The Quantum Capacity of Channels With Arbitrarily Correlated Noise

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Abstract—**We study optimal rates for quantum communication over a single use of a channel, which itself can correspond to a finite number of uses of a channel with arbitrarily correlated noise. The corresponding capacity is often referred to as the *one-shot* quantum capacity. In this paper, we prove bounds on the one-shot quantum capacity of an arbitrary channel. This allows us to compute the quantum capacity of a channel with arbitrarily correlated noise, in the limit of asymptotically many uses of the channel. In the memoryless case, we explicitly show that our results reduce to known expressions for the quantum capacity.**

Index Terms—**Quantum capacity, entanglement transmission, one-shot capacity, quasi-entropies, smooth Rényi entropies, information spectrum.**

I. Introduction

In contrast to a classical channel which has a unique capacity, a quantum channel has various distinct capacities. This is a consequence of the greater flexibility in the use of a quantum channel. As regards transmission of information through it, the different capacities arise from various factors: the nature of the transmitted information (classical or quantum), the nature of the input states (entangled or product states) the nature of the measurements done on the outputs of the channel (collective or individual), the absence or presence of any additional resource, e.g., prior shared entanglement between sender and receiver, and whether they are allowed to communicate classically with each other. The classical capacity of a quantum channel under the constraint of product state inputs was shown by Holevo [1], Schumacher and Westmoreland [2] to be given by the Holevo capacity of the channel. The capacity of a quantum channel to transmit quantum information, in the absence of classical communication and any additional resource, and without any constraint on the inputs and the measurements, is called the quantum capacity of the channel. It is known to be given by the regularized coherent information [3]–[5]. A quantum channel can also be used to generate entanglement between two parties, which can then be used as a resource for teleportation. The corresponding capacity is referred to as the entanglement generation capacity of the quantum channel and is equivalent to the capacity of the channel for transmitting quantum information [5].

All these capacities were originally evaluated in the limit of asymptotically many uses of the channel, under the assumption that the noise acting on successive inputs to the channel is uncorrelated, i.e., under the assumption that the channel is *memoryless*. In reality, however, this assumption, and the consideration of an asymptotic scenario, is not necessarily justified. It is, hence, of importance to evaluate both (i) bounds on the *one-shot capacities* of a quantum channel, that is its capacities for a finite number of uses or even a single use, as well as (ii) the capacity of an arbitrary sequence of channels, possibly with memory. Both these issues are addressed in this paper.

For an arbitrary quantum channel, it is not in general possible to achieve perfect information transmission or entanglement generation over a single use or a finite number of uses. Hence, one needs to allow for a nonzero probability of error. This leads us to consider the capacities under the constraint that the probability of error is at most ε, for a given $\varepsilon \geq 0$.

In this paper we consider the following protocol, which we call *entanglement transmission* [6]. Let Φ be a quantum channel, let $\mathcal{H}_M$ be a subspace of its input Hilbert space, and let ε be a fixed positive constant. Suppose Alice prepares a maximally entangled state $|\Psi^+\rangle \in \mathcal{H}_M \otimes \mathcal{H}_{M'}$, where $\mathcal{H}_{M'} \simeq \mathcal{H}_M$, and sends the part M through the channel Φ to Bob. Bob is allowed to do any decoding operation (completely positive trace-preserving map) on the state that he receives. The final objective is for Alice and Bob to end up with a shared state which is nearly maximally entangled over $\mathcal{H}_M \otimes \mathcal{H}_{M'}$, its overlap with $|\Psi^+\rangle$ being at least $(1-\varepsilon)$. In this protocol, there is no classical communication allowed between Alice and Bob. For a given $\varepsilon \geq 0$, let $Q_{\rm ent}(\Phi;\varepsilon)$ denote the *one-shot capacity of entanglement transmission*. In this paper, we prove that this capacity is expressible in terms of a generalization of the relative Rényi entropy of order 0. Our results also yield a characterization of the *one-shot quantum capacity* of the channel. This is because it can be shown that the one-shot capacity of transmission of *any* quantum state by the channel, evaluated under the condition that the minimum fidelity of the channel is at most $(1-\varepsilon)$, for a given $\varepsilon \geq 0$, is bounded above by $Q_{\rm ent}(\Phi;2\varepsilon)$, and bounded below by $Q_{\rm ent}(\Phi;\varepsilon/2) - 1$ (see Section V).

By the Stinespring Dilation Theorem [7], the action of a quantum channel creates correlations between the sender, the receiver, and the environment interacting with the input. Faithful transmission of quantum information requires a decoupling of the state of the environment from that of the sender (see the special issue [8]). In [9], a lower bound to the accuracy with which this decoupling can be achieved in a single use of the channel, was obtained. Here we go a step further and

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evaluate bounds on the one-shot capacity. In evaluating the lower bound, we employ an inequality, given by Lemma 4, relating the decoupling accuracy to the decoding fidelity. To obtain the upper bound we instead generalize the standard arguments relying on the quantum data-processing inequality [5], [12]. Moreover, in the limit of asymptotically many uses of a memoryless channel, we prove, without explicitly resorting to any typicality argument, that each of these bounds converge independently to the familiar expression of the quantum capacity given by the regularized coherent information [3]–[5]. For the important case of an arbitrary sequence of channels, possibly with memory, our one-shot result yields the asymptotic quantum capacity in the Information Spectrum framework [13], [31]–[33], [14].

We start the paper with some definitions and notations in Section II, including that of quasi-entropies, which play a pivotal role in our analysis. In Section III we introduce the protocol of entanglement transmission, and define its fidelity and the corresponding one-shot capacity. Our main result is given by Theorem 1 of Section IV. In Section V we relate the one-shot entanglement transmission capacity with the one-shot quantum capacity. The tools used for the proof of Theorem 1 are given in Section VI, with the proof itself presented in Section VII. Further, in Section VIII, we consider a sequence of arbitrary channels, with or without memory, and derive an expression for its asymptotic quantum capacity. When the channels in the sequence are memoryless, we recover known expressions for quantum capacity given in terms of the regularized coherent information. We conclude with a discussion of our results in Section IX.

II. Definitions and Notations

A. Mathematical Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of linear operators acting on a finite-dimensional Hilbert space $\mathcal{H}$ and let $\mathfrak{S}(\mathcal{H})$ denote the set of positive operators of unit trace (states) acting on $\mathcal{H}$. A quantum channel is given by a completely positive trace-preserving (CPTP) map $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the input and output Hilbert spaces of the channel. Moreover, for any given subspace $\mathcal{S} \subseteq \mathcal{H}_A$, we define the restriction of the channel Φ to the subspace $\mathcal{S}$ as $\Phi|_{\mathcal{S}}(\rho) := \Phi(\Pi_{\mathcal{S}}\rho\Pi_{\mathcal{S}})$, for any $\rho \in \mathcal{B}(\mathcal{H}_A)$, with $\Pi_{\mathcal{S}}$ being the projector onto $\mathcal{S}$. Notice that $\Phi|_{\mathcal{S}}$ is itself a CPTP-map $\Phi|_{\mathcal{S}} : \mathcal{B}(\mathcal{S}) \mapsto \mathcal{B}(\mathcal{H}_B)$. Throughout this paper we restrict our considerations to finite-dimensional Hilbert spaces, and we take the logarithm to base 2.

For given orthonormal bases $\{|i^A\rangle\}_{i=1}^d$ and $\{|i^B\rangle\}_{i=1}^d$ in isomorphic Hilbert spaces $\mathcal{H}_A \simeq \mathcal{H}_B \simeq \mathcal{H}$ of dimension d, we define a maximally entangled state (MES) of rank $m \leq d$ to be

$$\left|\Psi_m^{AB}\right\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i^A\rangle \otimes |i^B\rangle. \tag{1}$$

When $m = d$, for any given operator $A \in \mathcal{B}(\mathcal{H})$, the following relation can be shown by direct inspection:

$$(A \otimes \mathbb{1}) \left|\Psi_d^{AB}\right\rangle = (\mathbb{1} \otimes A^T) \left|\Psi_d^{AB}\right\rangle \tag{2}$$

where $\mathbb{1}$ denotes the identity operator, and A^T denotes the transposition with respect to the basis fixed by (1). Moreover, for any given pure state $|\phi\rangle$, we denote the projector $|\phi\rangle\langle\phi|$ simply as ϕ.

The trace distance between two operators A and B is given by

$$\|A - B\|_1 := \mathrm{Tr}[\{A \geq B\}(A - B)] - \mathrm{Tr}[\{A < B\}(A - B)]$$

where $\{A \geq B\}$ denotes the projector on the subspace where the operator $(A - B)$ is nonnegative, and $\{A < B\} := \mathbb{1} - \{A \geq B\}$. The fidelity of two states ρ and σ is defined as

$$F(\rho, \sigma) := \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = \|\sqrt{\rho}\sqrt{\sigma}\|_1. \tag{3}$$

The trace distance between two states ρ and σ is related to the fidelity $F(\rho, \sigma)$ as follows (see, e.g., [12]):

$$1 - F(\rho, \sigma) \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F^2(\rho, \sigma)} \tag{4}$$

where we use the notation $F^2(\rho, \sigma) = (F(\rho, \sigma))^2$. We also use the following results.

Lemma 1 ([15]): For self-adjoint operators A and B, and any positive operator $0 \leq P \leq \mathbb{1}$

$$\mathrm{Tr}[P(A - B)] \leq \mathrm{Tr}[\{A \geq B\}(A - B)]$$

and

$$\mathrm{Tr}[P(A - B)] \geq \mathrm{Tr}[\{A < B\}(A - B)].$$

□

Lemma 2 (Gentle Measurement Lemma [17], [16]): For a state $\rho \in \mathfrak{S}(\mathcal{H})$ and operator $0 \leq \Lambda \leq \mathbb{1}$, if $\mathrm{Tr}(\rho\,\Lambda) \geq 1 - \delta$, then

$$\|\rho - \sqrt{\Lambda}\rho\sqrt{\Lambda}\|_1 \leq 2\sqrt{\delta}.$$

The same holds if ρ is a subnormalized density operator. □

Lemma 3 ([18]): For any self-adjoint operator X and any positive operator $\xi > 0$, we have

$$\begin{aligned}\|X\|_1^2 &\leq \mathrm{Tr}[\xi]\mathrm{Tr}\left[X\xi^{-1/2}X\xi^{-1/2}\right] \\ &\leq \mathrm{Tr}[\xi]\mathrm{Tr}\left[X^2\xi^{-1}\right].\end{aligned} \tag{5}$$

□

Proof: The first inequality in (5) was proved in [18]. The second one simply follows as an application of the Cauchy-Schwarz inequality, that is

$$\begin{aligned}&\mathrm{Tr}[X\xi^{-1/2}X\xi^{-1/2}] \\ &\quad \leq \sqrt{\mathrm{Tr}[(X\xi^{-1/2})(X\xi^{-1/2})^\dagger]} \\ &\quad\quad \times \sqrt{\mathrm{Tr}[(X\xi^{-1/2})^\dagger(X\xi^{-1/2})]} \\ &\quad = \mathrm{Tr}[X^2\xi^{-1}].\end{aligned}$$

■

Lemma 4: Given a tripartite pure state $|\Omega^{RBE}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_B \otimes \mathcal{H}_E$, let ω^{RB}, ω^{RE}, ω^R, and ω^E be its reduced states. Then

$$F^2(\omega^{RE}, \omega^R \otimes \omega^E) \leq \max_{\mathcal{D}} F^2((\mathrm{id}_R \otimes \mathcal{D}_B)(\omega^{RB}), \Psi^{RA}) \quad (6)$$

where $|\Psi^{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$ is some fixed purification of ω^R and $\mathcal{D} : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_A)$ denotes a CPTP map. □

Proof: Fix some purification $|\chi^{EA'}\rangle \in \mathcal{H}_E \otimes \mathcal{H}_{A'}$ of ω^E. Then, for the fixed purification $|\Psi^{RA}\rangle$ of ω^R, we have, by Uhlmann's theorem [10], the monotonicity of the fidelity under partial trace, and Stinespring's Dilation Theorem [7]

$$\begin{aligned} &F^2(\omega^{RE}, \omega^R \otimes \omega^E) \\ &= \max_{\substack{|\varphi^{REAA'}\rangle \\ \mathrm{Tr}_{AA'}[\varphi^{REAA'}] = \omega^{RE}}} F^2(\varphi^{REAA'}, \Psi^{RA} \otimes \chi^{EA'}) \\ &= \max_{\substack{V: B \to AA' \\ V^\dagger V = \mathbb{1}_B}} F^2((\mathbb{1}^{RE} \otimes V_B)\Omega^{RBE}(\mathbb{1}^{RE} \otimes V_B^\dagger), \Psi^{RA} \otimes \chi^{EA'}) \\ &\leq \max_{\mathcal{D}} F^2((\mathrm{id}_R \otimes \mathcal{D}_B)(\omega^{RB}), \Psi^{RA}) \end{aligned} \quad (7)$$

where $\mathcal{D} : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_A)$ denotes a CPTP map. In the second equality of (7) we also used the well-known fact that all possible purifications of a given mixed state (ω^{RE}, in our case) are related by some local isometry acting on the purifying system only (i.e., subsystem B).$blackboxfill$

B. Quasi-Entropies and Coherent Information

For any $\rho, \sigma \geq 0$ and any $0 \leq P \leq \mathbb{1}$, the *quantum relative quasi-entropy of order* α [22], for $\alpha \in (0, \infty) \backslash \{1\}$, is defined as

$$S_\alpha^P(\rho \,\|\, \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}[\sqrt{P} \rho^\alpha \sqrt{P} \sigma^{1-\alpha}]. \quad (8)$$

Notice that for $P = \mathbb{1}$, the quasi-entropy defined above reduces to the well-known Rényi relative entropy of order α.

