Yarn-Level Cloth Simulation with Sliding Persistent Contacts

Supplementary Document: Forces and Jacobians

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This document defines in detail forces and Jacobians for internal yarn forces and inter-yarn contact. All notation refers to Fig. 1 and Fig. 2, and force terms are described for yarns passing through the yarn crossing $\mathbf{q}_0$. We also rewrite the energy potentials for better readability.

1 Preliminaries

Let us first define some auxiliary quantities and their derivatives.

The norm l_i of the vector connecting $\mathbf{q}_0$ and each of its adjacent crossing nodes $\mathbf{q}_i$:

$$l_i = \|\mathbf{x}_i - \mathbf{x}_0\|. \tag{1}$$

$$l_i^2 = (\mathbf{x}_i - \mathbf{x}_0)^T (\mathbf{x}_i - \mathbf{x}_0),$$

$$l_i \frac{\partial l_i}{\partial \mathbf{x}_i} = (\mathbf{x}_i - \mathbf{x}_0)^T,$$

$$\frac{\partial l_i}{\partial \mathbf{x}_i} = \frac{1}{l_i} (\mathbf{x}_i - \mathbf{x}_0)^T = \mathbf{d}_i^T. \tag{2}$$

$$\frac{\partial l_i}{\partial \mathbf{x}_0} = -\mathbf{d}_i^T. \tag{3}$$

The unit vector $\mathbf{d}_i$ connecting $\mathbf{q}_0$ and each of its adjacent crossing nodes $\mathbf{q}_i$:

$$\mathbf{d}_i = \frac{\mathbf{x}_i - \mathbf{x}_0}{l_i}. \tag{4}$$

$$l_i \mathbf{d}_i = \mathbf{x}_i - \mathbf{x}_0,$$

$$l_i \frac{\partial \mathbf{d}_i}{\partial \mathbf{x}_i} + \mathbf{d}_i \frac{\partial l_i}{\partial \mathbf{x}_i} = \mathbf{I},$$

$$\frac{\partial \mathbf{d}_i}{\partial \mathbf{x}_i} = \frac{1}{l_i} \left(\mathbf{I} - \mathbf{d}_i \mathbf{d}_i^T \right) = \frac{1}{l_i} \mathbf{P}_i. \tag{5}$$

$$\frac{\partial \mathbf{d}_i}{\partial \mathbf{x}_0} = -\frac{1}{l_i} \mathbf{P}_i. \tag{6}$$

It is also convenient to define the projection to the normal plane of $\mathbf{d}_i$:

$$\mathbf{P}_i = \mathbf{I} - \mathbf{d}_i \mathbf{d}_i^T. \tag{7}$$

The vector $\mathbf{w}_i$ between $\mathbf{q}_0$ and each of its adjacent crossing nodes $\mathbf{q}_i$, normalized by arc length:

$$\mathbf{w}_i = \frac{\mathbf{x}_i - \mathbf{x}_0}{\|u_i - u_0\|}. \tag{8}$$

2 Stretch

Energy of the warp segment $[\mathbf{q}_0, \mathbf{q}_1]$, assuming $\Delta u = u_1 - u_0 > 0$:

$$V = \frac{1}{2} k_s \Delta u \left(\|\mathbf{w}_1\| - 1\right)^2. \tag{9}$$

Forces on crossing points:

$$\mathbf{F}_{\mathbf{x}_1} = -\mathbf{F}_{\mathbf{x}_0} = -k_s \left(\|\mathbf{w}_1\| - 1\right) \mathbf{d}_1. \tag{10}$$

Forces on warp coordinates:

$$F_{u_1} = -F_{u_0} = \frac{1}{2} k_s \left(\|\mathbf{w}_1\|^2 - 1\right). \tag{11}$$

