfrak{Q}_{AB} is **m-sufficient for $\mathfrak{S}_{AB'}$** , in formula

$$\mathfrak{Q}_{AB} \succ_m \mathfrak{S}_{AB'}, \quad (14)$$

if and only if there exists a statistical morphism $\mathcal{L}_B : \mathfrak{S}_B(\mathfrak{Q}_{AB}) \rightarrow \mathfrak{S}_{B'}(\mathfrak{S}_{AB'})$ such that

$$\sigma_{AB'} = (\text{id}_A \otimes \mathcal{L}_B)(\rho_{AB}). \quad (15)$$

We say that \mathfrak{Q}_{AB} is **sufficient for $\mathfrak{S}_{AB'}$** , in formula

$$\mathfrak{Q}_{AB} \succ \mathfrak{S}_{AB'}, \quad (16)$$

if and only if there exists a completely positive, trace-preserving map $\mathcal{E}_B : \mathfrak{L}(\mathcal{H}_B) \rightarrow \mathfrak{L}(\mathcal{H}_{B'})$ such that

$$\sigma_{AB'} = (\text{id}_A \otimes \mathcal{E}_B)(\rho_{AB}). \quad (17)$$

Intuitively speaking, the idea of sufficiency is related with the fact that the transformation can be actually performed *physically*, as an open evolution. On the contrary, the notion of m-sufficiency introduced here just assumes the existence of a *formal* statistical procedure to map one strategy into another.

5. Extension Theorems for Statistical Morphisms

Even if the notion of statistical morphism is weaker than that of positive map, two famous extension theorems for positive maps, proved by Choi [18] and Arveson [19], can be generalized to statistical morphisms as well.

Definition 16 (Complete State Spaces). *A state space \mathfrak{S} on \mathcal{H} is called **complete** for $\mathfrak{L}(\mathcal{H})$ if and only if it contains $(\dim \mathcal{H})^2$ linearly independent density matrices.*

Definition 17 (Composition of State Spaces). *Given two state spaces \mathfrak{S}_α (on \mathcal{H}_α) and \mathfrak{S}_β (on \mathcal{H}_β), we define the set*

$$\mathfrak{S}_\alpha \times \mathfrak{S}_\beta := \{ \sigma_\alpha \otimes \tau_\beta \mid \sigma_\alpha \in \mathfrak{S}_\alpha, \tau_\beta \in \mathfrak{S}_\beta \}. \tag{18}$$

An operator $X \in \mathfrak{L}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$ is an effect on $\mathfrak{S}_\alpha \times \mathfrak{S}_\beta$ if and only if there exists an operator $P \in \mathfrak{L}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$, $0 \leq P \leq \mathbb{1}_\alpha \otimes \mathbb{1}_\beta$, such that $\text{Tr}[X(\sigma_\alpha \otimes \tau_\beta)] = \text{Tr}[P(\sigma_\alpha \otimes \tau_\beta)]$, for all $\sigma_\alpha \in \mathfrak{S}_\alpha$ and $\tau_\beta \in \mathfrak{S}_\beta$. In the same way we extend the notion of tests. Notice that effects or tests on $\mathfrak{S}_\alpha \times \mathfrak{S}_\beta$ need not be factorized.

Proposition 2. *Given two state spaces \mathfrak{S}_{in} and $\mathfrak{S}_{\text{out}}$, defined on \mathcal{H}_{in} and \mathcal{H}_{out} , respectively, and a third auxiliary complete state space \mathfrak{S}_0 , defined on $\mathcal{H}_0 \cong \mathcal{H}_{\text{out}}$, suppose that the linear map $\text{id} \otimes \mathcal{L} : \mathfrak{L}(\mathcal{H}_0) \otimes \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_0) \otimes \mathfrak{L}(\mathcal{H}_{\text{out}})$ induces a statistical morphism from $\mathfrak{S}_0 \times \mathfrak{S}_{\text{in}}$ to $\mathfrak{S}_0 \times \mathfrak{S}_{\text{out}}$. Then, there exists a completely positive, trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{out}})$ such that*

$$\mathcal{L}(\sigma) = \mathcal{E}(\sigma), \tag{19}$$

for all $\sigma \in \mathfrak{S}_{\text{in}}$.

Proof. Let $(B^i)_{i=1}^{d^2}$, where $d = \dim \mathcal{H}_0 = \dim \mathcal{H}_{\text{out}}$, be the POVM consisting of the d^2 generalized Bell projectors acting on $\mathcal{H}_0 \otimes \mathcal{H}_{\text{out}}$. By trace-duality:

$$\text{Tr} \left[B^i (\omega \otimes \mathcal{L}(\sigma)) \right] = \text{Tr} \left[(\text{id} \otimes \mathcal{L}^*)(B^i) (\omega \otimes \sigma) \right], \tag{20}$$

for all $\sigma \in \mathfrak{S}_{\text{in}}$ and all $\omega \in \mathfrak{S}_0$. The fact that $\text{id} \otimes \mathcal{L}$ is a statistical morphism implies, by definition, that the operators $((\text{id} \otimes \mathcal{L}^*)(B^i))_{i=1}^{d^2}$, even if not positive, yet induce a test on $\mathfrak{S}_0 \times \mathfrak{S}_{\text{in}}$. In other words, there exists a POVM $(\tilde{B}^i)_{i=1}^{d^2}$ on $\mathcal{H}_0 \otimes \mathcal{H}_{\text{in}}$ such that

$$\text{Tr} \left[(\text{id} \otimes \mathcal{L}^*)(B^i) (\omega \otimes \sigma) \right] = \text{Tr} \left[\tilde{B}^i (\omega \otimes \sigma) \right], \tag{21}$$

for all $\sigma \in \mathfrak{S}_{\text{in}}$, all $\omega \in \mathfrak{S}_0$, and every i . Due to the assumption that \mathfrak{S}_0 is complete, there always exist d^2 states in \mathfrak{S}_0 which form an operator basis for $\mathfrak{L}(\mathcal{H}_0)$. We can then extend Eq. (21) by linearity and obtain that, in fact,

$$\text{Tr} \left[B^i (X \otimes \mathcal{L}(\sigma)) \right] = \text{Tr} \left[\tilde{B}^i (X \otimes \sigma) \right], \tag{22}$$

for all $\sigma \in \mathfrak{S}_{\text{in}}$, all $X \in \mathfrak{L}(\mathcal{H}_0)$, and every i .