In this paper, in particular, the quasi-entropy of order 0, namely

$$S_0^P(\rho \,\|\, \sigma) := \lim_{\alpha \searrow 0} S_\alpha^P(\rho \,\|\, \sigma) \quad (9)$$

plays an important role. Note that

$$S_0^P(\rho \,\|\, \sigma) = -\log \mathrm{Tr}[\sqrt{P} \Pi_\rho \sqrt{P}\, \sigma] \quad (10)$$

where Π_ρ denotes the projector onto the support of ρ. Our main result, Theorem 1, is expressible in terms of two "smoothed" quantities, which are derived from the quasi-entropy of order 0, for any $\delta \geq 0$, as

$$I_{0,\delta}^c(\rho^{AB}) := \max_{\bar{\rho}^{AB} \in \mathfrak{b}(\rho^{AB}; \delta)} \min_{\sigma^B \in \mathfrak{S}(\mathcal{H}_B)} S_0^{\mathbb{1}}(\bar{\rho}^{AB} \,\|\, \mathbb{1}_A \otimes \sigma^B) \quad (11)$$

and

$$\tilde{I}_{0,\delta}^c(\rho^{AB}) := \max_{P \in \mathfrak{p}(\rho^{AB}; \delta)} \min_{\sigma^B \in \mathfrak{S}(\mathcal{H}_B)} S_0^P(\rho^{AB} \,\|\, \mathbb{1}_A \otimes \sigma^B) \quad (12)$$

where

$$\mathfrak{b}(\rho; \delta) := \{\sigma : \sigma \geq 0,\ \mathrm{Tr}[\sigma] \leq 1,\ F^2(\rho, \sigma) \geq 1 - \delta^2\} \quad (13)$$

and

$$\mathfrak{p}(\rho; \delta) := \{P : 0 \leq P \leq \mathbb{1},\ \mathrm{Tr}[P\rho] \geq 1 - \delta\}. \quad (14)$$

[Note that, in (13), the definition of fidelity (3) has been naturally extended to subnormalized density operators.] Such smoothed quantities are needed in order to allow for a finite accuracy (i.e., nonzero error) in the protocol, which is a natural requirement in the one-shot regime. Their properties are discussed in detail in Section VI-B.

III. THE PROTOCOL: ENTANGLEMENT TRANSMISSION

As mentioned in the Introduction, we consider the protocol of *entanglement transmission* [6]: Given a quantum channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$, let $\mathcal{H}_M$ be an m-dimensional subspace of its input Hilbert space, and let ε be a fixed positive constant. Alice prepares a maximally entangled state $|\Psi_m^{M'M}\rangle \in \mathcal{H}_{M'} \otimes \mathcal{H}_M$, where $\mathcal{H}_{M'} \simeq \mathcal{H}_M$, and sends the part M through the channel Φ to Bob. Bob is allowed to do any decoding operation (CPTP map) on the state that he receives. The final objective is for Alice and Bob to end up with a shared state which is nearly maximally entangled over $\mathcal{H}_{M'} \otimes \mathcal{H}_M$, its overlap with $|\Psi_m^{M'M}\rangle$ being at least $(1 - \varepsilon)$. There is no classical communication possible between Alice and Bob. Within this scenario, for any positive integer m, the efficiency of the channel Φ in transmitting entanglement, is given in terms of the fidelity defined here.

Definition 1 (Entanglement Transmission Fidelity): Let a channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ be given. For any given positive integer $m \leq \dim \mathcal{H}_A$, we define the entanglement transmission fidelity of Φ as

$$\begin{aligned} &\mathsf{F}_{\mathrm{ent}}(\Phi; m) \\ &\quad := \max_{\substack{\mathcal{H}_M \subseteq \mathcal{H}_A \\ \dim \mathcal{H}_M = m}} \max_{\mathcal{D}} \Big\langle \Psi_m^{M'M} \Big| \\ &\quad\quad (\mathrm{id} \otimes \mathcal{D} \circ \Phi) \\ &\quad\quad \times \left(\Psi_m^{M'M}\right) \Big| \Psi_m^{M'M} \Big\rangle \end{aligned} \quad (15)$$

where $\mathcal{D} : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_A)$ is a decoding CPTP-map. □

We can now define an achievable rate as follows.

Definition 2 (ε-Achievable Rate): Given a channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ and a real number $\varepsilon \geq 0$, any $R = \log m$, $m \in \mathbb{N}$, is an ε-achievable rate, if

$$\mathsf{F}_{\mathrm{ent}}(\Phi; m) \geq 1 - \varepsilon.$$

□

This leads to the definition of the one-shot capacity of entanglement transmission.

Definition 3 (One-Shot Capacity): Given a quantum channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ and a real number $\varepsilon \geq 0$, the one-shot capacity of entanglement transmission of Φ is defined as

$$Q_{\rm ent}(\Phi;\varepsilon) := \max\{R : R \text{ is } \varepsilon\text{−achievable}\}.$$

□

IV. Main Result: One-Shot Entanglement Transmission Capacity

Given a Hilbert space $\mathcal{H}_A$ with $d := \dim \mathcal{H}_A$, let $\mathcal{H}_R$ be isomorphic to $\mathcal{H}_A$, and fix a basis $\{|i^R\rangle\}_{i=1}^d$ for $\mathcal{H}_R$. Then, for any given subspace $\mathcal{S} \subseteq \mathcal{H}_A$ of dimension s, we construct the maximally entangled state of rank s in $\mathcal{H}_R \otimes \mathcal{H}_A$ as

$$\left|\Psi_{\mathcal{S}}^{RA}\right\rangle := \frac{1}{\sqrt{s}} \sum_{i=1}^{s} |i^R\rangle \otimes |\varsigma_i^A\rangle \tag{16}$$

where $\{|\varsigma_i^A\rangle\}_{i=1}^s$ is an orthonormal basis of $\mathcal{S}$. Now, given a channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$, let $V_\Phi^A : \mathcal{H}_A \mapsto \mathcal{H}_B \otimes \mathcal{H}_E$ be a Stinespring isometry realizing the channel Φ as

$$\Phi(\rho) = \mathrm{Tr}_E[V_\Phi \rho V_\Phi^\dagger]$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$. For any subspace $\mathcal{S} \subseteq \mathcal{H}_A$, from (16), we define the tripartite pure state

$$\left|\Omega_{\mathcal{S}}^{RBE}\right\rangle := (\mathbb{1}_R \otimes V_\Phi^A) \left|\Psi_{\mathcal{S}}^{RA}\right\rangle. \tag{17}$$

We then define $\omega_{\mathcal{S}}^{RB} := \mathrm{Tr}_E[\Omega_{\mathcal{S}}^{RBE}]$ and $\omega_{\mathcal{S}}^{RE} := \mathrm{Tr}_B[\Omega_{\mathcal{S}}^{RBE}]$ to be its reduced states. Our main result is stated in Theorem 1 below.

Theorem 1: For any $\varepsilon \geq 0$, the one-shot capacity of entanglement transmission for a quantum channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$, $Q_{\rm ent}(\Phi;\varepsilon)$, satisfies the following bounds:

$$\begin{aligned} &\max_{\mathcal{S}\subseteq\mathcal{H}_A} I_{0,\varepsilon/8}^c\left(\omega_{\mathcal{S}}^{RB}\right) + \log\left[\frac{1}{d} + \frac{\varepsilon^2}{4}\right] - \Delta \\ &\quad \leq Q_{\rm ent}(\Phi;\varepsilon) \\ &\quad \leq \max_{\mathcal{S}\subseteq\mathcal{H}_A} \tilde{I}_{0,2\sqrt{\varepsilon}}^c\left(\omega_{\mathcal{S}}^{RB}\right) \end{aligned} \tag{18}$$

where $d := \dim \mathcal{H}_A$, $I_{0,\varepsilon/8}^c(\omega_{\mathcal{S}}^{RB})$ and $\tilde{I}_{0,2\sqrt{\varepsilon}}^c(\omega_{\mathcal{S}}^{RB})$ are the smoothed 0-coherent informations defined, respectively, by (11) and (12), and $0 \leq \Delta \leq 1$ is included to ensure that the lower bound is equal to the logarithm of a positive integer. □

Remark: Given a positive real x, for $x - \Delta$ to be the logarithm of a positive integer, we must have $\Delta \equiv \Delta(x) := x - \log\lfloor 2^x \rfloor$, where $\lfloor y \rfloor$ denotes the largest integer less than or equal to y. It can be shown that $0 \leq \Delta(x) \leq 1$ for all $x \geq 0$, and that $\Delta(x)$ decreases rapidly as x increases.

V. One-Shot Quantum Capacity

It is interesting to compare the entanglement transmission fidelity of a quantum channel with the minimum output fidelity defined below.

Definition 4 (Minimum Output Fidelity): Let a channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ be given. For any given positive integer m, we define the minimum output fidelity of Φ as

$$\mathsf{F}_{\min}(\Phi;m) := \max_{\substack{\mathcal{H}_M\subseteq\mathcal{H}_A \\ \dim\mathcal{H}_M = m}} \max_{\mathcal{D}} \min_{|\phi\rangle\in\mathcal{H}_M} \langle\phi|(\mathcal{D}\circ\Phi)(\phi)|\phi\rangle$$

where $\mathcal{D} : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_A)$ is a decoding CPTP-map. □

Remark: Note that Definitions 1 and 4 include an optimization over all decoding operations. Hence, they provide a measure of how well the effect of the noise in the channel can be corrected. This is in contrast to the definitions of fidelities used in [23], [24] which provide a measure of the "distance" of a given channel from the trivial (identity) channel.

The minimum output fidelity is related to the entanglement transmission fidelity through the following lemma [23], [24].

Lemma 5 (Pruning Lemma): Let a channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ be given. Then, for any positive integer m

$$\mathsf{F}_{\min}(\Phi;m/2) \geq 1 - 2[1 - \mathsf{F}_{\rm ent}(\Phi;m)].$$

□

Analogously to what we did for the entanglement transmission fidelity, one could also define the one-shot capacity with respect to the fidelity $\mathsf{F}_{\min}$ as follows:

$$Q_{\min}(\Phi;\varepsilon) := \max\{\log m : \mathsf{F}_{\min}(\Phi;m) \geq 1 - \varepsilon\}. \tag{19}$$

Remark: Note that quantum capacity is traditionally defined with respect to the minimum output fidelity $\mathsf{F}_{\min}$ [5]. Hence, we define $Q_{\min}(\Phi;\varepsilon)$ to be the one-shot quantum capacity of a channel Φ, for any $\varepsilon \geq 0$.

The following corollary, derived from Lemma 5, allows us to relate the one-shot entanglement transmission capacity $Q_{\rm ent}(\Phi;\varepsilon)$ to the one-shot quantum capacity.

Corollary 1: Given a quantum channel $\Phi:\mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ and a real number $\varepsilon > 0$

$$Q_{\rm ent}(\Phi;\varepsilon) - 1 \leq Q_{\min}(\Phi;2\varepsilon) \leq Q_{\rm ent}(\Phi;4\varepsilon).$$

□

Proof: The lower bound follows directly from the Pruning Lemma. To prove the upper bound we resort to another frequently used fidelity, namely, the *average fidelity*

$$\mathsf{F}_{\rm avg}(\Phi;m) := \max_{\substack{\mathcal{H}_M\subseteq\mathcal{H}_A \\ \dim\mathcal{H}_M = m}} \max_{\mathcal{D}} \int \mathrm{d}\phi\, \langle\phi\,|\,(\mathcal{D}\circ\Phi)(\phi)|\phi\rangle$$

where $\mathrm{d}\phi$ is the normalized unitarily invariant measure over pure states in $\mathcal{H}_M$, and $\mathcal{D} : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_A)$ is a decoding CPTP-map.