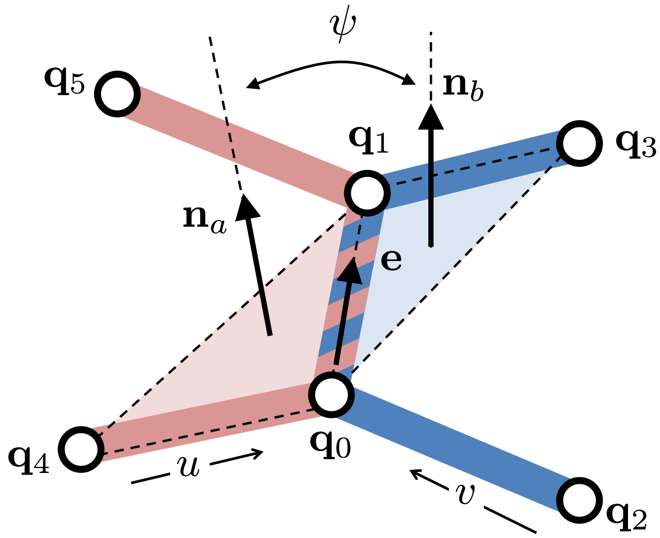

Figure 1: *a: Warp (u) and weft (v) yarns crossing at node $\mathbf{q}_0$, and the four adjacent yarn crossings. b: Bending angle θ between two adjacent warp segments. c: Forces producing normal compression at a crossing node. Subscripts s and b denote stretch and bending; superscripts $+$ and $-$ denote positive and negative yarn directions. d: Shear angle ϕ and shear jamming angle ϕ_j between two adjacent warp and weft yarns.*

Non-zero Jacobians:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = \frac{k}{l_1}\,\mathbf{P}_1 - \frac{k}{\Delta u}\,\mathbf{I}, \tag{12}$$

$$\frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_0}}{\partial u_0} = -\frac{\partial F_{u_1}}{\partial u_0} = -\frac{\partial F_{u_0}}{\partial u_1} = -k_s\,\frac{\|\mathbf{w}_1\|^2}{\Delta u}, \tag{13}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} = \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1} = k_s\,\frac{\|\mathbf{w}_1\|}{\Delta u}\,\mathbf{d}_1, \tag{14}$$

$$\frac{\partial F_{u_1}}{\partial \mathbf{x}_1} = \frac{\partial F_{u_0}}{\partial \mathbf{x}_0} = -\frac{\partial F_{u_1}}{\partial \mathbf{x}_0} = -\frac{\partial F_{u_0}}{\partial \mathbf{x}_1} = \frac{k_s}{\Delta u}\mathbf{w}_1^T. \tag{15}$$

3 Bending

Bending angle θ between warp segments $[\mathbf{q}_2, \mathbf{q}_0]$ and $[\mathbf{q}_0, \mathbf{q}_1]$, assuming $u_1 > u_0 > u_2$:

$$\theta = \arccos\left(-\mathbf{d}_1^T\mathbf{d}_2\right). \tag{16}$$

$$\cos\theta = -\mathbf{d}_1^T\mathbf{d}_2,$$

$$-\sin\theta\,\frac{\partial\theta}{\partial\mathbf{x}} = -\mathbf{d}_1^T\,\frac{\partial\mathbf{d}_2}{\partial\mathbf{x}} - \mathbf{d}_2^T\,\frac{\partial\mathbf{d}_1}{\partial\mathbf{x}},$$

$$\frac{\partial\theta}{\partial\mathbf{x}} = \frac{1}{\sin\theta}\left(\mathbf{d}_1^T\,\frac{\partial\mathbf{d}_2}{\partial\mathbf{x}} + \mathbf{d}_2^T\,\frac{\partial\mathbf{d}_1}{\partial\mathbf{x}}\right). \tag{17}$$

Energy of straight yarn:

$$V = k_b\,\frac{\theta^2}{u_1 - u_2}. \tag{18}$$

$$\frac{\partial V}{\partial\mathbf{x}} = \frac{2\,k_b\,\theta}{u_1 - u_2}\,\frac{\partial\theta}{\partial\mathbf{x}},$$

$$\frac{\partial V}{\partial\mathbf{x}} = \frac{2\,k_b\,\theta}{(u_1 - u_2)\,\sin\theta}\left(\mathbf{d}_1^T\,\frac{\partial\mathbf{d}_2}{\partial\mathbf{x}} + \mathbf{d}_2^T\,\frac{\partial\mathbf{d}_1}{\partial\mathbf{x}}\right),$$

$$\frac{\partial V}{\partial\mathbf{x}}^T = \frac{2\,k_b\,\theta}{(u_1 - u_2)\,\sin\theta}\left(\frac{\partial\mathbf{d}_2}{\partial\mathbf{x}}^T\mathbf{d}_1 + \frac{\partial\mathbf{d}_1}{\partial\mathbf{x}}^T\mathbf{d}_2\right). \tag{19}$$

$$\frac{\partial V}{\partial u} = -\frac{k_b\,\theta^2}{(u_1 - u_2)^2}\,\frac{\partial\,(u_1 - u_2)}{\partial u}. \tag{20}$$