Using the POVM $(\tilde{B}^i)_{i=1}^{d^2}$ (whose existence we proved above), we now consider the identity (via teleportation):

$$\begin{aligned} \mathcal{L}(\sigma) &= \sum_{i=1}^{d^2} \text{Tr}_{\beta\gamma} \left[\left(U_\alpha^i \otimes \mathbb{1}_{\beta\gamma} \right) \left(\mathbb{1}_\alpha \otimes B_{\beta\gamma}^i \right) \left(\Psi_{\alpha\beta}^+ \otimes \mathcal{L}_\gamma(\sigma_\gamma) \right) \left((U_\alpha^i)^\dagger \otimes \mathbb{1}_{\beta\gamma} \right) \right] \\ &= \sum_{i=1}^{d^2} \text{Tr}_{\beta\gamma} \left[\left(U_\alpha^i \otimes \mathbb{1}_{\beta\gamma} \right) \left(\mathbb{1}_\alpha \otimes \tilde{B}_{\beta\gamma}^i \right) \left(\Psi_{\alpha\beta}^+ \otimes \sigma_\gamma \right) \left((U_\alpha^i)^\dagger \otimes \mathbb{1}_{\beta\gamma} \right) \right], \end{aligned} \tag{23}$$

where $\Psi^+ = d^{-1} \sum_{i,j=1}^d |i\rangle\langle j| \otimes |i\rangle\langle j|$ is a maximally entangled state on $\mathcal{H}_0^{\otimes 2}$ and $(U^i)_{i=1}^{d^2}$ is an appropriate family of unitary matrices on \mathcal{H}_0 . The relation above holds for all $\sigma \in \mathfrak{S}_{\text{in}} \subseteq \mathfrak{S}(\mathcal{H}_{\text{in}})$. However, it is clear that the last term in Eq. (23) can be extended, by linearity, to a completely positive trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_0) \cong \mathfrak{L}(\mathcal{H}_{\text{out}})$ defined as:

$$\begin{aligned} \mathcal{E}(\rho) &:= \sum_{i=1}^{d^2} \text{Tr}_{\beta\gamma} \left[\left(U_\alpha^i \otimes \mathbb{1}_{\beta\gamma} \right) \left(\mathbb{1}_\alpha \otimes \tilde{B}_{\beta\gamma}^i \right) \left(\Psi_{\alpha\beta}^+ \otimes \rho_\gamma \right) \left((U_\alpha^i)^\dagger \otimes \mathbb{1}_{\beta\gamma} \right) \right], \end{aligned} \tag{24}$$

for all $\rho \in \mathfrak{S}(\mathcal{H}_{\text{in}})$. This hence concludes the proof that a completely positive trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{out}})$ exists, such that

$$\mathcal{E}(\sigma) = \mathcal{L}(\sigma), \tag{25}$$

for all $\sigma \in \mathfrak{S}_{\text{in}}$. \square

Another important case is when the output state space $\mathfrak{S}_{\text{out}}$ is abelian, namely, $[\rho, \sigma] = 0$, for all $\rho, \sigma \in \mathfrak{S}_{\text{out}}$. This condition, in particular, implies that there exists an orthonormal basis $\{|i\rangle\}_{i=1}^d$ for \mathcal{H}_{out} that diagonalizes all $\rho \in \mathfrak{S}_{\text{out}}$.

Proposition 3. *Given two state spaces \mathfrak{S}_{in} and $\mathfrak{S}_{\text{out}}$, defined on \mathcal{H}_{in} and \mathcal{H}_{out} , respectively, let $\mathfrak{S}_{\text{out}}$ be abelian. If there exists a linear map $\mathcal{L} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{out}})$ inducing a statistical morphism from \mathfrak{S}_{in} to $\mathfrak{S}_{\text{out}}$, then there exists a completely positive, trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{out}})$ such that*

$$\mathcal{L}(\rho) = \mathcal{E}(\rho), \tag{26}$$

for all $\rho \in \mathfrak{S}_{\text{in}}$.

Proof. For $d = \dim \mathcal{H}_{\text{out}}$, let $\{|i\rangle\}_{i=1}^d$ be the basis for \mathcal{H}_{out} that simultaneously diagonalizes every $\sigma \in \mathfrak{S}_{\text{out}}$, and denote by $\Pi_i \in \mathfrak{L}(\mathcal{H}_{\text{out}})$ each projector $|i\rangle\langle i|$. Then, for any $\sigma \in \mathfrak{S}_{\text{out}}$,

$$\sigma = \sum_{i=1}^d \text{Tr}[\Pi_i \sigma] \Pi_i. \tag{27}$$

Next, we note that, by definition of the trace-dual map \mathcal{L}^* ,

$$\text{Tr} \left[\Pi^i \mathcal{L}(\rho) \right] = \text{Tr} \left[\mathcal{L}^*(\Pi^i) \rho \right], \tag{28}$$

for all $\rho \in \mathfrak{S}_{\text{in}}$. The fact that \mathcal{L} is a statistical morphism implies, by definition, that the operators $(\mathcal{L}^*(\Pi^i))_{i=1}^d$, even if not positive, yet induce a test on \mathfrak{S}_{in} . In other words, there exists a POVM $(\tilde{\Pi}^i)_{i=1}^d$ such that

$$\text{Tr} \left[\mathcal{L}^*(\Pi^i) \rho \right] = \text{Tr} \left[\tilde{\Pi}^i \rho \right], \tag{29}$$

for all $\rho \in \mathfrak{S}_{\text{in}}$ and every i .

Using the POVM $(\tilde{\Pi}^i)_{i=1}^d$ (whose existence we proved above), we recall Eq. (27) above and consider the identity:

$$\begin{aligned} \mathcal{L}(\rho) &= \sum_{i=1}^d \text{Tr} \left[\Pi^i \mathcal{L}(\rho) \right] \Pi^i \\ &= \sum_{i=1}^d \text{Tr} \left[\tilde{\Pi}^i \rho \right] \Pi^i, \end{aligned} \tag{30}$$

The relation above holds for all $\rho \in \mathfrak{S}_{\text{in}} \subseteq \mathfrak{S}(\mathcal{H}_{\text{in}})$. However, it is clear that the last term in Eq. (30) can be extended, by linearity, to a completely positive trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{out}})$ defined as:

$$\mathcal{E}(\rho) := \sum_{i=1}^d \text{Tr} \left[\tilde{\Pi}^i \rho \right] \Pi^i, \tag{31}$$

for all $\rho \in \mathfrak{S}(\mathcal{H}_{\text{in}})$. This hence concludes the proof that a completely positive trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}_{\text{in}}) \rightarrow \mathfrak{L}(\mathcal{H}_{\text{out}})$ exists, such that

$$\mathcal{E}(\rho) = \mathcal{L}(\rho), \tag{32}$$

for all $\rho \in \mathfrak{S}_{\text{in}}$. \square

6. A Fundamental Equivalence Relation

In this section, we prove our main result:

Theorem 2. *Given two quantum information structures $\mathfrak{Q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ and $\mathfrak{S}_{AB'} = (\mathcal{H}_A, \mathcal{H}_{B'}, \sigma_{AB'})$,*

$$\mathfrak{Q}_{AB} \succ_m \mathfrak{S}_{AB'} \Leftrightarrow \mathfrak{Q}_{AB} \supset_A \mathfrak{S}_{AB'}. \tag{33}$$

Moreover, the linear map inducing the statistical morphism between \mathfrak{Q}_{AB} and $\mathfrak{S}_{AB'}$ can always be chosen to be trace-preserving on the whole space $\mathfrak{L}(\mathcal{H}_B)$.