In [25] and [34], the relation of the above fidelity to the entanglement transmission fidelity was shown to be given by

$$\mathsf{F}_{\rm avg}(\Phi;m) = \frac{m \cdot \mathsf{F}_{\rm ent}(\Phi;m) + 1}{m+1}$$

while clearly, by definition $\mathsf{F}_{\min}(\Phi;m) \le \mathsf{F}_{\rm avg}(\Phi;m)$. Hence, if $\mathsf{F}_{\min}(\Phi;m) \ge 1-\varepsilon'$, then

$$\mathsf{F}_{\rm ent}(\Phi;m) \ge \frac{(m+1)(1-\varepsilon')-1}{m} = 1-\frac{m+1}{m}\varepsilon' \ge 1-2\varepsilon'.$$

■

Note that, due to Corollary 1, Theorem 1 provides bounds on the one-shot quantum capacity of a channel as well.

VI. Tools Used in the Proof

The proof of Theorem 1 relies on the properties of various entropic quantities derived from the relative quasi-entropies defined in Section II-B.

A. Quantum Entropies

Let us first consider the relative Rényi entropy of order α, which as mentioned before, is obtained from the quasi-entropy (8) by setting $P=\mathbb{1}$. (In the following, when $P=\mathbb{1}$, we will drop the exponent in writing relative Rényi entropies, for sake of notational simplicity.) It is known that

$$S_1(\rho\,\|\,\sigma) := \lim_{\alpha\nearrow 1} S_\alpha(\rho\,\|\,\sigma) = S(\rho\,\|\,\sigma)$$

where $S(\rho\,\|\,\sigma)$ is the usual quantum relative entropy defined as

$$S(\rho\,\|\,\sigma) := \begin{cases} \mathrm{Tr}[\rho\log\rho-\rho\log\sigma], & \text{if } \mathsf{supp}\,\rho\subseteq\mathsf{supp}\,\sigma \\ +\infty, & \text{otherwise.}\end{cases} \quad (20)$$

From this, one derives the von Neumann entropy $S(\rho)$ of a state ρ as $S(\rho)=-S(\rho\,\|\,\mathbb{1})$. We make use of the following lemma in the sequel.

Lemma 6: Given a state $\rho^{AB}\in\mathcal{H}_A\otimes\mathcal{H}_B$, let $\rho^A := \mathrm{Tr}_B[\rho^{AB}]$ and $\rho^B := \mathrm{Tr}_A[\rho^{AB}]$. Then, for any operator $\sigma^A\ge 0$ with $\mathsf{supp}\sigma^A \supseteq \mathsf{supp}\rho^A$,

$$\min_{\xi^B\ge 0} S(\rho^{AB}\,\|\,\sigma^A\otimes\xi^B) = S(\rho^{AB}\,\|\,\sigma^A\otimes\rho^B).$$

This implies, in particular, that, for any state ρ^{AB}

$$\min_{\xi^B\ge 0} S(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\xi^B) = S(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\rho^B)$$

and

$$\min_{\omega^A,\xi^B\ge 0} S(\rho^{AB}\,\|\,\omega^A\otimes\xi^B) = S(\rho^{AB}\,\|\,\rho^A\otimes\rho^B). \quad \square \quad (21)$$

Proof: Here we only prove (21). The rest of the lemma can be proved exactly along the same lines. By definition, we have that

$$S(\rho^{AB}\,\|\,\omega^A\otimes\xi^B) = \mathrm{Tr}[\rho^{AB}\log\rho^{AB}] - \mathrm{Tr}[\rho^{AB}\log(\omega^A\otimes\xi^B)].$$

Since $\log(\omega^A\otimes\xi^B) = (\log\omega^A)\otimes\mathbb{1}_B + \mathbb{1}_A\otimes(\log\xi^B)$, we can rewrite

$$S(\rho^{AB}\,\|\,\omega^A\otimes\xi^B) = \mathrm{Tr}[\rho^{AB}\log\rho^{AB}] - \mathrm{Tr}[\rho^A\log\omega^A] - \mathrm{Tr}[\rho^B\log\xi^B].$$

Now, since for all ρ and σ

$$0\le S(\rho\,\|\,\sigma) = \mathrm{Tr}[\rho\log\rho]-\mathrm{Tr}[\rho\log\sigma]$$

we have that

$$\mathrm{Tr}[\rho\log\rho]\ge\mathrm{Tr}[\rho\log\sigma]$$

which implies that

$$\begin{aligned} S(\rho^{AB}\,\|\,\omega^A\otimes\xi^B) &\ge \mathrm{Tr}[\rho^{AB}\log\rho^{AB}] - \mathrm{Tr}[\rho^A\log\rho^A] - \mathrm{Tr}[\rho^B\log\rho^B] \\ &= S(\rho^{AB}\,\|\,\rho^A\otimes\rho^B). \end{aligned}$$

■

Recently, a generalized relative entropy, namely the max-relative entropy $D_{\max}$, was introduced in [19]. For a state ρ and an operator $\sigma\ge 0$

$$\begin{aligned} D_{\max}(\rho\,\|\,\sigma) &:= \log\min\{\lambda:\rho\le\lambda\sigma\} \\ &= \log\lambda_{\max}(\sigma^{-1/2}\rho\sigma^{-1/2}) \end{aligned}$$

$\lambda_{\max}(X)$ denoting the maximum eigenvalue of the operator X. Even though for commuting ρ and σ $D_{\max}(\rho\,\|\,\sigma) = \lim_{\alpha\to\infty} S_\alpha(\rho\,\|\,\sigma)$, this identity does not hold in general [20]. We can however easily prove the following property.

Lemma 7: For any $\rho,\sigma\ge 0$ with $\mathrm{Tr}[\rho]\le 1$, we have

$$S_2(\rho\|\sigma)\le D_{\max}(\rho\|\sigma).$$

□

Proof: By definition, $2^{S_2(\rho\,\|\,\sigma)} = \mathrm{Tr}[\rho^2\sigma^{-1}]$. By noticing that, for any Hermitian operator X and any subnormalized state ρ, $\mathrm{Tr}[\rho X]\le\lambda_{\max}(X)$, we obtain that $\mathrm{Tr}[\rho^2\sigma^{-1}] = \mathrm{Tr}[\rho(\rho^{1/2}\sigma^{-1}\rho^{1/2})] \le \lambda_{\max}(\rho^{1/2}\sigma^{-1}\rho^{1/2}) = \lambda_{\max}(\sigma^{-1/2}\rho\sigma^{-1/2}) = 2^{D_{\max}(\rho\,\|\,\sigma)}$, where, in the last passage, we used the fact that $\lambda_{\max}(A^\dagger A)=\lambda_{\max}(AA^\dagger)$. ■

Given an α-relative Rényi entropy $S_\alpha(\rho\,\|\,\sigma)$, for a bipartite $\rho=\rho^{AB}$, we define the corresponding α-conditional entropy as

$$H_\alpha(\rho^{AB}\,|\,\sigma^B) := -S_\alpha(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\sigma^B) \quad (22)$$

and

$$\begin{aligned} H_\alpha(\rho^{AB}\,|\,B) &:= \max_{\sigma^B\in\mathfrak{S}(\mathcal{H}_B)} H_\alpha(\rho^{AB}\,|\,\sigma^B) \\ &= -\min_{\sigma^B\in\mathfrak{S}(\mathcal{H}_B)} S_\alpha(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\sigma^B). \quad (23) \end{aligned}$$

For a bipartite state $\rho^{AB} \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the conditional min-entropy of ρ^{AB} given $\mathcal{H}_B$, denoted by $H_{\min}(\rho^{AB} \,|\, B)$ and introduced by Renner [18], is relevant for the proof of our main result. It is obtainable from the max-relative entropy as follows:

$$H_{\min}(\rho^{AB} \,|\, B) := -\min_{\sigma^B \in \mathfrak{S}(\mathcal{H}_B)} D_{\max}(\rho^{AB} \,\|\, \mathbb{1}_A \otimes \sigma^B).$$

Further, from the quantum relative entropy (20), we define the quantum conditional entropy as

$$H(\rho^{AB} \,|\, B) = -\min_{\sigma^B \in \mathfrak{S}(\mathcal{H}_B)} S(\rho^{AB} \,\|\, \mathbb{1}_A \otimes \sigma^B)$$

which, by Lemma 6, satisfies $H(\rho^{AB} \,|\, B) = H(\rho^{AB} \,|\, \rho^B) = S(\rho^{AB}) - S(\rho^B)$. Finally, given a bipartite state ρ^{AB}, its coherent information $I^c(\rho^{AB})$ is defined as

$$I^c(\rho^{AB}) := -H(\rho^{AB} \,|\, B) = S(\rho^B) - S(\rho^{AB}) \tag{24}$$

and, by analogy

$$I^c_\alpha(\rho^{AB}) := -H_\alpha(\rho^{AB} \,|\, B)$$

for any $\alpha \in [0, \infty)$. Clearly, $I^c_1(\rho^{AB}) = I^c(\rho^{AB})$.

B. Smoothed Entropies

As first noticed by Renner [18], in order to allow for a finite accuracy in one-shot protocols, it is necessary to introduce smoothed entropies. We consider two different classes of smoothed entropies, namely the *state-smoothed* and the *operator-smoothed* entropies. The former was introduced by Renner [18], while the latter arises naturally from the consideration of quasi-entropies.

1) State-Smoothed Quantum Entropies: For any bipartite state $\rho^{AB} \in \mathfrak{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, smoothed conditional entropies $H^\delta_{\min}(\rho^{AB} \,|\, B)$ and $H^\delta_0(\rho^{AB} \,|\, B)$ are defined for any $\delta \geq 0$ as

$$H^\delta_{\min}(\rho^{AB} \,|\, B) := \max_{\bar{\rho}^{AB} \in \mathfrak{b}(\rho^{AB};\delta)} H_{\min}(\bar{\rho}^{AB} \,|\, B)$$
$$H^\delta_0(\rho^{AB} \,|\, B) := \min_{\bar{\rho}^{AB} \in \mathfrak{b}(\rho^{AB};\delta)} H_0(\bar{\rho}^{AB} \,|\, B)$$

where $\mathfrak{b}(\rho^{AB};\delta)$ is the set defined in (13). For a bipartite ρ^{AB}, the smoothed α-conditional entropies $H^\delta_\alpha(\rho^{AB} \,|\, B)$ are then defined, using (22) and (23), as follows:

$$H^\delta_\alpha(\rho^{AB} \,|\, B) := \begin{cases} \min_{\bar{\rho}^{AB} \in \mathfrak{b}(\rho^{AB};\delta)} H_\alpha(\bar{\rho}^{AB} \,|\, B), & \text{for } 0 \leq \alpha < 1 \\ \max_{\bar{\rho}^{AB} \in \mathfrak{b}(\rho^{AB};\delta)} H_\alpha(\bar{\rho}^{AB} \,|\, B), & \text{for } 1 < \alpha \end{cases} \tag{25}$$

and the corresponding smoothed α-coherent information is defined as

$$I^c_{\alpha,\delta}(\rho^{AB}) := -H^\delta_\alpha(\rho^{AB} \,|\, B). \tag{26}$$

For $\alpha = 0$, this is identical to (11).

2) Operator-Smoothed Quasi-Entropies: Given $\rho, \sigma \geq 0$ and an operator $0 \leq P \leq \mathbb{1}$, let us consider the quantity

$$\psi^P_\alpha(\rho \,\|\, \sigma) := \log \mathrm{Tr}[\sqrt{P}\rho^\alpha\sqrt{P}\sigma^{1-\alpha}], \qquad \alpha > 0.$$

Note that $\psi^P_\alpha(\rho \,\|\, \sigma)$ is well-defined as long as $\sigma^{1-\alpha}$ and $\sqrt{P}\rho^\alpha\sqrt{P}$ do not have orthogonal supports. In the following, we shall assume this to be true.