Forces on crossing points:

$$\mathbf{F}_{\mathbf{x}_1} = -\frac{\partial V}{\partial \mathbf{x}_1}^T = -\frac{2\,k_b\,\theta}{(u_1-u_2)\,\sin\theta}\left(\frac{\partial \mathbf{d}_2}{\partial \mathbf{x}_1}^T \mathbf{d}_1 + \frac{\partial \mathbf{d}_1}{\partial \mathbf{x}_1}^T \mathbf{d}_2\right),$$

$$\mathbf{F}_{\mathbf{x}_1} = -\frac{2\,k_b\,\theta}{l_1\,(u_1-u_2)\,\sin\theta}\,\mathbf{P}_1\,\mathbf{d}_2. \tag{21}$$

$$\mathbf{F}_{\mathbf{x}_2} = -\frac{\partial V}{\partial \mathbf{x}_2}^T = -\frac{2\,k_b\,\theta}{(u_1-u_2)\,\sin\theta}\left(\frac{\partial \mathbf{d}_2}{\partial \mathbf{x}_2}^T \mathbf{d}_1 + \frac{\partial \mathbf{d}_1}{\partial \mathbf{x}_2}^T \mathbf{d}_2\right),$$

$$\mathbf{F}_{\mathbf{x}_2} = -\frac{2\,k_b\,\theta}{l_2\,(u_1-u_2)\,\sin\theta}\,\mathbf{P}_2\,\mathbf{d}_1. \tag{22}$$

$$\mathbf{F}_{\mathbf{x}_0} = -\frac{\partial V}{\partial \mathbf{x}_0}^T = -\frac{2\,k_b\,\theta}{(u_1-u_2)\,\sin\theta}\left(\frac{\partial \mathbf{d}_2}{\partial \mathbf{x}_0}^T \mathbf{d}_1 + \frac{\partial \mathbf{d}_1}{\partial \mathbf{x}_0}^T \mathbf{d}_2\right),$$

$$\mathbf{F}_{\mathbf{x}_0} = -\frac{2\,k_b\,\theta}{(u_1-u_2)\,\sin\theta}\left(-\frac{1}{l_2}\,\mathbf{P}_2\,\mathbf{d}_1 - \frac{1}{l_1}\,\mathbf{P}_1\,\mathbf{d}_2\right),$$

$$\mathbf{F}_{\mathbf{x}_0} = -\left(\mathbf{F}_{\mathbf{x}_1} + \mathbf{F}_{\mathbf{x}_2}\right). \tag{23}$$

Forces on warp coordinates:

$$F_{u_1} = -\frac{\partial V}{\partial u_1} = \frac{k_b\,\theta^2}{(u_1-u_2)^2}\,\frac{\partial\,(u_1-u_2)}{\partial u_1}, \tag{24}$$

$$F_{u_1} = \frac{k_b\,\theta^2}{(u_1-u_2)^2}.$$

$$F_{u_2} = -\frac{\partial V}{\partial u_2} = \frac{k_b\,\theta^2}{(u_1-u_2)^2}\,\frac{\partial\,(u_1-u_2)}{\partial u_2}, \tag{25}$$

$$F_{u_2} = -\frac{k_b\,\theta^2}{(u_1-u_2)^2} = -F_{u_1}.$$

$$F_{u_0} = -\frac{\partial V}{\partial u_0} = 0. \tag{26}$$

Non-zero Jacobians:

$$\mathbf{F}_{\mathbf{x}_1} = -\frac{2\,k_b\,\theta}{l_1\,(u_1-u_2)\,\sin\theta}\,\mathbf{P}_1\,\mathbf{d}_2,$$

$$l_1\,\sin\theta\,\mathbf{F}_{\mathbf{x}_1} = -\frac{2\,k_b\,\theta}{u_1-u_2}\,\mathbf{P}_1\,\mathbf{d}_2, = -\frac{2\,k_b\,\theta}{u_1-u_2}\left(\mathbf{I} - \mathbf{d}_1\,\mathbf{d}_1^T\right)\mathbf{d}_2,$$