Remark 11. Shmaya, in Remark 7 of his Ref. [15], asks the question whether $\mathfrak{Q}_{AB} \supset_A \mathfrak{S}_{AB'}$ is equivalent to the existence of a positive trace-preserving map \mathcal{P} such that $\sigma_{AB'} = (\text{id} \otimes \mathcal{P})\rho_{AB}$. The above theorem shows that Shmaya’s question is equivalent to asking whether any trace-preserving statistical morphism always admits a trace-preserving positive extension (about this point, see Remark 8 above).

For the sake of clarity, we divide the proof of Theorem 2 in two parts. The first part is a lemma proved by Shmaya in Ref. [15], as a direct consequence of the Separation Theorem for convex sets (see, e. g., Ref. [20]).

Before stating the lemma, we introduce the following notation: given a quantum information structure $\mathfrak{Q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$, a decision set \mathcal{X} , and a test $\mathfrak{M}_B = (M_B^i; i \in \mathcal{X})$ on the state space $\mathfrak{S}_B(\mathfrak{Q}_{AB})$, we define the following operators:

$$\rho_{A|\mathfrak{M}}^i := \text{Tr}_B \left[(\mathbb{1}_A \otimes M_B^i) \rho_{AB} \right], \tag{34}$$

for each $i \in \mathcal{X}$. In Eq. (34), we can replace the family of operators \mathfrak{M}_B by any other family of operators which is statistically equivalent (in the sense of Remark 6) to \mathfrak{M}_B on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$ ⁴. In particular, we can replace the operators M_B^i by the elements P_B^i of any POVM $P_B = (P_B^i; i \in \mathcal{X})$ on \mathcal{H}_B realizing the test \mathfrak{M}_B on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$.

We are now ready to state the following:

Lemma 1 (Shmaya [15]). *Given two quantum information structures $\mathfrak{Q}_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ and $\mathfrak{S}_{AB'} = (\mathcal{H}_A, \mathcal{H}_{B'}, \sigma_{AB'})$, if $\mathfrak{Q}_{AB} \supset_A \mathfrak{S}_{AB'}$, then, for any finite decision set \mathcal{X} and any test $\mathfrak{N}_{B'} = (N_{B'}^i; i \in \mathcal{X})$ on $\mathfrak{S}_{B'}(\mathfrak{S}_{AB'})$, there exists a test $\overline{\mathfrak{M}}_B = (\overline{M}_B^i; i \in \mathcal{X})$ on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$ such that*

$$\rho_{A|\overline{\mathfrak{M}}}^i = \sigma_{A|\mathfrak{N}}^i \tag{35}$$

for all $i \in \mathcal{X}$.

Proof. For the reader’s convenience, we reformulate here Shmaya’s proof according to our notation. For any finite decision set \mathcal{X} , let us consider the set $\mathcal{C}_A(\mathfrak{Q}_{AB}, \mathcal{X})$ of all $|\mathcal{X}|$ -tuples

$$\left(\rho_{A|\mathfrak{M}}^1, \rho_{A|\mathfrak{M}}^2, \dots, \rho_{A|\mathfrak{M}}^{|\mathcal{X}|} \right), \tag{36}$$

where \mathfrak{M}_B varies over all possible \mathcal{X} -tests on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$. Clearly, $\mathcal{C}_A(\mathfrak{Q}_{AB}, \mathcal{X})$ is a closed and bounded convex subset of the (real) linear space of $|\mathcal{X}|$ -tuples $(T^i; i \in \mathcal{X})$ of self-adjoint matrices on \mathcal{H}_A , since it inherits its structure from the convex structure of the set of \mathcal{X} -tests on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$.

The proof then proceeds by *reductio ad absurdum*. Suppose in fact that, for some decision set \mathcal{X} , there exists a test $\mathfrak{N}_{B'} = (N_{B'}^i; i \in \mathcal{X})$ on $\mathfrak{S}_{B'}(\mathfrak{S}_{AB'})$ such that the corresponding $|\mathcal{X}|$ -tuple

$$\left(\sigma_{A|\mathfrak{N}}^1, \sigma_{A|\mathfrak{N}}^2, \dots, \sigma_{A|\mathfrak{N}}^{|\mathcal{X}|} \right) \notin \mathcal{C}_A(\mathfrak{Q}_{AB}, \mathcal{X}). \tag{37}$$

Then, by the so-called Separation Theorem between convex sets (see, e. g., Ref. [20], Cor. 11.4.2), there exists a $|\mathcal{X}|$ -tuple of self-adjoint operators $(\tilde{T}_A^i; i \in \mathcal{X})$ on \mathcal{H}_A , such that

$$\max_{\mathfrak{M}_B} \sum_{i \in \mathcal{X}} \text{Tr} \left[\rho_{A|\mathfrak{M}}^i \tilde{T}_A^i \right] < \sum_{i \in \mathcal{X}} \text{Tr} \left[\sigma_{A|\mathfrak{N}}^i \tilde{T}_A^i \right], \tag{38}$$

⁴ This fact can be proved by noticing that the joint probability distribution $p_{\mathcal{Y}, \mathcal{X}}(j, i) := \text{Tr}[(F_A^j \otimes M_B^i) \rho_{AB}]$, where $(F_A^j; j \in \mathcal{Y})$ is an informationally complete POVM on \mathcal{H}_A , equals, for all $j \in \mathcal{Y}$ and all $i \in \mathcal{X}$, that computed as $\text{Tr}[(F_A^j \otimes X_B^i) \rho_{AB}]$, whenever $(X_B^i; i \in \mathcal{X}) \sim_{\mathfrak{S}_B(\mathfrak{Q}_{AB})} (M_B^i; i \in \mathcal{X})$. By the completeness of $(F_A^j; j \in \mathcal{Y})$, we conclude that, in fact, $\text{Tr}_B[(\mathbb{1}_A \otimes M_B^i) \rho_{AB}] = \text{Tr}_B[(\mathbb{1}_A \otimes X_B^i) \rho_{AB}]$, for all $i \in \mathcal{X}$.

where the maximization is taken over all tests $\mathfrak{M}_B = (M_B^i; i \in \mathcal{X})$ on $\mathfrak{S}_B(\varrho_{AB})$. This contradicts the assumption $\varrho_{AB} \supset_A \varsigma_{AB'}$. \square

Proof of Theorem 2. One direction of the theorem, that is $\varrho_{AB} \succ_m \varsigma_{AB'} \Rightarrow \varrho_{AB} \supset_A \varsigma_{AB'}$, simply follows from the definition of m-sufficiency given in Definition 15.