Lemma 8: For any $\rho, \sigma \geq 0$, and any $0 \leq P \leq \mathbb{1}$, the function

$$\alpha \mapsto \psi^P_\alpha(\rho \,\|\, \sigma)$$

is convex for $\alpha > 0$. □

Proof: Let $\rho = \sum_k a_k |\gamma_k\rangle\langle\gamma_k|$ and $\sigma = \sum_l b_l |\beta_l\rangle\langle\beta_l|$. Then

$$\psi^P_\alpha(\rho \,\|\, \sigma) = \log \sum_{k,l} |c_{kl}|^2 b_l \left(\frac{a_k}{b_l}\right)^\alpha,$$

where $|c_{kl}|^2 := |\langle\gamma_k|\sqrt{P}|\beta_l\rangle|^2$. By direct inspection then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\psi^P_\alpha(\rho \,\|\, \sigma) = \sum_{k,l} p_{kl}(\log a_k - \log b_l)$$

where p_{kl} is the probability distribution defined as

$$p_{kl} := \frac{|c_{kl}|^2 a_k^\alpha b_l^{1-\alpha}}{\sum_{k',l'} |c_{k'l'}|^2 a_{k'}^\alpha b_{l'}^{1-\alpha}},$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\psi^P_\alpha(\rho \,\|\, \sigma) = \sum_{k,l} p_{kl}(\log a_k - \log b_l)^2 - \left(\sum_{k,l} p_{kl}(\log a_k - \log b_l)\right)^2 \geq 0.$$

Due to the positivity of its second derivative hence, the function $\alpha \mapsto \psi^P_\alpha(\rho \,\|\, \sigma)$ is convex. ■

Note that the quantum relative quasi-entropy of order $\alpha, S^P_\alpha(\rho \,\|\, \sigma)$, can be equivalently written as

$$S^P_\alpha(\rho \,\|\, \sigma) = \frac{\psi^P_\alpha(\rho \,\|\, \sigma)}{\alpha - 1}. \tag{27}$$

It satisfies the following property.

Lemma 9: For any $\rho, \sigma \geq 0$, and any $0 \leq P \leq \mathbb{1}$, $S^P_\alpha(\rho \,\|\, \sigma)$ is monotonically increasing in α. □

Proof: Due to convexity of $\psi^P_\alpha(\rho \,\|\, \sigma)$, the function

$$\alpha \mapsto \frac{\psi^P_\alpha(\rho \,\|\, \sigma) - \psi^P_1(\rho \,\|\, \sigma)}{\alpha - 1}$$

is monotonically increasing in α. Let us write, for our convenience, $f(\alpha) := \psi^P_\alpha(\rho \,\|\, \sigma) - \psi^P_1(\rho \,\|\, \sigma)$, and, since $\psi_1^P(\rho\,\|\,\sigma) = \log\mathrm{Tr}[\sqrt{P}\rho\sqrt{P}\;\Pi_\sigma] \leq 0$, let us put $-c := \psi_1^P(\rho\,\|\,\sigma) \leq 0$. Then, from monotonicity of $\frac{f(x)+c}{x-1}$, we know that

$$\begin{aligned} 0 &\leq \frac{f'(x)(x-1)-(f(x)+c)}{(x-1)^2} \\ &\leq \frac{f'(x)(x-1)-f(x)}{(x-1)^2}. \end{aligned}$$

Since the second line is nothing but the derivative of (27), we proved the monotonicity of $S_\alpha^P(\rho\,\|\,\sigma)$. ■

Let us now compute $S_1^P(\rho\,\|\,\sigma) := \lim_{\alpha\to 1} S_\alpha^P(\rho\,\|\,\sigma)$: by l'Hôpital's rule

$$\begin{aligned} &\lim_{\alpha\to 1} S_\alpha^P(\rho\,\|\,\sigma) \\ &\quad= \frac{\mathrm{d}}{\mathrm{d}\alpha}\psi_\alpha^P(\rho\,\|\,\sigma)\Big|_{\alpha=1} \\ &\quad= \frac{\mathrm{Tr}\left[\sqrt{P}\rho\log\rho\sqrt{P}\Pi_\sigma - \sqrt{P}\rho\sqrt{P}\log\sigma\right]}{\mathrm{Tr}[\sqrt{P}\rho\sqrt{P}\;\Pi_\sigma]}. \end{aligned} \tag{28}$$

This leads to the definition of the corresponding smoothed coherent information

$$\tilde{I}_{1,\delta}^c(\rho^{AB}) := -\tilde{H}_1^\delta(\rho^{AB}\,|\,B) \tag{29}$$

where

$$\begin{aligned} &\tilde{H}_1^\delta(\rho^{AB}\,|\,B) \\ &\qquad:= \min_{P\in\mathfrak{p}(\rho^{AB};\delta)} \max_{\sigma^B\in\mathfrak{S}(\mathcal{H}_B)} \left[-S_1^P(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\sigma^B)\right]. \end{aligned}$$

Analogously, for any bipartite state ρ^{AB} and any $\delta \geq 0$, the quantity $\tilde{I}_{0,\delta}^c(\rho^{AB})$, given by (12), is referred to as the operator-smoothed 0-coherent information. It is equivalently expressed as

$$\begin{aligned} &-\tilde{I}_{0,\delta}^c(\rho^{AB}) \\ &\quad= \tilde{H}_0^\delta(\rho^{AB}\,|\,B) \\ &\quad:= \min_{P\in\mathfrak{p}(\rho^{AB};\delta)} \max_{\sigma^B\in\mathfrak{S}(\mathcal{H}_B)} \log\mathrm{Tr}\left[\sqrt{P}\Pi_{\rho^{AB}}\sqrt{P}\,(\mathbb{1}_A\otimes\sigma^B)\right]. \end{aligned} \tag{30}$$

The relation between $\tilde{I}_{0,\delta}^c(\rho^{AB})$ defined in (12) and $\tilde{I}_{1,\delta}^c(\rho^{AB})$ defined in (29) is provided by the following lemma.

Lemma 10: For any $\rho^{AB} \in \mathfrak{S}(\mathcal{H}_A\otimes\mathcal{H}_B)$ and any $\delta \geq 0$,

$$\tilde{I}_{0,\delta}^c(\rho^{AB}) \leq \tilde{I}_{1,\delta}^c(\rho^{AB}). \qquad \square$$

Proof: Let $\bar{P} \in \mathfrak{p}(\rho^{AB};\delta)$ be the operator achieving $\tilde{I}_{0,\delta}^c(\rho^{AB})$, and let $\bar{\sigma}^B$ be the state achieving $\min_{\sigma^B} S_1^{\bar{P}}(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\sigma^B)$. Then

$$\begin{aligned} \tilde{I}_{1,\delta}^c(\rho^{AB}) &\geq S_1^{\bar{P}}(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\bar{\sigma}^B) \\ &\geq S_0^{\bar{P}}(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\bar{\sigma}^B) \\ &\geq \min_{\sigma^B\in\mathfrak{S}(\mathcal{H}_B)} S_0^{\bar{P}}(\rho^{AB}\,\|\,\mathbb{1}_A\otimes\sigma^B) \\ &= \tilde{I}_{0,\delta}^c(\rho^{AB}) \end{aligned} \tag{31}$$

where in the second line we used Lemma 9. ■

VII. Proof of Theorem 1

A. Proof of the Lower Bound in Theorem 1

The lower bound on the one-shot entanglement transmission capacity $Q_{\mathrm{ent}}(\Phi;\varepsilon)$, for any fixed value $\varepsilon \geq 0$ of accuracy, is obtained by exploiting a lower bound on the entanglement transmission fidelity, which is derived below by the random coding method.

1) Lower Bound on Entanglement Transmission Fidelity: The lower bound on the entanglement transmission fidelity is given by the following lemma.

Lemma 11: Given a channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ and an s-dimensional subspace $\mathcal{S} \subseteq \mathcal{H}_A$, consider the channel $\Phi|_\mathcal{S} : \mathcal{B}(\mathcal{S}) \mapsto \mathcal{B}(\mathcal{H}_B)$ obtained by restricting Φ onto $\mathcal{S}$, i.e., $\Phi|_\mathcal{S}(\rho) := \Phi(\Pi_\mathcal{S}\rho\Pi_\mathcal{S})$ for any $\rho\in\mathcal{B}(\mathcal{H}_A)$, where $\Pi_\mathcal{S}$ denotes the projector onto $\mathcal{S}$. Then, for any $\delta \geq 0$ and any positive integer $m \leq s$,

$$\mathsf{F}_{\mathrm{ent}}(\Phi|_\mathcal{S};m) \geq 1 - 4\delta - \sqrt{m\left\{2^{I_{2,\delta}^c(\omega_\mathcal{S}^{RE})} - \frac{1}{s}\right\}} \tag{32}$$

where $I_{2,\delta}^c(\omega_\mathcal{S}^{RE})$ is given by (26) for $\alpha = 2$. □

Remark: From the theory of quantum error correction [12], it is known that, for a channel noiseless on $\mathcal{S}$, $\omega_\mathcal{S}^{RE}$, defined by (17) is a factorized state. Moreover, in our case, $\omega_\mathcal{S}^R := \mathrm{Tr}_E[\omega_\mathcal{S}^{RE}] = s^{-1}\sum_{i=1}^s |i\rangle\langle i|^R$. As shown in [11] by direct inspection, these two conditions imply that $H_{\min}(\omega_\mathcal{S}^{RE}|E) = \log s$. On the other hand, from (23), it follows that $H_0(\omega_\mathcal{S}^{RE}|E) = \log s$. These two calculations, together with the fact that $H_{\min}(\omega_\mathcal{S}^{RE}|E) \leq H_2(\omega_\mathcal{S}^{RE}|E) \leq H_0(\omega_\mathcal{S}^{RE}|E)$, see [19], lead us to conclude that also $H_2(\omega_\mathcal{S}^{RE}|E) = \log s$, i.e., $I_2^c(\omega_\mathcal{S}^{RE}) = -\log s$. Therefore, for any channel acting noiselessly in $\mathcal{S}$, $\mathsf{F}_{\mathrm{ent}}(\Phi|_\mathcal{S};m) = 1$ for all $m \leq s$, as expected.

Proof of Lemma 11: Fix the value of the positive integer $m \leq s$. Then, starting from the pure state $|\Omega_\mathcal{S}^{RBE}\rangle$ given by (17), let us define

$$\left|\Omega_{m,g}^{RBE}\right\rangle := \sqrt{\frac{s}{m}}\left(P_m^R U_g^R\otimes\mathbb{1}_B\otimes\mathbb{1}_E\right)\left|\Omega_\mathcal{S}^{RBE}\right\rangle$$

where U_g^R is a unitary representation of the element g of the group $\mathbb{SU}(s)$, and let

$$P_m^R = \sum_{i=1}^m |i^R\rangle\langle i^R|$$

the vectors $|i^R\rangle$, $i = 1,\ldots,s$, being the same as in (16). The reduced state $\mathrm{Tr}_B[\Omega_{m,g}^{RBE}]$ will be denoted as $\omega_{m,g}^{RE}$ (and analogously the others). Notice that, by construction

$$\omega_{m,g}^R = \tau_m^R := \frac{P_m^R}{m}.$$

The lower bound (32) would follow if there exists a subspace $\mathcal{H}_M \subseteq \mathcal{S}$ of dimension m which is transmitted with fidelity greater or equal to the right-hand side (RHS) of (32). One way to prove the existence of such a subspace is to show that the *group-averaged* fidelity, $\bar{\mathsf{F}}(\mathcal{S},m)$ (defined below), is larger than that value

$$\bar{\mathsf{F}}(\mathcal{S},m) := \int \mathrm{d}g \max_{\mathcal{D}} F^2\left((\mathrm{id}_R\otimes\mathcal{D}_B)\left(\omega^{RB}_{m,g}\right),\Psi^{RA}_{m,g}\right) \quad (33)$$

where $|\Psi^{RA}_{m,g}\rangle := \sqrt{\frac{s}{m}}(P^R_m U^R_g\otimes \mathbb{1}_A)|\Psi^{RA}_{\mathcal{S}}\rangle$, which is a MES of rank m due to (2). It is hence sufficient to compute a lower bound to $\bar{\mathsf{F}}(\mathcal{S},m)$.