$$l_1\,\sin\theta\,\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}} + l_1\,\cos\theta\,\mathbf{F}_{\mathbf{x}_1}\,\frac{\partial\theta}{\partial \mathbf{x}} + \sin\theta\,\mathbf{F}_{\mathbf{x}_1}\,\frac{\partial l_1}{\partial \mathbf{x}} =$$
$$-\frac{2\,k_b}{u_1-u_2}\,\mathbf{P}_1\,\mathbf{d}_2\,\frac{\partial\theta}{\partial \mathbf{x}} - \frac{2\,k_b\,\theta}{u_1-u_2}\,\mathbf{P}_1\,\frac{\partial \mathbf{d}_2}{\partial \mathbf{x}} + \frac{2\,k_b\,\theta}{u_1-u_2}\left(\mathbf{d}_1\,\mathbf{d}_2^T + \mathbf{d}_1^T\,\mathbf{d}_2\,\mathbf{I}\right)\frac{\partial \mathbf{d}_1}{\partial \mathbf{x}},$$

$$l_1\,\sin\theta\,\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}} = -\sin\theta\,\mathbf{F}_{\mathbf{x}_1}\,\frac{\partial l_1}{\partial \mathbf{x}} - \left(l_1\,\cos\theta\,\mathbf{F}_{\mathbf{x}_1} + \frac{2\,k_b}{u_1-u_2}\,\mathbf{P}_1\,\mathbf{d}_2\right)\frac{\partial\theta}{\partial \mathbf{x}}$$
$$-\frac{2\,k_b\,\theta}{u_1-u_2}\,\mathbf{P}_1\,\frac{\partial \mathbf{d}_2}{\partial \mathbf{x}} + \frac{2\,k_b\,\theta}{u_1-u_2}\left(\mathbf{d}_1\,\mathbf{d}_2^T + \mathbf{d}_1^T\,\mathbf{d}_2\,\mathbf{I}\right)\frac{\partial \mathbf{d}_1}{\partial \mathbf{x}}. \tag{27}$$

$$l_1\,\sin\theta\,\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = -\sin\theta\,\mathbf{F}_{\mathbf{x}_1}\,\frac{\partial l_1}{\partial \mathbf{x}_1} - \left(l_1\,\cos\theta\,\mathbf{F}_{\mathbf{x}_1} + \frac{2\,k_b}{u_1-u_2}\,\mathbf{P}_1\,\mathbf{d}_2\right)\frac{\partial\theta}{\partial \mathbf{x}_1} + \frac{2\,k_b\,\theta}{u_1-u_2}\left(\mathbf{d}_1\,\mathbf{d}_2^T + \mathbf{d}_1^T\,\mathbf{d}_2\,\mathbf{I}\right)\frac{\partial \mathbf{d}_1}{\partial \mathbf{x}_1},$$

$$l_1\,\sin\theta\,\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \sin\theta\,\frac{2\,k_b\,\theta}{l_1\,(u_1-u_2)\,\sin\theta}\,\mathbf{P}_1\,\mathbf{d}_2\,\mathbf{d}_1^T - \left(-l_1\,\cos\theta\,\frac{2\,k_b\,\theta}{l_1\,(u_1-u_2)\,\sin\theta}\,\mathbf{P}_1\,\mathbf{d}_2 + \frac{2\,k_b}{u_1-u_2}\,\mathbf{P}_1\,\mathbf{d}_2\right)\frac{1}{\sin\theta}\,\mathbf{d}_2^T\,\frac{1}{l_1}\,\mathbf{P}_1$$
$$+\frac{2\,k_b\,\theta}{u_1-u_2}\left(\mathbf{d}_1\,\mathbf{d}_2^T + \mathbf{d}_1^T\,\mathbf{d}_2\,\mathbf{I}\right)\frac{1}{l_1}\,\mathbf{P}_1,$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \frac{2\,k_b}{l_1^2\,\sin\theta\,(u_1-u_2)}\left(\theta\left(\mathbf{P}_1\,\mathbf{d}_2\,\mathbf{d}_1^T + \mathbf{d}_1\,\mathbf{d}_2^T\,\mathbf{P}_1 + \mathbf{d}_1^T\,\mathbf{d}_2\,\mathbf{P}_1\right) - \frac{1}{\sin\theta}\left(1 + \frac{\theta}{\sin\theta}\,\mathbf{d}_1^T\,\mathbf{d}_2\right)\mathbf{P}_1\,\mathbf{d}_2\,\mathbf{d}_2^T\,\mathbf{P}_1\right). \tag{28}$$