Only the converse direction, i.e. $\varrho_{AB} \supset_A \varsigma_{AB'} \Rightarrow \varrho_{AB} \succ_m \varsigma_{AB'}$, is hence non-trivial. In order to construct a statistical morphism $\mathcal{L}_B : \mathfrak{S}_B(\varrho_{AB}) \rightarrow \mathfrak{S}_{B'}(\varsigma_{AB'})$, consider the decision set $\mathcal{X} = \{1, 2, \dots, (\dim \mathcal{H}_{B'})^2\}$ and an informationally complete \mathcal{X} -POVM $(F_{B'}^i; i \in \mathcal{X})$ on $\mathcal{H}_{B'}$, with self-adjoint dual operators $(\theta_{B'}^i; i \in \mathcal{X})$. The following identity holds:

$$T_{B'} = \sum_{i \in \mathcal{X}} \text{Tr}[T_{B'} F_{B'}^i] \theta_{B'}^i, \tag{39}$$

for all operators $T_{B'} \in \mathfrak{L}(\mathcal{H}_{B'})$. By linearity then

$$T_{AB'} = \sum_{i \in \mathcal{X}} \text{Tr}_{B'} \left[T_{AB'} (\mathbb{1}_A \otimes F_{B'}^i) \right] \otimes \theta_{B'}^i, \tag{40}$$

for all operators $T_{AB'} \in \mathfrak{L}(\mathcal{H}_A \otimes \mathcal{H}_{B'})$.

Let us now put, in Eq. (40), $T_{AB'} = \sigma_{AB'}$. Since we assume $\varrho_{AB} \supset_A \varsigma_{AB'}$, by Lemma 1, there exists a \mathcal{X} -POVM $(\tilde{F}_B^i; i \in \mathcal{X})$ on \mathcal{H}_B such that

$$\text{Tr}_B \left[\rho_{AB} (\mathbb{1}_A \otimes \tilde{F}_B^i) \right] = \text{Tr}_{B'} \left[\sigma_{AB'} (\mathbb{1}_A \otimes F_{B'}^i) \right], \tag{41}$$

for all $i \in \mathcal{X}$. Fixed any such POVM $(\tilde{F}_B^i; i \in \mathcal{X})$, we define a linear map $\mathcal{L}_B : \mathfrak{L}(\mathcal{H}_B) \rightarrow \mathfrak{L}(\mathcal{H}_{B'})$ via the relation

$$\mathcal{L}_B(T_B) := \sum_{i \in \mathcal{X}} \text{Tr}[T_B \tilde{F}_B^i] \theta_{B'}^i, \tag{42}$$

for all operators $T_B \in \mathfrak{L}(\mathcal{H}_B)$. Equivalently, the linear map \mathcal{L}_B can be defined by the relations $\mathcal{L}_B^*(F_{B'}^i) = \tilde{F}_B^i$, for all $i \in \mathcal{X}$. This guarantees $\mathcal{L}_B^*(\mathbb{1}_{B'}) = \mathbb{1}_B$, i.e. the linear map \mathcal{L}_B is trace-preserving. Now, we have to check that the linear map $\mathcal{L}_B : \mathfrak{L}(\mathcal{H}_B) \rightarrow \mathfrak{L}(\mathcal{H}_{B'})$ so constructed in fact satisfies both conditions in Definition 13 and induces a statistical morphism from $\mathfrak{S}_B(\varrho_{AB})$ to $\mathfrak{S}_{B'}(\varsigma_{AB'})$.

We begin by noting that, as a consequence of Eqs. (40), (41), and (42), $(\text{id}_A \otimes \mathcal{L}_B)(\rho_{AB}) = \sigma_{AB'}$. This can be shown as follows:

$$\begin{aligned} \sigma_{AB'} &= \sum_{i \in \mathcal{X}} \text{Tr}_{B'} \left[\sigma_{AB'} (\mathbb{1}_A \otimes F_{B'}^i) \right] \otimes \theta_{B'}^i \\ &= \sum_{i \in \mathcal{X}} \text{Tr}_B \left[\rho_{AB} (\mathbb{1}_A \otimes \tilde{F}_B^i) \right] \otimes \theta_{B'}^i \\ &\stackrel{\text{def}}{=} (\text{id}_A \otimes \mathcal{L}_B)(\rho_{AB}). \end{aligned} \tag{43}$$

This also ensures that $\mathcal{L}_B(\mathfrak{S}_B(\varrho_{AB})) \subseteq \mathfrak{S}_{B'}(\varsigma_{AB'})$.

Let now \mathcal{X} be an arbitrary (finite) decision set, and let $\mathfrak{N}_{B'} := (N_{B'}^i; i \in \mathcal{X})$ be any \mathcal{X} -test on $\mathfrak{S}_{B'}(\mathfrak{S}_{AB'})$. We will now check, by applying Lemma 1, that the operators $X_B^i := \mathcal{L}^*(N_{B'}^i)$ indeed constitute a test on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$. The proof goes as follows: for every $\omega_B \in \mathfrak{S}_B(\mathfrak{Q}_{AB})$, let $R_A^\omega \in \mathfrak{L}(\mathcal{H}_A)$ be the positive operator such that $\omega_B = \text{Tr}_A [(R_A^\omega \otimes \mathbb{1}_B) \rho_{AB}]$. Consider now, for all $i \in \mathcal{X}$, the trace

$$\begin{aligned} \text{Tr}[X_B^i \omega_B] &= \text{Tr} \left[(R_A^\omega \otimes X_B^i) \rho_{AB} \right] \\ &= \text{Tr} \left[R_A^\omega \text{Tr}_B \left[(\mathbb{1}_A \otimes X_B^i) \rho_{AB} \right] \right] \\ &= \text{Tr} \left[R_A^\omega \text{Tr}_B \left[(\mathbb{1}_A \otimes \mathcal{L}_B^*(N_{B'}^i)) \rho_{AB} \right] \right] \\ &= \text{Tr} \left[R_A^\omega \text{Tr}_{B'} \left[(\mathbb{1}_A \otimes N_{B'}^i) (\text{id}_A \otimes \mathcal{L}_B)(\rho_{AB}) \right] \right] \\ &= \text{Tr} \left[R_A^\omega \text{Tr}_{B'} \left[(\mathbb{1}_A \otimes N_{B'}^i) \sigma_{AB'} \right] \right]. \end{aligned} \tag{44}$$