Using Lemma 4, we have

$$\bar{\mathsf{F}}(\mathcal{S},m) \geq \int \mathrm{d}g\, F^2\left(\omega^{RE}_{m,g},\tau^R_m\otimes\omega^E_{m,g}\right).$$

Further, using $F^2(\rho,\sigma)\geq 1-\|\rho-\sigma\|_1$, we have that

$$\bar{\mathsf{F}}(\mathcal{S},m) \geq 1-\int \mathrm{d}g\, \left\|\omega^{RE}_{m,g}-\tau^R_m\otimes\omega^E_{m,g}\right\|_1.$$

Now, for any fixed $\delta\geq 0$, let $\bar{\omega}^{RE}\in\mathfrak{b}(\omega^{RE}_{\mathcal{S}};\delta)$. Let us, moreover, define $\bar{\omega}^{RE}_{m,g} := \frac{s}{m}(P^R_M U^R_g\otimes\mathbb{1}_E)\bar{\omega}^{RE}(P^R_M U^R_g\otimes\mathbb{1}_E)^\dagger$. By the triangle inequality, we have that

$$\begin{aligned}&\left\|\omega^{RE}_{m,g}-\tau^R_m\otimes\omega^E_{m,g}\right\|_1\\ &\quad\leq \left\|\bar{\omega}^{RE}_{m,g}-\tau^R_m\otimes\bar{\omega}^E_{m,g}\right\|_1\\ &\qquad+\left\|\omega^{RE}_{m,g}-\bar{\omega}^{RE}_{m,g}\right\|_1\\ &\qquad+\left\|\tau^R_m\otimes\bar{\omega}^E_{m,g}-\tau^R_m\otimes\omega^E_{m,g}\right\|_1\\ &\quad\leq \left\|\bar{\omega}^{RE}_{m,g}-\tau^R_m\otimes\bar{\omega}^E_{m,g}\right\|_1\\ &\qquad+2\left\|\omega^{RE}_{m,g}-\bar{\omega}^{RE}_{m,g}\right\|_1\end{aligned}$$

which, in turns, implies that

$$\begin{aligned}\bar{\mathsf{F}}(\mathcal{S},m) \geq 1 &- \int \mathrm{d}g \left\|\bar{\omega}^{RE}_{m,g}-\tau^R_m\otimes\bar{\omega}^E_{m,g}\right\|_1\\ &-2\int \mathrm{d}g\left\|\omega^{RE}_{m,g}-\bar{\omega}^{RE}_{m,g}\right\|_1\end{aligned}$$

for any choice of $\bar{\omega}^{RE}$ in $\mathfrak{b}(\omega^{RE}_{\mathcal{S}};\delta)$. Now, thanks to [9, Lemma 3.2] and (4), we know that

$$\int \mathrm{d}g\left\|\omega^{RE}_{m,g}-\bar{\omega}^{RE}_{m,g}\right\|_1 \leq \left\|\bar{\omega}^{RE}-\omega^{RE}_{\mathcal{S}}\right\|_1\leq 2\delta$$

which leads us to the estimate

$$\bar{\mathsf{F}}(\mathcal{S},m) \geq 1-4\delta-\int \mathrm{d}g\left\|\bar{\omega}^{RE}_{m,g}-\tau^R_m\otimes\bar{\omega}^E_{m,g}\right\|_1.$$

We are, hence, left with estimating the last group average.

In order to do so, we exploit a technique used by Renner [18] and Berta [26]: by applying Lemma 3, for any given state σ^E invertible on supp $\bar{\omega}^E$, we obtain the estimate

$$\begin{aligned}&\left\|\bar{\omega}^{RE}_{m,g}-\tau^R_m\otimes\bar{\omega}^E_{m,g}\right\|_1^2\\ &\quad\leq m\mathrm{Tr}\left[\left(\bar{\omega}^{RE}_{m,g}-\tau^R_m\otimes\bar{\omega}^E_{m,g}\right)\ A^{RE}_{m,g}\right]\\ &\quad:= m\left\|\tilde{\rho}^{RE}_{m,g}-\tau^R_m\otimes\tilde{\rho}^E_{m,g}\right\|_2^2\end{aligned}$$

where $A^{RE}_{m,g} := (P^R_m\otimes\sigma^E)^{-1/2}(\bar{\omega}^{RE}_{m,g}-\tau^R_m\otimes\bar{\omega}^E_{m,g})(P^R_m\otimes\sigma^E)^{-1/2}$, $\|X\|_2 := \sqrt{\mathrm{Tr}[X^\dagger X]}$ denotes the Hilbert-Schmidt norm, and

$$\tilde{\rho}^{RE}_{m,g} := \left(P^R_m\otimes\sigma^E\right)^{-1/4}\bar{\omega}^{RE}_{m,g}\left(P^R_m\otimes\sigma^E\right)^{-1/4}$$

and, correspondingly, $\tilde{\rho}^E_{m,g} := \mathrm{Tr}_R[\tilde{\rho}^{RE}_{m,g}] = (\sigma^E)^{-1/4}\bar{\omega}^E_{m,g}(\sigma^E)^{-1/4}$. It is easy to check that

$$\left\|\tilde{\rho}^{RE}_{m,g}-\tau^R_m\otimes\tilde{\rho}^E_{m,g}\right\|_2^2=\left\|\tilde{\rho}^{RE}_{m,g}\right\|_2^2-\frac{1}{m}\left\|\tilde{\rho}^E_{m,g}\right\|_2^2.$$

Further, using the concavity of the function $f(x)=\sqrt{x}$, we have

$$\begin{aligned}\bar{\mathsf{F}}(\mathcal{S},m) &\geq 1-4\delta\\ &-\sqrt{\left\{m\int \mathrm{d}g\ \left\|\tilde{\rho}^{RE}_{m,g}\right\|_2^2-\int\mathrm{d}g\ \left\|\tilde{\rho}^E_{m,g}\right\|_2^2\right\}}.\end{aligned} \quad (34)$$

Standard calculations, similar to those reported in [9] and [26], lead to

$$\begin{aligned}&\int \mathrm{d}g\ \left\|\tilde{\rho}^{RE}_{m,g}\right\|_2^2\\ &\quad=\frac{s}{m}\frac{s-m}{s^2-1}\|\tilde{\rho}^E\|_2^2+\frac{s}{m}\frac{ms-1}{s^2-1}\|\tilde{\rho}^{RE}\|_2^2\end{aligned}$$

and

$$\begin{aligned}&\int \mathrm{d}g\ \left\|\tilde{\rho}^{E}_{m,g}\right\|_2^2\\ &\quad=\frac{s}{m}\frac{ms-1}{s^2-1}\left\|\tilde{\rho}^E\right\|_2^2+\frac{s}{m}\frac{s-m}{s^2-1}\left\|\tilde{\rho}^{RE}\right\|_2^2\end{aligned}$$

where

$$\tilde{\rho}^{RE} := (\mathbb{1}_R\otimes\sigma^E)^{-1/4}\bar{\omega}^{RE}(\mathbb{1}_R\otimes\sigma^E)^{-1/4}$$

and $\tilde{\rho}^E_{\mathcal{S}} := \mathrm{Tr}_R[\tilde{\rho}^{RE}_{\mathcal{S}}]$. By simple manipulations, we arrive at

$$\begin{aligned}&m\int\mathrm{d}g\ \left\|\tilde{\rho}^{RE}_{m,g}\right\|_2^2-\int\mathrm{d}g\ \left\|\tilde{\rho}^E_{m,g}\right\|_2^2\\ &\quad=\frac{s^2(m^2-1)}{m(s^2-1)}\left\{\left\|\tilde{\rho}^{RE}\right\|_2^2-\frac{1}{s}\left\|\tilde{\rho}^E\right\|_2^2\right\}.\end{aligned}$$

Since $m\leq s$,

$$\frac{s^2(m^2-1)}{m(s^2-1)}=m\frac{1-\frac{1}{m^2}}{1-\frac{1}{s^2}}\leq m,$$

so that (34) can be rewritten as

$$\bar{\mathsf{F}}(\mathcal{S},m)\geq 1-4\delta-\sqrt{m\left\{\|\tilde{\rho}^{RE}\|_2^2-\frac{1}{s}\|\tilde{\rho}^E\|_2^2\right\}}$$

for any choice of the states $\bar{\omega}^{RE}\in\mathfrak{b}(\omega^{RE}_{\mathcal{S}};\delta)$ and σ^E invertible on supp $\bar{\omega}^E$.

Now, notice that

$$\|\tilde{\rho}^{RE}\|_2^2\leq 2^{S_2(\bar{\omega}^{RE}\|\mathbb{1}_R\otimes\sigma^E)}.$$

This inequality easily follows from (5), i.e.

$$\begin{aligned}\mathrm{Tr}[(\omega^{-1/4}\rho\omega^{-1/4})^2] &= \mathrm{Tr}[\omega^{-1/2}\rho\omega^{-1/2}\rho] \\ &\le \mathrm{Tr}[\rho^2\omega^{-1}] = 2^{S_2(\rho\,\|\,\omega)}.\end{aligned}$$

Moreover, from Lemma 3, $\|\tilde{\rho}^E\|_2^2 \ge 1$. Thus

$$\bar{\mathsf{F}}(\mathcal{S},m) \ge 1 - 4\delta - \sqrt{m\left\{2^{S_2(\bar{\omega}^{RE}\,\|\,\mathbb{1}_R\otimes\sigma^E)} - \frac{1}{s}\right\}}$$

for any choice of states $\bar{\omega}^{RE} \in \mathfrak{b}(\omega_{\mathcal{S}}^{RE};\delta)$ and σ^E, the latter strictly positive on supp $\bar{\omega}^R$. In order to tighten the bound, we first optimize (i.e., minimize) $S_2(\bar{\omega}^{RE}\,\|\,\mathbb{1}_R\otimes\sigma^E)$ over σ^E for any $\bar{\omega}^{RE}$, obtaining $I_2^c(\bar{\omega}^{RE})$. We further optimize (i.e., minimize) $I_2^c(\bar{\omega}^{RE})$ over $\bar{\omega}^{RE} \in \mathfrak{b}(\omega_{\mathcal{S}}^{RE};\delta)$, eventually obtaining $I_{2,\delta}^c(\omega_{\mathcal{S}}^{RE})$. □

2) Proof of the Lower Bound in (18): By Lemma 11, we have what follows.

Corollary 2: Given a channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$, an s-dimensional subspace $\mathcal{S} \subseteq \mathcal{H}_A$, and any $\delta \in [0,\varepsilon/4]$, a nonnegative real number $R = \log m$, $m \in \mathbb{N}$, is an ε-achievable rate for entanglement transmission through $\Phi\,|_{\mathcal{S}}$ if

$$4\delta + \sqrt{m\left\{2^{I_{2,\delta}^c(\omega_{\mathcal{S}}^{RE})} - \frac{1}{s}\right\}} \le \varepsilon.$$

□

In particular, since $s \le d := \dim\mathcal{H}_A$, a positive real number $R = \log m$ is an ε-achievable rate for $\Phi|_{\mathcal{S}}$ if, for any $\delta \in [0,\varepsilon/4]$,

$$m2^{I_{2,\delta}^c(\omega_{\mathcal{S}}^{RE})} \le \frac{1}{d} + (\varepsilon - 4\delta)^2$$

or, equivalently, if

$$\log m \le \log\left[\frac{1}{d} + (\varepsilon-4\delta)^2\right] - I_{2,\delta}^c\left(\omega_{\mathcal{S}}^{RE}\right).$$

This, together with (26), implies the following lower bound to the one-shot capacity of entanglement transmission through $\Phi\,|_{\mathcal{S}}$, for any $\delta \in [0,\varepsilon/4]$:

$$Q_{\mathrm{ent}}(\Phi\,|_{\mathcal{S}};\varepsilon) \ge \log\left[\frac{1}{d} + (\varepsilon-4\delta)^2\right] + H_2^\delta\left(\omega_{\mathcal{S}}^{RE}\,|\,E\right) - \Delta$$

where $\Delta \le 1$ is a positive quantity included to make the RHS of the above inequality equal to the logarithm of a positive integer (see the Remark after Theorem 1). This in turn implies the following lower bound to the one-shot capacity of entanglement transmission through Φ:

$$Q_{\mathrm{ent}}(\Phi;\varepsilon) \ge \log\left[\frac{1}{d} + (\varepsilon-4\delta)^2\right] + \max_{\mathcal{S}\subseteq\mathcal{H}_A} H_2^\delta\left(\omega_{\mathcal{S}}^{RE}|E\right) - \Delta.$$

As a consequence of Lemma 7, we have

$$\begin{aligned}&Q_{\mathrm{ent}}(\Phi;\varepsilon)\\ &\ge \log\left[\frac{1}{d} + (\varepsilon-4\delta)^2\right]\\ &\quad + \max_{\mathcal{S}\subseteq\mathcal{H}_A} H_{\min}^\delta\left(\omega_{\mathcal{S}}^{RE}\,|\,E\right) - \Delta\\ &\ge \log\left[\frac{1}{d} + (\varepsilon-4\delta)^2\right]\\ &\quad + \max_{\mathcal{S}\subseteq\mathcal{H}_A} H_{\min}^\delta\left(\omega_{\mathcal{S}}^{RE}\,|\,\omega_{\mathcal{S}}^{E}\right) - \Delta\end{aligned}$$

where

$$H_{\min}^\delta\left(\omega_{\mathcal{S}}^{RE}\,|\,\omega_{\mathcal{S}}^{E}\right) := -\min_{\bar{\omega}^{RE}\in\mathfrak{b}(\omega_{\mathcal{S}}^{RE};\delta)D_{\max}\left(\bar{\omega}^{RE}\,|\,\mathbb{1}_R\otimes\bar{\omega}^E\right)}$$

for

$$\bar{\omega}^E = \mathrm{Tr}_R[\bar{\omega}^{RE}].$$

In [26], it is proved that $H_{\min}(\rho^{AB}\,|\,\rho^B) = -H_0(\rho^{AC}\,|\,C)$, if ρ^{AB} and ρ^{AC} are both reduced states of the same tripartite pure state. This fact, together with arguments analogous to those used in [21] to prove Lemma 3 there, leads to the identity $H_{\min}^\delta(\omega_{\mathcal{S}}^{RE}\,|\,\omega_{\mathcal{S}}^{E}) = -H_0^\delta(\omega_{\mathcal{S}}^{RB}\,|\,B)$, implying, via (26), the desired lower bound to the one-shot capacity of entanglement transmission

$$Q_{\mathrm{ent}}(\Phi;\varepsilon) \ge \log\left[\frac{1}{d} + (\varepsilon-4\delta)^2\right] + \max_{\mathcal{S}\subseteq\mathcal{H}_A} I_{0,\delta}^c\left(\omega_{\mathcal{S}}^{RB}\right) - \Delta \tag{35}$$

for any $\delta \in [0,\varepsilon/4]$, and, in particular, for $\delta = \varepsilon/8$.