$$l_1\,\sin\theta\,\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_2}=-\left(l_1\,\cos\theta\,\mathbf{F}_{\mathbf{x}_1}+\frac{2\,k_b}{u_1-u_2}\,\mathbf{P}_1\,\mathbf{d}_2\right)\,\frac{\partial\theta}{\partial\mathbf{x}_2}-\frac{2\,k_b\,\theta}{u_1-u_2}\,\mathbf{P}_1\,\frac{\partial\mathbf{d}_2}{\partial\mathbf{x}_2},$$
$$l_1\,\sin\theta\,\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_2}=\left(-\left(-l_1\,\cos\theta\,\frac{2\,k_b\,\theta}{l_1\,(u_1-u_2)\,\sin\theta}\,\mathbf{P}_1\,\mathbf{d}_2+\frac{2\,k_b}{u_1-u_2}\,\mathbf{P}_1\,\mathbf{d}_2\right)\,\frac{1}{\sin\theta}\,\mathbf{d}_1^T-\frac{2\,k_b\,\theta}{u_1-u_2}\,\mathbf{P}_1\right)\,\frac{1}{l_2}\,\mathbf{P}_2,$$
$$\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_2}=-\frac{2\,k_b}{l_2\,l_1\,\sin\theta\,(u_1-u_2)}\,\left(\frac{1}{\sin\theta}\,\left(1+\frac{\theta}{\sin\theta}\,\mathbf{d}_1^T\,\mathbf{d}_2\right)\,\mathbf{P}_1\,\mathbf{d}_2\,\mathbf{d}_1^T+\theta\,\mathbf{P}_1\right)\,\mathbf{P}_2. \tag{29}$$

$$\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_1}=-\frac{2\,k_b}{l_1\,l_2\,\sin\theta\,(u_1-u_2)}\,\left(\frac{1}{\sin\theta}\,\left(1+\frac{\theta}{\sin\theta}\,\mathbf{d}_1^T\,\mathbf{d}_2\right)\,\mathbf{P}_2\,\mathbf{d}_1\,\mathbf{d}_2^T+\theta\,\mathbf{P}_2\right)\,\mathbf{P}_1, \tag{30}$$
$$\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_2}=\frac{2\,k_b}{l_2^2\,\sin\theta\,(u_1-u_2)}\,\left(\theta\,\left(\mathbf{P}_2\,\mathbf{d}_1\,\mathbf{d}_2^T+\mathbf{d}_2\,\mathbf{d}_1^T\,\mathbf{P}_2+\mathbf{d}_2^T\,\mathbf{d}_1\,\mathbf{P}_2\right)-\frac{1}{\sin\theta}\,\left(1+\frac{\theta}{\sin\theta}\,\mathbf{d}_1^T\,\mathbf{d}_2\right)\,\mathbf{P}_2\,\mathbf{d}_1\,\mathbf{d}_1^T\,\mathbf{P}_2\right), \tag{31}$$
$$\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_0}=-\left(\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_1}+\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_2}\right),\qquad\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_0}=-\left(\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_1}+\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_2}\right), \tag{32}$$
$$\frac{\partial\mathbf{F}_{\mathbf{x}_0}}{\partial\mathbf{x}_1}=-\left(\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_1}+\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_1}\right),\qquad\frac{\partial\mathbf{F}_{\mathbf{x}_0}}{\partial\mathbf{x}_2}=-\left(\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_2}+\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_2}\right),\qquad\frac{\partial\mathbf{F}_{\mathbf{x}_0}}{\partial\mathbf{x}_0}=-\left(\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial\mathbf{x}_0}+\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial\mathbf{x}_0}\right). \tag{33}$$