Lemma 1 provides the existence of a POVM $(\bar{P}_B^i; i \in \mathcal{X})$ on \mathcal{H}_B such that

$$\text{Tr}_B \left[(\mathbb{1}_A \otimes \bar{P}_B^i) \rho_{AB} \right] = \text{Tr}_{B'} \left[(\mathbb{1}_A \otimes N_{B'}^i) \sigma_{AB'} \right], \tag{45}$$

for all $i \in \mathcal{X}$. Plugging such POVM into Eq. (44), we obtain

$$\begin{aligned} \text{Tr}[X_B^i \omega_B] &= \text{Tr} \left[R_A^\omega \text{Tr}_{B'} \left[(\mathbb{1}_A \otimes N_{B'}^i) \sigma_{AB'} \right] \right] \\ &= \text{Tr} \left[R_A^\omega \text{Tr}_B \left[(\mathbb{1}_A \otimes \bar{P}_B^i) \rho_{AB} \right] \right] \\ &= \text{Tr} \left[(R_A^\omega \otimes \bar{P}_B^i) \rho_{AB} \right] \\ &= \text{Tr} \left[\bar{P}_B^i \omega_B \right], \end{aligned} \tag{46}$$

for all $i \in \mathcal{X}$. Since this holds for every $\omega_B \in \mathfrak{S}_B(\mathfrak{Q}_{AB})$, we proved that, for any finite \mathcal{X} and any \mathcal{X} -test $(N_{B'}^i; i \in \mathcal{X})$ on $\mathfrak{S}_{B'}(\mathfrak{S}_{AB'})$, the operators $X_B^i := \mathcal{L}_B^*(N_{B'}^i)$ indeed constitute a test on $\mathfrak{S}_B(\mathfrak{Q}_{AB})$. This shows that \mathcal{L}_B is a well-defined statistical morphism from $\mathfrak{S}_B(\mathfrak{Q}_{AB})$ to $\mathfrak{S}_{B'}(\mathfrak{S}_{AB'})$, as requested. \square

7. The Blackwell-Sherman-Stein Theorem in the Quantum Case

The BSS Theorem (see Theorem 1) is about the comparison of classical statistical models. According to Postulate 1, however, we can actually identify the notion of classical statistical models with that of abelian quantum statistical models, so that the BSS Theorem becomes a statement about comparison of abelian quantum statistical models. In this sense then, we call a “non-commutative (or quantum) BSS theorem” a statement characterizing equivalent conditions for the comparison of general quantum statistical models, recovering Theorem 1 in the abelian case. In the following, we will show how Theorem 2 can be used to prove such a generalized statement.

Definition 18. Given a quantum statistical model $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$, the associated state space $\mathfrak{S}(\mathbf{R}) \subset \mathfrak{S}(\mathcal{H})$ is defined as the set of states $\mathfrak{S}(\mathbf{R}) = \{\rho_\theta : \theta \in \Theta\}$.

Remark 12. As already noticed in Remark 10, it is irrelevant whether the maximum in Eq. (7) is taken over POVM's on \mathcal{H} or over tests on $\mathfrak{S}(\mathbf{R})$. For what follows, however, it is convenient to consider the expected payoff as maximized over tests, rather than POVM's.

As it happens for quantum information structures (see Definition 15), also for quantum statistical models we have two different notions of sufficiency:

Definition 19 (Sufficiency and m-sufficiency). *The quantum statistical model $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ is said to be **m-sufficient** for $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$, in formula*

$$\mathbf{R} \succ_m \mathbf{S}, \tag{47}$$

if and only if there exists a statistical morphism $\mathcal{L} : \mathfrak{S}(\mathbf{R}) \rightarrow \mathfrak{S}(\mathbf{S})$ such that

$$\sigma_\theta = \mathcal{L}(\rho_\theta), \quad \forall \theta \in \Theta. \tag{48}$$

*The quantum statistical model $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ is said to be **sufficient** for $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$, in formula*

$$\mathbf{R} \succ \mathbf{S}, \tag{49}$$

if and only if there exists a completely positive, trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}) \rightarrow \mathfrak{L}(\mathcal{H}')$ such that

$$\sigma_\theta = \mathcal{E}(\rho_\theta), \quad \forall \theta \in \Theta. \tag{50}$$

Theorem 2, via the correspondence exhibited in Eq. (8), directly implies the following:

Theorem 3 (Non-commutative BSS Theorem). *Given two quantum statistical models $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ and $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$,*

$$\mathbf{R} \succ_m \mathbf{S} \Leftrightarrow \mathbf{R} \supset \mathbf{S}. \tag{51}$$

Proof. Given the quantum statistical model $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$, let us construct the quantum information structure $\mathfrak{Q}_{AB}^{\mathbf{R}} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$, as done in Eq. (8). Let us repeat the same construction (using the same basis for $\mathcal{H}_A \cong \mathcal{H}_\Theta$) to obtain $\mathfrak{S}_{AB'}^{\mathbf{S}} = (\mathcal{H}_A, \mathcal{H}_{B'}, \sigma_{AB'})$ from $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$. Keeping in mind Remark 4, it is easy to verify that

$$\mathbf{R} \succ_m \mathbf{S} \Leftrightarrow \mathfrak{Q}_{AB}^{\mathbf{R}} \succ_m \mathfrak{S}_{AB'}^{\mathbf{S}}, \tag{52}$$

and that

$$\mathbf{R} \supset \mathbf{S} \Leftrightarrow \mathfrak{Q}_{AB}^{\mathbf{R}} \supset_A \mathfrak{S}_{AB'}^{\mathbf{S}}. \tag{53}$$

We then obtain the statement by direct application of Theorem 2. \square

Further, by applying Proposition 3, we obtain the following:

Proposition 4 (Semi-classical case). *Given two quantum statistical models $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ and $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$, if \mathbf{S} is abelian,*

$$\mathbf{R} \succ \mathbf{S} \Leftrightarrow \mathbf{R} \supset \mathbf{S}. \tag{54}$$

Proof. By definition, \mathbf{S} is an abelian quantum statistical model if and only if $\mathfrak{S}(\mathbf{S})$ is an abelian state space. Then, due to Proposition 3, we know that, whenever \mathbf{S} is an abelian quantum statistical model, $\mathbf{R} \succ_m \mathbf{S}$ if and only if $\mathbf{R} \succ \mathbf{S}$. With these remarks at hand, the statement is finally proved as a simple consequence of Theorem 3 above. \square