B. Proof of the Upper Bound in Theorem 1

In this section, we prove the upper bound

$$Q_{\mathrm{ent}}(\Phi;\varepsilon) \le \max_{\mathcal{S}\subseteq\mathcal{H}_A} \tilde{I}_{0,2\sqrt{\varepsilon}}^c\left(\omega_{\mathcal{S}}^{RB}\right)$$

where $\tilde{I}_{0,2\sqrt{\varepsilon}}^c(\omega_{\mathcal{S}}^{RB})$ is defined in (30).

We start by proving the following monotonicity relation.

Lemma 12 (Quantum Data-Processing Inequality): For any bipartite state ρ^{AB}, any channel $\Phi : B \mapsto C$, and any $\delta \ge 0$, we have

$$\tilde{I}_{0,2\sqrt{\delta}}^c(\rho^{AB}) \ge \tilde{I}_{0,\delta}^c((\mathrm{id}\otimes\Phi)(\rho^{AB})).$$

□

Proof: Let $P \in \mathfrak{p}((\mathrm{id}\otimes\Phi)(\rho^{AB});\delta)$ and $\bar{\sigma}^C$ be the pair achieving $\tilde{H}_0^\delta((\mathrm{id}\otimes\Phi)(\rho^{AB})\,|\,C)$, that is

$$\begin{aligned}&\tilde{H}_0^\delta((\mathrm{id}\otimes\Phi)(\rho^{AB})\,|\,C)\\ &\qquad = \log\mathrm{Tr}\left[\sqrt{P}\Pi_{(\mathrm{id}\otimes\Phi)(\rho^{AB})}\sqrt{P}(\mathbb{1}_A\otimes\bar{\sigma}^C)\right].\end{aligned}$$

Consider now the operator

$$Q := (\mathrm{id}_A \otimes \Phi^*) \left(\sqrt{P} \Pi_{(\mathrm{id}\otimes\Phi)(\rho^{AB})} \sqrt{P} \right)$$

where $\Phi^* : C \mapsto B$ denotes the identity-preserving adjoint map associated with the trace-preserving map $\Phi : B \mapsto C$. It clearly satisfies $0 \le Q \le \mathbb{1}$. Let us now put, for sake of clarity, $\gamma^{AC} := (\mathrm{id} \otimes \Phi)(\rho^{AB})$. Then

$$\begin{aligned}
&\mathrm{Tr}[Q\,\rho^{AB}] \\
&\quad = \mathrm{Tr}[(\sqrt{P}\Pi_{\gamma^{AC}}\sqrt{P})\,\gamma^{AC}] \\
&\quad = 1 + \mathrm{Tr}[\Pi_{\gamma^{AC}}(\sqrt{P}\gamma^{AC}\sqrt{P} - \gamma^{AC})] \\
&\quad \ge 1 + \mathrm{Tr}[\{\sqrt{P}\gamma^{AC}\sqrt{P} < \gamma^{AC}\} \\
&\qquad \times (\sqrt{P}\gamma^{AC}\sqrt{P} - \gamma^{AC})]
\end{aligned}$$

where in the last line we used Lemma 1. Due to Gentle Measurement Lemma 2, we have that

$$\|\sqrt{P}\gamma^{AC}\sqrt{P} - \gamma^{AC}\|_1 \le 2\sqrt{\delta}$$

which, together with the formula $\|A-B\|_1 = \mathrm{Tr}[\{A \ge B\}(A-B)] - \mathrm{Tr}[\{A < B\}(A-B)]$, implies

$$\mathrm{Tr}[\{\sqrt{P}\gamma^{AC}\sqrt{P} < \gamma^{AC}\}(\sqrt{P}\gamma^{AC}\sqrt{P} - \gamma^{AC})] \ge -2\sqrt{\delta}.$$

This leads to the estimate

$$\mathrm{Tr}[Q\,\rho^{AB}] \ge 1 - 2\sqrt{\delta}.$$

In other words, $Q \in \mathfrak{p}(\rho^{AB}; 2\sqrt{\delta})$. Now, let $\bar{\sigma}^B$ be the state achieving $\max_{\sigma^B \in \mathfrak{S}(\mathcal{H}_B)} \log \mathrm{Tr}[\sqrt{Q}\Pi_{\rho^{AB}}\sqrt{Q}\,(\mathbb{1}_A \otimes \sigma^B)]$. We then have the following chain of inequalities:

$$\begin{aligned}
&\tilde{H}_0^\delta((\mathrm{id}\otimes\Phi)(\rho^{AB})\,|\,C) \\
&\quad = \log \mathrm{Tr}\left[\sqrt{P}\Pi_{(\mathrm{id}\otimes\Phi)(\rho^{AB})}\sqrt{P}(\mathbb{1}_A \otimes \bar{\sigma}^C)\right] \\
&\quad \ge \log \mathrm{Tr}\left[\sqrt{P}\Pi_{(\mathrm{id}\otimes\Phi)(\rho^{AB})}\sqrt{P}(\mathbb{1}_A \otimes \Phi(\bar{\sigma}^B))\right] \\
&\quad = \log \mathrm{Tr}[Q\,(\mathbb{1}_A \otimes \bar{\sigma}^B)] \\
&\quad \ge \log \mathrm{Tr}[\sqrt{Q}\Pi_{\rho^{AB}}\sqrt{Q}\,(\mathbb{1}_A \otimes \bar{\sigma}^B)] \\
&\quad = \max_{\sigma^B \in \mathfrak{S}(\mathcal{H}_B)} \log \mathrm{Tr}[\sqrt{Q}\Pi_{\rho^{AB}}\sqrt{Q}\,(\mathbb{1}_A \otimes \sigma^B)] \\
&\quad \ge \tilde{H}_0^{2\sqrt{\delta}}(\rho^{AB}\,|\,B).
\end{aligned}$$

The statement of the Lemma is finally obtained by (30). □

With Lemma 12 in hand, it is now easy, by the following standard arguments, to prove the upper bound in Theorem 1.

In fact, suppose now that R_0 is the maximum of all ε-achievable rates, i.e., $R_0 = Q_{\mathrm{ent}}(\Phi;\varepsilon)$. By Definition 2, the integer $s := 2^{R_0}$ is such that

$$\mathsf{F}_{\mathrm{ent}}(\Phi; s) \ge 1 - \varepsilon.$$

This is equivalent to saying that there exists an s-dimensional subspace $\mathcal{S} \subseteq \mathcal{H}_A$ such that

$$\max_{\mathcal{D}} F^2((\mathrm{id}_R \otimes \mathcal{D}_B)\left(\omega_{\mathcal{S}}^{RB}\right), \Psi_{\mathcal{S}}^{RA}) \ge 1 - \varepsilon$$

or, equivalently, that there exists a decoding operation $\bar{\mathcal{D}} : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_A)$ such that $\Psi_{\mathcal{S}}^{RA} := |\Psi_{\mathcal{S}}^{RA}\rangle\langle\Psi_{\mathcal{S}}^{RA}| \in \mathfrak{p}((\mathrm{id}_R \otimes \bar{\mathcal{D}}_B)(\omega_{\mathcal{S}}^{RB}); \varepsilon)$. Then, by exploiting Lemma 12, we have that

$$\begin{aligned}
&\tilde{I}_{0,2\sqrt{\varepsilon}}^c\left(\omega_{\mathcal{S}}^{RB}\right) \\
&\quad \ge \tilde{I}_{0,\varepsilon}^c((\mathrm{id}_R \otimes \bar{\mathcal{D}}_B)\left(\omega_{\mathcal{S}}^{RB}\right) \\
&\quad \ge -\max_{\sigma^A} \log \mathrm{Tr}\left[\Psi_{\mathcal{S}}^{RA}\Pi_{(\mathrm{id}_R\otimes\bar{\mathcal{D}}_B)(\omega_{\mathcal{S}}^{RB})}\Psi_{\mathcal{S}}^{RA}(\mathbb{1}_R \otimes \sigma^A)\right] \\
&\quad \ge -\max_{\sigma^A} \log \mathrm{Tr}\left[\Psi_{\mathcal{S}}^{RA}(\mathbb{1}_R \otimes \sigma^A)\right].
\end{aligned}$$

The claim is finally proved by noticing that the last line in the equation above equals $I_0^c(\Psi_{\mathcal{S}}^{RA})$, so that $\tilde{I}_{0,2\sqrt{\varepsilon}}^c(\omega_{\mathcal{S}}^{RB}) \ge I_0^c(\Psi_{\mathcal{S}}^{RA}) = \log s = R_0 = Q_{\mathrm{ent}}(\Phi;\varepsilon)$.

VIII. Quantum Capacity of a Sequence of Channels

Let $\{\mathcal{H}_A^{\otimes n}\}_{n=1}^\infty$ and $\{\mathcal{H}_B^{\otimes n}\}_{n=1}^\infty$ be two sequences of Hilbert spaces, and let $\hat{\Phi} := \{\Phi_n\}_{n=1}^\infty$ be a sequence of quantum channels such that, for each n

$$\Phi_n : \mathcal{B}\left(\mathcal{H}_A^{\otimes n}\right) \mapsto \mathcal{B}\left(\mathcal{H}_B^{\otimes n}\right).$$

For any given $\varepsilon > 0$ and any fixed finite n, the one-shot quantum capacity of Φ_n, with respect to the fidelity F_x, where $x \in \{\mathrm{ent}, \min\}$, is given by $Q_x(\Phi_n;\varepsilon)$. However, since Φ_n itself could be the CPTP-map describing n uses of an arbitrary channel, possibly with memory, it is meaningful to introduce the quantity

$$\frac{1}{n} Q_x(\Phi_n;\varepsilon)$$

which can be interpreted as the capacity *per use* of the channel. This quantity is of relevance in all practical situations because, instead of considering an asymptotically large number of uses of the channel, it is more realistic to consider using a channel a large *but finite* number of times, in order to achieve reliable transmission of quantum information. Theorem 1 provides the following bounds on this quantity:

$$\begin{aligned}
&\frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} I_{0,\varepsilon/8}^c\left(\omega_{\mathcal{S}}^{R_nB_n}\right) + \frac{1}{n}\log\left[\frac{1}{d^n} + \frac{\varepsilon^2}{4}\right] - \frac{\Delta}{n} \\
&\le \frac{1}{n} Q_{\mathrm{ent}}(\Phi_n;\varepsilon) \\
&\le \frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} \tilde{I}_{0,2\sqrt{\varepsilon}}^c\left(\omega_{\mathcal{S}}^{R_nB_n}\right)
\end{aligned}$$

where $\omega_{\mathcal{S}_n}^{R_nB_n} = \mathrm{Tr}_{E_n}[\Omega_{\mathcal{S}_n}^{R_nB_nE_n}]$, the pure state $|\Omega_{\mathcal{S}_n}^{R_nB_nE_n}\rangle$ being defined through (17). Note that the second and third terms in the lower bound decrease rapidly as n increases, resulting in sharp bounds on the capacity for entanglement transmission per use, even for finite n. Moreover, due to Corollary 1, the difference between $Q_{\mathrm{ent}}(\Phi_n;\varepsilon)/n$ and $Q_{\min}(\Phi_n;\varepsilon)/n$ also decreases as n increases.