$$\frac{\partial F_{u_1}}{\partial u_1}=\frac{\partial F_{u_2}}{\partial u_2}=-\frac{\partial F_{u_1}}{\partial u_2}=-\frac{\partial F_{u_2}}{\partial u_1}=-\frac{2\,k_b\,\theta^2}{(u_1-u_2)^3}, \tag{34}$$
$$\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial u_1}=-\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial u_2}=\frac{2\,k_b\,\theta}{l_1\,(u_1-u_2)^2\,\sin\theta}\,\mathbf{P}_1\,\mathbf{d}_2, \tag{35}$$
$$\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial u_1}=-\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial u_2}=\frac{2\,k_b\,\theta}{l_2\,(u_1-u_2)^2\,\sin\theta}\,\mathbf{P}_2\,\mathbf{d}_1, \tag{36}$$
$$\frac{\partial\mathbf{F}_{\mathbf{x}_0}}{\partial u_1}=-\frac{\partial\mathbf{F}_{\mathbf{x}_0}}{\partial u_2}=-\left(\frac{\partial\mathbf{F}_{\mathbf{x}_1}}{\partial u_1}+\frac{\partial\mathbf{F}_{\mathbf{x}_2}}{\partial u_1}\right), \tag{37}$$
$$\frac{\partial F_{u_1}}{\partial\mathbf{x}_1}=-\frac{\partial F_{u_2}}{\partial\mathbf{x}_1}=\frac{2\,k_b\,\theta}{l_1\,(u_1-u_2)^2\,\sin\theta}\,\mathbf{d}_2^T\,\mathbf{P}_1, \tag{38}$$
$$\frac{\partial F_{u_1}}{\partial\mathbf{x}_2}=-\frac{\partial F_{u_2}}{\partial\mathbf{x}_2}=\frac{2\,k_b\,\theta}{l_2\,(u_1-u_2)^2\,\sin\theta}\,\mathbf{d}_1^T\,\mathbf{P}_2, \tag{39}$$
$$\frac{\partial F_{u_1}}{\partial\mathbf{x}_0}=-\frac{\partial F_{u_2}}{\partial\mathbf{x}_0}=-\left(\frac{\partial F_{u_1}}{\partial\mathbf{x}_1}+\frac{\partial F_{u_1}}{\partial\mathbf{x}_2}\right). \tag{40}$$

4 Shear

Shear angle ϕ between warp segment $[\mathbf{q}_0,\mathbf{q}_1]$ and weft segment $[\mathbf{q}_0,\mathbf{q}_3]$, assuming $u_1>u_0$ and $v_3>v_0$:

$$\phi=\arccos(\mathbf{d}_1^T\mathbf{d}_3). \tag{41}$$

Energy:

$$V=\frac{1}{2}\,k_x\,L\,\left(\phi-\frac{\pi}{2}\right)^2. \tag{42}$$

Forces on warp and weft coordinates are all zero. Forces on crossing points:

$$\mathbf{F}_{\mathbf{x}_1}=\frac{k_x\,L\,\left(\phi-\frac{\pi}{2}\right)}{l_1\,\sin\phi}\,\mathbf{P}_1\,\mathbf{d}_3, \tag{43}$$
$$\mathbf{F}_{\mathbf{x}_3}=\frac{k_x\,L\,\left(\phi-\frac{\pi}{2}\right)}{l_3\,\sin\phi}\,\mathbf{P}_3\,\mathbf{d}_1, \tag{44}$$
$$\mathbf{F}_{\mathbf{x}_0}=-(\mathbf{F}_{\mathbf{x}_1}+\mathbf{F}_{\mathbf{x}_3}). \tag{45}$$