Notice that Proposition 4 is still more general than the BSS Theorem, since commutativity is required only for \mathbf{S} , whereas the classical case is equivalent to the situation in which both \mathbf{R} and \mathbf{S} are abelian. Proposition 4 hence describes a “semi-classical” scenario. In the case in which also \mathbf{R} is an abelian quantum statistical model, it is easy to prove that any completely positive, trace-preserving map \mathcal{E} such that $\sigma_\theta = \mathcal{E}(\rho_\theta)$ can be in fact written as a transition matrix $M_{\mathcal{E}}$, mapping the vectors \vec{p}_θ of eigenvalues of ρ_θ into the vectors \vec{q}_θ of eigenvalues of σ_θ , for all $\theta \in \Theta$, in complete accordance with the notion of sufficiency used in the BSS Theorem 1. We leave the proof of this to the reader.

Next, we show that Theorem 3, together with Proposition 2, provides an equivalent characterization of the sufficiency relation \succ for quantum statistical models. We first need the following definitions:

Definition 20 (Composition of Quantum Statistical Models). *Given any two quantum statistical models $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$, with $\rho = (\rho_\theta; \theta \in \Theta)$, and $\mathbf{T} = (\Xi, \mathcal{K}, \tau)$, with $\tau = (\tau_\xi; \xi \in \Xi)$, the composition $\mathbf{T} \times \mathbf{R}$ is defined as the quantum statistical model $(\Xi \times \Theta, \mathcal{K} \otimes \mathcal{H}, \tau \times \rho)$, where $\tau \times \rho := (\tau_\xi \otimes \rho_\theta; \xi \in \Xi, \theta \in \Theta)$. Moreover, $\mathfrak{S}(\mathbf{T} \times \mathbf{R}) = \mathfrak{S}(\mathbf{T}) \times \mathfrak{S}(\mathbf{R})$.*

Definition 21 (Complete Quantum Statistical Models). *A quantum statistical model $\mathbf{T} = (\Xi, \mathcal{K}, \tau)$ is said to be **complete** if and only if $\mathfrak{S}(\mathbf{T})$ is a complete state space.*

Proposition 5 (Equivalent condition for sufficiency). *Given two quantum statistical models $\mathbf{R} = (\Theta, \mathcal{H}, \rho)$ and $\mathbf{S} = (\Theta, \mathcal{H}', \sigma)$, the following are equivalent:*

1.

$$\mathbf{R} \succ \mathbf{S}; \tag{55}$$

2.

$$\mathbf{T} \times \mathbf{R} \supset \mathbf{T} \times \mathbf{S}, \tag{56}$$

for every auxiliary quantum statistical model $\mathbf{T} = (\Xi, \mathcal{K}, \tau)$;

3.

$$\mathbf{T} \times \mathbf{R} \supset \mathbf{T} \times \mathbf{S}, \tag{57}$$

for some complete quantum statistical model $\mathbf{T} = (\Xi, \mathcal{K}, \tau)$ with $\mathcal{K} \cong \mathcal{H}'$.

Proof. The implications “1 \Rightarrow 2” and “2 \Rightarrow 3” are trivial. In order to prove the implication “3 \Rightarrow 1”, let us consider an auxiliary quantum statistical model $\mathbf{T} = (\Xi, \mathcal{H}', \tau)$, such that $\mathfrak{S}(\mathbf{T})$ is complete for $\mathfrak{L}(\mathcal{H}')$, according to Definition 16. The condition $\mathbf{T} \times \mathbf{R} \supset \mathbf{T} \times \mathbf{S}$ implies, by Theorem 3, the existence of a statistical morphism $\mathcal{L} : \mathfrak{S}(\mathbf{T} \times \mathbf{R}) \rightarrow \mathfrak{S}(\mathbf{T} \times \mathbf{S})$ such that $\mathcal{L}(\tau_\xi \otimes \rho_\theta) = \tau_\xi \otimes \sigma_\theta$, for all $\xi \in \Xi$ and all $\theta \in \Theta$. By the completeness of $\mathfrak{S}(\mathbf{T})$, this implies that the linear map $\mathcal{L} : \mathfrak{L}(\mathcal{H}') \otimes \mathfrak{L}(\mathcal{H}) \rightarrow \mathfrak{L}(\mathcal{H}') \otimes \mathfrak{L}(\mathcal{H})$ must in fact have the form $\text{id} \otimes \mathcal{L}'$. We are hence in the position to apply Proposition 2, which proves the existence of a completely positive, trace-preserving map $\mathcal{E} : \mathfrak{L}(\mathcal{H}) \rightarrow \mathfrak{L}(\mathcal{H}')$ such that $\sigma_\theta = \mathcal{E}(\rho_\theta)$, for all $\theta \in \Theta$, i.e. $\mathbf{R} \succ \mathbf{S}$. \square

The corollary above makes it apparent that complete positivity is always related with the possibility of *extending* a quantum system (in this case, a quantum statistical model) by composing it with an auxiliary one.

8. Sufficiency of Quantum Information Structures, without Entanglement

We begin this section with the following definition:

Definition 22 (Composition of Quantum Information Structures). *Given two quantum information structures $\varrho_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ and $\omega_{XY} = (\mathcal{H}_X, \mathcal{H}_Y, \omega_{XY})$, the composition $\varrho_{AB} \otimes \omega_{XY}$ is defined as the triple $(\mathcal{H}_A \otimes \mathcal{H}_X, \mathcal{H}_B \otimes \mathcal{H}_Y, \rho_{AB} \otimes \omega_{XY})$.*

Remark 13. From Definitions 14, 17, and 22, it simply follows that

$$\mathfrak{S}_{BY}(\varrho_{AB} \otimes \omega_{XY}) \supseteq \mathfrak{S}_B(\varrho_{AB}) \times \mathfrak{S}_Y(\omega_{XY}). \tag{58}$$

Definition 23 (Complete Information Structures). *A quantum information structure $\omega_{XY} = (\mathcal{H}_X, \mathcal{H}_Y, \omega_{XY})$ is **complete** if and only if:*

1. *the local state space $\mathfrak{S}_Y(\omega_{XY})$ is complete (see Definition 16), and,*
2. *for any given linear map $\mathcal{L}_Y : \mathfrak{L}(\mathcal{H}_Y) \rightarrow \mathfrak{L}(\mathcal{H}_Y)$, $(\text{id}_X \otimes \mathcal{L}_Y)(\omega_{XY}) = \omega_{XY}$ if and only if $\mathcal{L}_Y = \text{id}_Y$.*