If the sequence is infinite, we define the corresponding asymptotic capacity of the channel Φ as

$$Q_x^\infty(\hat{\Phi}) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} Q_x(\Phi_n;\varepsilon).$$

Due to the equivalence relations stated in Corollary 1, we see that the different fidelities yield the same asymptotic quantum capacity, so that

$$Q^{\infty}_{\rm ent}(\hat{\Phi}) = Q^{\infty}_{\min}(\hat{\Phi}) := Q^{\infty}(\hat{\Phi}). \tag{36}$$

A. Multiple Uses of a Memoryless Channel

Here, we prove that the asymptotic quantum capacity of a memoryless channel, sometimes referred to as the "LSD Theorem" [3]–[5], can be obtained from Theorem 1. For a memoryless channel, the sequence $\hat{\Phi}$ is given by $\{\Phi^{\otimes n}\}_{n=1}^{\infty}$, and hence its capacity can simply be labelled by Φ. The LSD Theorem, strictly speaking, gives an expression for $Q^{\infty}_{\min}(\Phi)$, whereas our method gives an expression for $Q^{\infty}_{\rm ent}(\Phi)$. However, by (36), these expressions are equivalent.

Here, we prove the following theorem, which can be seen as an alternative formulation of the LSD theorem.

Theorem 2 (Memoryless Channels): For a memoryless channel $\Phi : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$

$$Q^{\infty}(\Phi) = \lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} I^c(\mathcal{S}, \Phi^{\otimes n}) \tag{37}$$

where $I^c(\mathcal{S}, \Phi)$ denotes the coherent information of the channel Φ with respect to an input subspace $\mathcal{S}$, and is defined through (24) as follows:

$$I^c(\mathcal{S}, \Phi) := I^c\left(\omega_{\mathcal{S}}^{RB}\right)$$

where $\omega_{\mathcal{S}}^{RB}$ is the reduced state of the pure state $|\Omega_{\mathcal{S}}^{RBE}\rangle$ defined in (17). □

Notice that in (37) liminf has been replaced by lim, since the limit exists [28].

1) Direct Part of Theorem 2: Here we prove that

$$Q^{\infty}(\Phi) \geq \lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} I^c(\mathcal{S}, \Phi^{\otimes n}).$$

From Theorem 1

$$Q^{\infty}(\Phi) \geq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \left\{ \log\left[\frac{1}{d^n} + \frac{\varepsilon^2}{4}\right] - \Delta \right.$$
$$\left. + \max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} I^c_{0,\varepsilon/8}\left(\omega_{\mathcal{S}}^{R_nB_n}\right)\right\}.$$

The first two terms clearly vanish. We are hence left with the evaluation of the third term. First of all, we recall that [see arguments before (35)]

$$I^c_{0,\varepsilon/8}\left(\omega_{\mathcal{S}}^{R_nB_n}\right) = H_{\min}^{\varepsilon/8}\left(\omega_{\mathcal{S}}^{R_nE_n} \,|\, \omega_{\mathcal{S}}^{E_n}\right).$$

This implies that

$$\begin{aligned} Q^{\infty}(\Phi) &\geq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} H_{\min}^{\varepsilon/8}\left(\omega_{\mathcal{S}}^{R_nE_n}\middle|\, \omega_{\mathcal{S}}^{E_n}\right) \\ &\geq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A} H_{\min}^{\varepsilon/8}\left(\left(\omega_{\mathcal{S}}^{RE}\right)^{\otimes n}\middle|\, \left(\omega_{\mathcal{S}}^{E}\right)^{\otimes n}\right). \end{aligned}$$

As shown in [18], we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A} H_{\min}^{\varepsilon/8}\left(\left(\omega_{\mathcal{S}}^{RE}\right)^{\otimes n}\middle|\, \left(\omega_{\mathcal{S}}^{E}\right)^{\otimes n}\right) \\ &\quad = \max_{\mathcal{S}\subseteq\mathcal{H}_A} H\left(\omega_{\mathcal{S}}^{RE}\middle|\, \omega_{\mathcal{S}}^{E}\right) \\ &\quad := \max_{\mathcal{S}\subseteq\mathcal{H}_A} \left[-I^c\left(\omega_{\mathcal{S}}^{RE}\right)\right] \\ &\quad = \max_{\mathcal{S}\subseteq\mathcal{H}_A} I^c(\mathcal{S}, \Phi) \end{aligned}$$

where in the last line we used the fact that $I^c(\omega_{\mathcal{S}}^{RB}) = -I^c(\omega_{\mathcal{S}}^{RE})$, since $\Omega_{\mathcal{S}}^{RBE}$ is pure. Therefore

$$Q^{\infty}(\Phi) \geq \max_{\mathcal{S}\subseteq\mathcal{H}_A} I^c(\mathcal{S}, \Phi).$$

As in [28], we can then achieve the RHS of (37) by the usual blocking argument.

2) Weak Converse of Theorem 2: In order to obtain the upper bound, it suffices to evaluate the asymptotic behavior of the upper bound on $Q^{\infty}(\Phi)$ which, by Theorem 1, is given by

$$Q^{\infty}(\Phi) \leq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} \tilde{I}^c_{0,2\sqrt{\varepsilon}}\left(\omega_{\mathcal{S}}^{R_nB_n}\right). \tag{38}$$

The following two lemmas are essential for the evaluation of this bound, and are also of independent interest. The first one relates $S_1^P(\rho\,|\,\sigma)$ to the quantum relative entropy $S(\rho\,|\,\sigma)$, while the second one relates the operator-smoothed 0-coherent information to the usual coherent information.

Lemma 13: Consider two states $\rho, \sigma \in \mathfrak{S}(\mathcal{H})$, with $\mathrm{supp}\rho \subseteq \mathrm{supp}\sigma$, and a positive operator $0 \leq P \leq \mathbb{1}$ such that $P \in \mathfrak{p}(\rho; \delta)$ for some given $\delta \geq 0$. Then we have

$$S_1^P(\rho\,|\,\sigma) \leq \frac{S(\sqrt{P}\rho\sqrt{P}\,\|\,\sigma) + 2\delta' \log d + 2}{1 - \delta'} \tag{39}$$

where $\delta' := 2\sqrt{\delta}$, and $d := \dim \mathcal{H}$. □

Proof: The main ingredients of the proof of this lemma are the monotonicity property of the operator-smoothed conditional entropy (Lemma 10), the matrix convexity of the function $t \log t$, and the Fannes' inequality. From (28), we have

$$\begin{aligned} &S_1^P(\rho\,\|\,\sigma) \\ &= \frac{\mathrm{Tr}\left[\sqrt{P}\rho\log\rho\sqrt{P}\,\Pi_\sigma - \sqrt{P}\rho\sqrt{P}\log\sigma\right]}{\mathrm{Tr}[\sqrt{P}\rho\sqrt{P}\,\Pi_\sigma]} \\ &= \frac{\mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[(\mathbb{1} - P')(\rho\log\rho)] - \mathrm{Tr}[\sqrt{P}\rho\sqrt{P}\log\sigma]}{\mathrm{Tr}[P'\rho]} \end{aligned} \tag{40}$$

where, for our convenience, we have put $P' := \sqrt{P}\Pi_\sigma\sqrt{P}$. Since $P \in \mathfrak{p}(\rho; \delta)$, due to Lemma 2

$$\left\|\rho - \sqrt{P}\rho\sqrt{P}\right\|_1 \leq \delta' \tag{41}$$

where $\delta' := 2\sqrt{\delta}$. Obviously $0 \le P' \le \mathbb{1}$. Using (41), the fact that $\mathbb{1} \ge \Pi_\sigma \ge \Pi_\rho$, and the cyclicity of the trace, we have

$$\begin{aligned}\mathrm{Tr}[P'\rho] &= \mathrm{Tr}[\sqrt{P}\Pi_\sigma\sqrt{P}\,\rho] \\ &\ge \mathrm{Tr}[\Pi_\rho\sqrt{P}\rho\sqrt{P}] \\ &= \mathrm{Tr}[\Pi_\rho\,\rho] - \mathrm{Tr}\left[\Pi_\rho(\rho - \sqrt{P}\rho\sqrt{P})\right] \\ &\ge 1-\delta'. \end{aligned} \tag{42}$$

Hence, $P' \in \mathfrak{p}(\rho;\delta')$.

Since $t\log t$ is a matrix convex function, it is known that

$$-\mathrm{Tr}[K^\dagger(\rho\log\rho)K] \le -\mathrm{Tr}[(K^\dagger\rho K)\log(K^\dagger\rho K)]$$

for any contraction K, [29]. Let $K = K^\dagger = \sqrt{\mathbb{1}-P'}$. Then

$$\mathrm{Tr}[(\mathbb{1}-P')(-\rho\log\rho)] \le S(\bar{\rho})$$

where $\bar{\rho}$ is the subnormalized density matrix defined as $\bar{\rho} := \sqrt{\mathbb{1}-P'}\rho\sqrt{\mathbb{1}-P'}$. It is clear that, since $P' \in \mathfrak{p}(\rho;\delta')$, $\mathrm{Tr}[\bar{\rho}] \le \delta'$. Moreover, by simple algebra, $S(\bar{\rho}) \le \delta'\log d + 1$. This implies that

$$\begin{aligned}&S_1^P(\rho\,\|\,\sigma)\\ &\quad\le \frac{\mathrm{Tr}[\rho\log\rho] - \mathrm{Tr}[\sqrt{P}\rho\sqrt{P}\log\sigma] + \delta'\log d + 1}{1-\delta'}.\end{aligned} \tag{43}$$

By (41) and Fannes' continuity property of the von Neumann entropy [30], we have that

$$\begin{aligned}&\mathrm{Tr}[\rho\log\rho]\\ &\qquad\le \mathrm{Tr}[\sqrt{P}\rho\sqrt{P}\log(\sqrt{P}\rho\sqrt{P})] + \delta'\log d + 1\end{aligned}$$

which in turn yields (39). ■

Lemma 14: For any bipartite state $\rho^{AB} \in \mathfrak{S}(\mathcal{H}_A\otimes\mathcal{H}_B)$, and any given $\delta \ge 0$, we have

$$\tilde{I}^c_{0,\delta}(\rho^{AB}) \le \frac{I^c(\rho^{AB})}{1-\delta'} + \frac{4(\delta'\log(d_Ad_B)+1)}{1-\delta'} \tag{44}$$

where $d_A := \dim\mathcal{H}_A$, $d_B := \dim\mathcal{H}_B$, and $\delta' := 2\sqrt{\delta}$. □

Proof: By Lemma 10 we have

$$\begin{aligned}\tilde{I}^c_{0,\delta}(\rho^{AB}) &\le \tilde{I}^c_{1,\delta}(\rho^{AB}) \\ &= \max_{P\in\mathfrak{p}(\rho^{AB};\delta)}\min_{\sigma^B} S_1^P(\rho^{AB}\,\|\,\tau^A\otimes\sigma^B) \\ &\quad - \log d_A\end{aligned} \tag{45}$$

where $\tau^A := \mathbb{1}_A/d_A$, namely, the completely mixed state. In the above, we have made use of the following identity, which is easily obtained from (28): for two states ρ and σ, and any constant $c > 0$, $S_1^P(\rho\,\|\,c\sigma) = S_1^P(\rho\,\|\,\sigma) - \log c$. Using Lemma 13, and the analogous identity, $S(\rho\,\|\,c\sigma) = S(\rho\,\|\,\sigma) - \log c$, we have: for $\delta' = 2\sqrt{\delta}$

$$\begin{aligned}&S_1^P(\rho^{AB}\,\|\,\tau^A\otimes\sigma^B)\\ &\quad\le \frac{S(\sqrt{P}\rho^{AB}\sqrt{P}\,\|\,\tau^A\otimes\sigma^B) + 2\delta'\log(d_Ad_B)+2}{1-\delta'}\\ &\quad= \frac{S(\sqrt{P}\rho^{AB}\sqrt{P}\,\|\,\mathbb{1}^A\otimes\sigma^B) + 2\delta'\log(d_Ad_B)+2}{1-\delta'}\\ &\qquad+\frac{1}{1-\delta'}\log d_A.\end{aligned} \tag{46}$$

Then by (45), (46), and Lemma 6, we obtain

$$\begin{aligned}\tilde{I}^c_{0,\delta}(\rho^{AB}) \le \frac{I^c(\sqrt{P}\rho^{AB}\sqrt{P})}{1-\delta'} &+ \frac{2\delta'\log(d_Ad_B)+2}{1-\delta'}\\ &+\frac{\delta'}{1-\delta'}\log d_A.\end{aligned} \tag{47}$$

Finally, applying Fannes' inequality to each of the terms on the RHS of the identity $I^c(\omega^{AB}) = S(\omega^B) - S(\omega^{AB})$, where $\omega^{AB} := \sqrt{P}\rho^{AB}\sqrt{P}$, and $\omega^B := \mathrm{Tr}_A\omega^{AB}$, we obtain (44). ■

From (38) and Lemma 14 we obtain

$$\begin{aligned}Q^\infty(\Phi) &\le \lim_{n\to\infty}\frac{1}{n}\max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} I^c\left(\omega_{\mathcal{S}}^{R_nB_n}\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\max_{\mathcal{S}\subseteq\mathcal{H}_A^{\otimes n}} I^c(\mathcal{S},\Phi^{\otimes n})\end{aligned}$$

as claimed.