Remark 14. In order to explicitly show the existence of a complete information structure $\omega_{XY} = (\mathcal{H}_X, \mathcal{H}_Y, \omega_{XY})$, let us consider the family of information structures $\omega_{XY}^p = (\mathcal{H}_X, \mathcal{H}_Y, \omega_{XY}^p)$, for $p \in [0, 1]$, where $\dim \mathcal{H}_X = \dim \mathcal{H}_Y = d$ and ω_{XY}^p is an isotropic state, that is,

$$\omega_{XY}^p := p\Psi_{XY}^+ + (1 - p)\frac{\mathbb{1}_{XY}}{d^2}, \tag{59}$$

with Ψ_{XY}^+ denoting a maximally entangled state in $\mathcal{H}_X \otimes \mathcal{H}_Y$. These states are known to satisfy the second condition in Definition 23 for $p \neq 0$ [23]. Moreover, a simple calculation shows that

$$\mathfrak{S}_Y(\omega_{XY}^p) = \left\{ p\sigma_Y + (1 - p)\frac{\mathbb{1}_Y}{d} \mid \sigma_Y \in \mathfrak{S}(\mathcal{H}_Y) \right\}, \tag{60}$$

meaning that, for $p \neq 0$, $\mathfrak{S}_Y(\omega_{XY}^p)$ is complete.

We are now able to state the following:

Proposition 6 (Comparison of quantum information structures). *Given two quantum information structures $\varrho_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ and $\varsigma_{AB'} = (\mathcal{H}_A, \mathcal{H}_{B'}, \sigma_{AB'})$, the following are equivalent:*

- 1.

$$\varrho_{AB} \succ \varsigma_{AB'}; \tag{61}$$

- 2.

$$[\omega_{XY} \otimes \varrho_{AB}] \supset_{XA} [\omega_{XY} \otimes \varsigma_{AB'}], \tag{62}$$

for every auxiliary quantum information structure $\omega_{XY} = (\mathcal{H}_X, \mathcal{H}_Y, \omega_{XY})$;

- 3.

$$[\psi_{XY}^+ \otimes \varrho_{AB}] \supset_{XA} [\psi_{XY}^+ \otimes \varsigma_{AB'}], \tag{63}$$

for some auxiliary quantum information structure $\psi_{XY}^+ = (\mathcal{H}_X, \mathcal{H}_Y, \Psi_{XY}^+)$, such that Ψ_{XY}^+ is a maximally entangled pure state and $\mathcal{H}_X \cong \mathcal{H}_Y \cong \mathcal{H}_{B'}$;

4.

$$[\omega_{XY} \otimes \varrho_{AB}] \supset_{XA} [\omega_{XY} \otimes \varsigma_{AB'}], \tag{64}$$

for some auxiliary complete quantum information structure $\omega_{XY} = (\mathcal{H}_X, \mathcal{H}_Y, \omega_{XY})$ with $\mathcal{H}_Y \cong \mathcal{H}_{B'}$.

Proof. The implications “1 \Rightarrow 2” and “2 \Rightarrow 3” are trivial. The implication “3 \Rightarrow 4” follows from the fact that, from Eq. (59), any maximally entangled information structure is, in particular, complete. We hence prove only the implications “4 \Rightarrow 1”.

Starting from (64), Theorem 2 guarantees the existence of a statistical morphism $\mathcal{L}_{YB} : \mathfrak{S}_{YB}(\omega_{XY} \otimes \varrho_{AB}) \rightarrow \mathfrak{S}_{YB'}(\omega_{XY} \otimes \varsigma_{AB'})$ such that

$$\omega_{XY} \otimes \sigma_{AB'} = (\text{id}_{XA} \otimes \mathcal{L}_{YB})(\omega_{XY} \otimes \rho_{AB}). \tag{65}$$

Since ω_{XY} is a complete state, Eq. (65) implies that the linear map \mathcal{L}_{YB} must in fact have the form

$$\mathcal{L}_{YB} \equiv \text{id}_Y \otimes \mathcal{L}_B. \tag{66}$$

Further, the fact that $\text{id}_Y \otimes \mathcal{L}_B$ is a statistical morphism from $\mathfrak{S}_{YB}(\omega_{XY} \otimes \varrho_{AB})$ to $\mathfrak{S}_{YB'}(\omega_{XY} \otimes \varsigma_{AB'})$ implies that $\text{id}_Y \otimes \mathcal{L}_B$ is also a statistical morphism, in particular, from $\mathfrak{S}_Y(\omega_{XY}) \times \mathfrak{S}_B(\varrho_{AB})$ to $\mathfrak{S}_Y(\omega_{XY}) \times \mathfrak{S}_{B'}(\varsigma_{AB'})$, because of Eq. (58). Finally, since we assumed that $\mathfrak{S}_Y(\omega_{XY})$ is a complete state space, we can apply Proposition 2 to show that, indeed, $\varrho_{AB} \succ \varsigma_{AB'}$. \square

Remark 15. In Ref. [15], the statement “3 \Leftrightarrow 1” is proved. Proposition 6 shows that the hypotheses can in fact be relaxed so that only the property of *completeness*, rather than entanglement, is required. Let us consider, as an example, the set of isotropic states defined in (59). Such states are known to be separable for $p \leq \frac{1}{d+1}$. Hence, by fixing a value $p_* \in (0, \frac{1}{d+1}]$, we have that $\omega_{XY}^{p_*}$ is complete, induces a complete state space on Y , and, yet, it is a separable state. This fact recalls the results of Ref. [23], where it was first noted how completeness (there referred to as “faithfulness”) can replace entanglement, although in a different context (namely, quantum process tomography).

Remark 16. In Remark 4 we described how quantum statistical models can be identified with those quantum information structures, for which a decomposition like that in Eq. (8) exists. One should hence expect that Proposition 6 implies Proposition 5, whenever $\varrho_{AB} = (\mathcal{H}_A, \mathcal{H}_B, \rho_{AB})$ and $\varsigma_{AB'} = (\mathcal{H}_A, \mathcal{H}_{B'}, \sigma_{AB'})$ can be written in the form of Eq. (8). In such a case, indeed, the fourth statement of Proposition 6 can be used to re-derive Proposition 5 simply by considering an auxiliary quantum information structure $\omega_{XY} = (\mathcal{H}_X, \mathcal{H}_Y, \omega_{XY})$ of the form

$$\omega_{XY} := \frac{1}{|\Xi|} \sum_{\xi \in \Xi} |\xi\rangle \langle \xi|_X \otimes \tau_Y^\xi. \tag{67}$$

The crucial observation is that the above quantum information structure is complete if and only if the corresponding quantum statistical model $\mathbf{T}_\omega := (\Xi, \mathcal{H}_Y, \boldsymbol{\tau})$, with $\boldsymbol{\tau} = (\tau_\xi; \xi \in \Xi)$, is complete. The rest of the proof is left to the interested reader.

9. Conclusions

We extended some results from the theory of comparison of statistical models to quantum statistical decision theory. This has been done by relaxing Petz’s definition of coarse-grainings to that of statistical morphisms. By using such generalized notion, we introduced comparison criteria for quantum statistical models and quantum information structures, which are the direct generalization to a non-commutative setting of the comparison criteria used in classical decision theory. The framework we described turned out to be general enough to encompass both the classical and the quantum case. We showed how results that previously were independent, like the Blackwell-Sherman-Stein theorem for statistical models and Shmaya’s result for quantum information structures, can be in fact recovered as special cases of a single, unifying comparison theorem, which also sheds new light on both: the BSS Theorem has been extended to a quantum-classical scenario, and Shmaya’s comparison criterion has been strengthened by removing the need of auxiliary entangled resources.

As a final remark, the reader might have noticed that, as long as the states of a statistical theory can be represented by self-adjoint matrices (not necessarily positive) of unit trace, the definitions of information ordering and m -sufficiency proposed here can be straightforwardly extended to consider such cases as well. For such *generalized probabilistic theories*, an extension of the BSS Theorem can also be proved, along the same lines described in the present work.

Acknowledgements. The author is grateful to Masanao Ozawa for illuminating conversations and clarifying suggestions. Discussions with Giacomo Mauro D’Ariano and Madalin Guta greatly contributed in improving the presentation of the work. This research was supported by the Program for Improvement of Research Environment for Young Researchers from Special Coordination Funds for Promoting Science and Technology (SCF) commissioned by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. Part of this work has been done when the author was visiting the Statistical Laboratory of the University of Cambridge.

A. Proof of Proposition 1

Proposition 1. *For any two given statistical models $\mathcal{E} = (\Theta, \Delta, \alpha)$ and $\mathcal{F} = (\Theta, \Delta', \beta)$, $\mathcal{E} \supset \mathcal{F}$ if and only if $\mathcal{E} \supset_{\text{Bayes}} \mathcal{F}$.*

Proof. The statement can be proved by using the Separation Theorem between convex sets [20] as follows. (Notice that in our case all convex sets are closed and bounded, so that we can proceed without paying attention to too many technical details.)

Generally speaking, the convex set $\mathcal{C}_1 \subset \mathbb{R}^N$ is not contained in the convex set $\mathcal{C}_2 \subset \mathbb{R}^N$ if and only if there exists a point $\vec{v} \in \mathcal{C}_1$ such that $\vec{v} \notin \mathcal{C}_2$. Then, the Separation Theorem (Corollary 11.4.2 of Ref. [20]), applied to the convex set \mathcal{C}_2 and the single-point (hence convex) set $\{\vec{v}\}$, states that, for such \vec{v} , there exists a vector $\vec{b} \in \mathbb{R}^N$ such that

$$\max_{\vec{w} \in \mathcal{C}_2} \sum_{n=1}^N b^n w^n < \sum_{n=1}^N b^n v^n. \tag{68}$$

Equivalently, we can say that the convex set $\mathcal{C}_1 \subset \mathbb{R}^N$ is contained in the convex set $\mathcal{C}_2 \subset \mathbb{R}^N$ if and only if, for all vectors $\vec{b} \in \mathbb{R}^N$,

$$\max_{\vec{w} \in \mathcal{C}_2} \sum_{n=1}^N b^n w^n \geq \max_{\vec{v} \in \mathcal{C}_1} \sum_{n=1}^N b^n v^n. \tag{69}$$

Moreover, for any given non-vanishing probability distribution $\pi(n)$, $\sum_n \pi(n) = 1$, the convex set $\mathcal{C}_1 \subset \mathbb{R}^N$ is contained in the convex set $\mathcal{C}_2 \subset \mathbb{R}^N$ if and only if, for all vectors $\vec{b} \in \mathbb{R}^N$,

$$\max_{\vec{w} \in \mathcal{C}_2} \sum_{n=1}^N \pi(n) b^n w^n \geq \max_{\vec{v} \in \mathcal{C}_1} \sum_{n=1}^N \pi(n) b^n v^n. \tag{70}$$

This follows from the fact that the above equation has to hold for all $\vec{b} \in \mathbb{R}^N$, so that the non-vanishing probabilities $\pi(n)$ can be absorbed in the definition of \vec{b} . In particular, there is no loss of generality in considering $\pi(n) = 1/N$, for all n .

We now turn to the case of $\mathcal{C}(\mathcal{E}, \mathcal{X}, \ell)$ and $\mathcal{C}(\mathcal{F}, \mathcal{X}, \ell)$, choosing the *a priori* probability on Θ as $\pi(\theta) = 1/|\Theta|$, for all θ . Then, for every $\vec{b} \in \mathbb{R}^{|\Theta|}$,

$$\max_{\phi: \mathcal{X}\text{-r.d.f.}} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} b^\theta v^\theta(\phi; \mathcal{E}, \mathcal{X}, \ell) = \max_{\phi: \mathcal{X}\text{-r.d.f.}} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} v^\theta(\phi; \mathcal{E}, \mathcal{X}, \tilde{\ell}), \tag{71}$$

where the function $\tilde{\ell}$ at the left-hand side is another payoff function with such that $\tilde{\ell}(\theta, i) = \ell(\theta, i)b^\theta$. In other words, the vector \vec{b} can be absorbed in the definition of the payoff function. This means that, for any finite set of decisions \mathcal{X} and any payoff function $\underline{\ell} : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$, $\mathcal{C}(\mathcal{E}, \mathcal{X}, \ell) \supseteq \mathcal{C}(\mathcal{F}, \mathcal{X}, \ell)$ if and only if, for every payoff function $\tilde{\ell} : \Theta \times \mathcal{X} \rightarrow \mathbb{R}$,

$$\max_{\phi: \mathcal{X}\text{-r.d.f.}} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} v^\theta(\phi; \mathcal{E}, \mathcal{X}, \tilde{\ell}) \geq \max_{\phi': \mathcal{X}\text{-r.d.f.}} \frac{1}{|\Theta|} \sum_{\theta \in \Theta} v^\theta(\phi'; \mathcal{F}, \mathcal{X}, \tilde{\ell}), \tag{72}$$

where the maxima are taken over all possible \mathcal{X} -r.d.f. ϕ on Δ and ϕ' on Δ' . This, in turns, implies the statement. \square

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Communicated by M. B. Ruskai