B. Multiple Uses of an Arbitrary Channel

To evaluate the quantum capacity of an arbitrary sequence of channels, we employ the well-known Quantum Information Spectrum Method [13], [31]–[33], [14]. Two fundamental quantities used in this approach are the *quantum spectral sup- and inf-divergence rates*, defined as follows.

Definition 5 (Spectral Divergence Rates): Given a sequence of states $\hat{\rho} = \{\rho_n\}_{n=1}^\infty$ and a sequence of positive operators $\hat{\sigma} = \{\sigma_n\}_{n=1}^\infty$, the quantum spectral sup- (inf-)divergence rates are defined in terms of the difference operators $\Pi_n(\gamma) = \rho_n - 2^{n\gamma}\sigma_n$ as

$$\begin{aligned}\bar{D}(\hat{\rho}\,\|\,\hat{\sigma}) := \inf\Big\{\gamma : &\limsup_{n\to\infty}\\ &\times\mathrm{Tr}[\{\Pi_n(\gamma)\ge 0\}\Pi_n(\gamma)] = 0\Big\}\end{aligned} \tag{48}$$

$$\begin{aligned}\underline{D}(\hat{\rho}\,\|\,\hat{\sigma}) := \sup\Big\{\gamma : &\liminf_{n\to\infty}\\ &\times\mathrm{Tr}[\{\Pi_n(\gamma)\ge 0\}\Pi_n(\gamma)] = 1\Big\}\end{aligned} \tag{49}$$

respectively. □

It is known that (see, e.g., [15])

$$\bar{D}(\hat{\rho}\,\|\,\hat{\sigma}) \ge \underline{D}(\hat{\rho}\,\|\,\hat{\sigma}). \tag{50}$$

In analogy with the usual definition of the coherent information (24), we moreover define the *spectral sup-* and *inf-coherent information rates*, respectively, as follows:

$$\bar{I}^c(\hat{\rho}^{RB}) := \min_{\hat{\sigma}^B} \bar{D}(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B), \quad (51)$$

$$\underline{I}^c(\hat{\rho}^{RB}) := \min_{\hat{\sigma}^B} \underline{D}(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B) \quad (52)$$

where $\hat{\rho}^{RB} := \{\rho^{R_nB_n} \in \mathfrak{S}(\mathcal{H}_R^{\otimes n} \otimes \mathcal{H}_B^{\otimes n})\}_{n=1}^{\infty}$, $\hat{\sigma}^B := \{\sigma^{B_n} \in \mathfrak{S}(\mathcal{H}_B^{\otimes n})\}_{n=1}^{\infty}$, and $\hat{\mathbb{1}}_R := \{\mathbb{1}_{R_n}\}_{n=1}^{\infty}$. Inequality (50) ensures that

$$\bar{I}^c(\hat{\rho}^{RB}) \geq \underline{I}^c(\hat{\rho}^{RB}). \quad (53)$$

Note that in (51) and (52) we could write minimum instead of infimum due to Lemma 1 in [14]. The same remark applies also to the following.

Theorem 3 (Arbitrary Channels): The quantum capacity of $\hat{\Phi}$ is given by

$$Q^{\infty}(\hat{\Phi}) = \max_{\hat{\mathcal{S}}} \underline{I}^c\left(\hat{\omega}_{\hat{\mathcal{S}}}^{RB}\right)$$

where $\hat{\mathcal{S}} := \{\mathcal{S}_n : \mathcal{S}_n \subseteq \mathcal{H}_A^{\otimes n}\}_{n=1}^{\infty}$, and $\hat{\omega}_{\hat{\mathcal{S}}}^{RB} := \{\omega_{\mathcal{S}_n}^{R_nB_n}\}_{n=1}^{\infty}$, with $\omega_{\mathcal{S}_n}^{R_nB_n} = \mathrm{Tr}_{E_n}[\Omega_{\mathcal{S}_n}^{R_nB_nE_n}]$, the pure state $|\Omega_{\mathcal{S}_n}^{R_nB_nE_n}\rangle$ being defined through (19). □

The above theorem follows directly from Theorem 1 and Lemma 15 and Lemma 16 given below.

Lemma 15 (Direct Part): Given a sequence of bipartite states $\hat{\rho}^{RB}$

$$\lim_{\delta\to 0}\liminf_{n\to\infty} \times \max_{\bar{\rho}_n^{R_nB_n} \in \mathfrak{b}(\rho_n^{R_nB_n};\delta)} \min_{\sigma_n^{B_n}} \frac{1}{n} S_0\left(\bar{\rho}_n^{R_nB_n} \| \mathbb{1}_{R_n} \otimes \sigma_n^{B_n}\right) \geq \min_{\hat{\sigma}^B} \underline{D}(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B).$$

□

Proof: This follows directly from [19, Theorem 3]. ■

Lemma 16 (Weak Converse): Given a sequence of bipartite states $\hat{\rho}^{RB}$,

$$\lim_{\delta\to 0}\liminf_{n\to\infty} \times \max_{P_n \in \mathfrak{p}(\rho_n^{R_nB_n};\delta)} \min_{\sigma_n^{B_n}} \frac{1}{n} S_0^{P_n}\left(\rho_n^{R_nB_n} \,\|\, \mathbb{1}_{R_n} \otimes \sigma_n^{B_n}\right) \leq \min_{\hat{\sigma}^B} \underline{D}(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B).$$

□

Proof: The proof is by *reductio ad absurdum*: we will assume that

$$\lim_{\delta\to 0}\liminf_{n\to\infty} \times \max_{P_n \in \mathfrak{p}(\rho_n^{R_nB_n};\delta)} \min_{\sigma_n^{B_n}} \frac{1}{n} S_0^{P_n}\left(\rho_n^{R_nB_n} \,\|\, \mathbb{1}_{R_n} \otimes \sigma_n^{B_n}\right) > \min_{\hat{\sigma}^B} \underline{D}(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B) \quad (54)$$

and show that such an assumption leads to a contradiction, hence proving the statement of the lemma.

Let $\hat{\bar{\sigma}} := \{\bar{\sigma}_n^{B_n}\}_{n=1}^{\infty}$ be the sequence achieving $\min_{\hat{\sigma}^B} \underline{D}(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B)$. Moreover, for any $\delta > 0$ fixed but arbitrary, let $\{\bar{P}_n\}_{n=1}^{\infty}$ be the sequence of operators, satisfying both $0 \leq \bar{P}_n \leq \mathbb{1}_{R_nB_n}$ and $\mathrm{Tr}[\bar{P}_n\rho_n^{R_nB_n}] \geq 1-\delta$, achieving the maximum over P_n of $\min_{\sigma_n^{B_n}} S_0^{P_n}(\rho_n^{R_nB_n} \,\|\, \mathbb{1}_{R_n} \otimes \sigma_n^{B_n})$, for all n. Then, (54) implies that

$$\lim_{\delta\to 0}\liminf_{n\to\infty} \frac{1}{n} S_0^{\bar{P}_n}\left(\rho_n^{R_nB_n} \,\|\, \mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}\right) > \underline{D}\left(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\bar{\sigma}}^B\right). \quad (55)$$

By arguments analogous to those used in the proof of Lemma 12, we can see that

$$\mathrm{Tr}\left[\sqrt{\bar{P}_n}\Pi_{\rho_n^{R_nB_n}}\sqrt{\bar{P}_n}\,\rho_n^{R_nB_n}\right] \geq 1 - 2\sqrt{\delta}. \quad (56)$$

For our convenience, let us put

$$\beta_\delta := \liminf_{n\to\infty} \frac{1}{n} S_0^{\bar{P}_n}\left(\rho_n^{R_nB_n} \,\|\, \mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}\right), \quad \gamma := \lim_{\delta\to 0} \beta_\delta.$$

It is clear that $\beta_\delta \geq \gamma$. Now, (54) implies (55), which is in turn equivalent to

$$\gamma > \min_{\hat{\sigma}^B} \underline{D}(\hat{\rho}^{RB} \,\|\, \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B).$$

Let then γ_0 be such that

$$\beta_\delta > \gamma_0 > \min_{\hat{\sigma}^B} \underline{D}(\hat{\rho}^{RB} \mid \hat{\mathbb{1}}_R \otimes \hat{\sigma}^B).$$

Moreover, by the definition of $\liminf$, there exists an n_0 such that, for all $n \geq n_0$,

$$\frac{1}{n} S_0^{\bar{P}_n}(\rho_n^{R_nB_n} \mid \mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}) \geq \beta_\delta.$$

The above equation can be rewritten as

$$\mathrm{Tr}\left[\sqrt{\bar{P}_n}\Pi_{\rho_n^{R_nB_n}}\sqrt{\bar{P}_n}\left(\mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}\right)\right] \leq 2^{-n\beta_\delta}$$

for all $n \geq n_0$. Now, for all $n \geq n_0$

$$\begin{aligned}
&\mathrm{Tr}\left[\sqrt{\bar{P}_n}\Pi_{\rho_n^{R_nB_n}}\sqrt{\bar{P}_n}\,\rho_n^{R_nB_n}\right] \\
&= \mathrm{Tr}\Big[\sqrt{\bar{P}_n}\Pi_{\rho_n^{R_nB_n}}\sqrt{\bar{P}_n} \\
&\quad \times \left(\rho_n^{R_nB_n} - 2^{n\gamma_0}\left(\mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}\right)\right)\Big] \\
&\quad + 2^{n\gamma_0}\,\mathrm{Tr}\left[\sqrt{\bar{P}_n}\Pi_{\rho_n^{R_nB_n}}\sqrt{\bar{P}_n}\left(\mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}\right)\right] \\
&\leq \mathrm{Tr}\Big[\left\{\rho_n^{R_nB_n} \geq 2^{n\gamma_0}\left(\mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}\right)\right\} \\
&\quad \times \left(\rho_n^{R_nB_n} - 2^{n\gamma_0}\left(\mathbb{1}_{R_n} \otimes \bar{\sigma}_n^{B_n}\right)\right)\Big] \\
&\quad + 2^{-n(\beta_\delta - \gamma_0)}
\end{aligned}$$

where, in the last step, we used Lemma 1. The second term in the sum goes to 0 as $n \to \infty$, since we chose $\gamma_0 < \beta_\delta$. The first term, on the other hand, has to be bounded away from 1 due to (49), since $\gamma_0 > \underline{D}(\hat{\rho}^{RB} \,|\, \hat{\mathbb{1}}_R \otimes \hat{\tilde{\sigma}}^B)$. Hence, (54) leads to

$$\liminf_{n\to\infty} \mathrm{Tr}\left[\sqrt{\bar{P}_n}\Pi_{\rho_n^{R_nB_n}}\sqrt{\bar{P}_n}\,\rho_n^{R_nB_n}\right] \le 1 - c_0$$

where $c_0 > 0$ is a constant independent of δ. This is clearly in contradiction with (56), which holds for all n and any arbitrary $\delta > 0$. ■

IX. Discussion

In this paper, we obtained bounds on the one-shot entanglement transmission capacity of an arbitrary quantum channel, which itself could correspond to a finite number of uses of a channel with arbitrarily correlated noise. Our result, in turn, yielded bounds on the one-shot quantum capacity of the channel. Further, for multiple uses of a memoryless channel, our results led to an expression for the asymptotic quantum capacity of the channel, in terms of the regularized coherent information. This provided an alternative form of the LSD theorem, which was however known to be equivalent to it [24]. Finally, by employing the Quantum Information Spectrum Method, we obtained an expression for the quantum capacity of an arbitrary infinite sequence of channels.

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