Non-zero Jacobians:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \frac{k_x\, L}{l_1^2\, \sin\phi} \left(\left(\phi - \frac{\pi}{2}\right) \left(-\mathbf{P}_1\, \mathbf{d}_3\, \mathbf{d}_1^T + \frac{\cos\phi}{\sin^2\phi}\, \mathbf{P}_1\, \mathbf{d}_3\, \mathbf{d}_3^T\, \mathbf{P}_1 - \cos\phi\, \mathbf{P}_1 - \mathbf{d}_1\, \mathbf{d}_3^T\, \mathbf{P}_1 \right) - \frac{1}{\sin\phi}\, \mathbf{P}_1\, \mathbf{d}_3\, \mathbf{d}_3^T\, \mathbf{P}_1 \right), \tag{46}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} = \frac{k_x\, L}{l_3\, l_1\, \sin\phi} \left(\left(\phi - \frac{\pi}{2}\right) \left(\frac{\cos\phi}{\sin^2\phi}\, \mathbf{P}_1\, \mathbf{d}_3\, \mathbf{d}_1^T + \mathbf{P}_1 \right) - \frac{1}{\sin\phi}\, \mathbf{P}_1\, \mathbf{d}_3\, \mathbf{d}_1^T \right) \mathbf{P}_3, \tag{47}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} = \frac{k_x\, L}{l_1\, l_3\, \sin\phi} \left(\left(\phi - \frac{\pi}{2}\right) \left(\frac{\cos\phi}{\sin^2\phi}\, \mathbf{P}_3\, \mathbf{d}_1\, \mathbf{d}_3^T + \mathbf{P}_3 \right) - \frac{1}{\sin\phi}\, \mathbf{P}_3\, \mathbf{d}_1\, \mathbf{d}_3^T \right) \mathbf{P}_1, \tag{48}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3} = \frac{k_x\, L}{l_3^2\, \sin\phi} \left(\left(\phi - \frac{\pi}{2}\right) \left(-\mathbf{P}_3\, \mathbf{d}_1\, \mathbf{d}_3^T + \frac{\cos\phi}{\sin^2\phi}\, \mathbf{P}_3\, \mathbf{d}_1\, \mathbf{d}_1^T\, \mathbf{P}_3 - \cos\phi\, \mathbf{P}_3 - \mathbf{d}_3\, \mathbf{d}_1^T\, \mathbf{P}_3 \right) - \frac{1}{\sin\phi}\, \mathbf{P}_3\, \mathbf{d}_1\, \mathbf{d}_1^T\, \mathbf{P}_3 \right), \tag{49}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} \right), \qquad \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3} \right), \tag{50}$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} \right), \qquad \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_3} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3} \right), \qquad \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_0} \right). \tag{51}$$

5 Twist

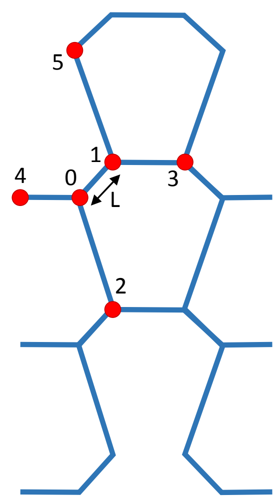

Figure 2

Two adjacent triangles $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_5)$ and $(\mathbf{q}_0, \mathbf{q}_3, \mathbf{q}_1)$ have normal vectors $\mathbf{n}_a$ and $\mathbf{n}_b$.

$$\mathbf{v}_a = (\mathbf{x}_5 - \mathbf{x}_1) \times (\mathbf{x}_0 - \mathbf{x}_1),$$

$$\mathbf{n}_a = \frac{\mathbf{v}_a}{\|\mathbf{v}_a\|}. \tag{52}$$

$$\frac{\partial \mathbf{v}_a}{\partial \mathbf{x}_0} = (\mathbf{x}_5 - \mathbf{x}_1)^* = \mathbf{x}_{a0}^*, \quad \frac{\partial \mathbf{v}_a}{\partial \mathbf{x}_1} = (\mathbf{x}_0 - \mathbf{x}_5)^* = \mathbf{x}_{a1}^*, \quad \frac{\partial \mathbf{v}_a}{\partial \mathbf{x}_5} = (\mathbf{x}_1 - \mathbf{x}_0)^* = \mathbf{x}_{a5}^*. \tag{53}$$

$$\mathbf{v}_b = (\mathbf{x}_0 - \mathbf{x}_1) \times (\mathbf{x}_3 - \mathbf{x}_1),$$

$$\mathbf{n}_b = \frac{\mathbf{v}_b}{\|\mathbf{v}_b\|}. \tag{54}$$

$$\frac{\partial \mathbf{v}_b}{\partial \mathbf{x}_0} = (\mathbf{x}_1 - \mathbf{x}_3)^* = \mathbf{x}_{b0}^*, \quad \frac{\partial \mathbf{v}_b}{\partial \mathbf{x}_1} = (\mathbf{x}_3 - \mathbf{x}_0)^* = \mathbf{x}_{b1}^*, \quad \frac{\partial \mathbf{v}_b}{\partial \mathbf{x}_3} = (\mathbf{x}_0 - \mathbf{x}_1)^* = \mathbf{x}_{b3}^*. \tag{55}$$

$$\frac{\partial \mathbf{x}_{ai}}{\partial \mathbf{x}_j} \in \{\mathbf{I}, -\mathbf{I}, \mathbf{0}\}, \text{ and the same for } \frac{\partial \mathbf{x}_{bi}}{\partial \mathbf{x}_j}. \tag{56}$$

Derivative of a unit vector:

$$\mathbf{n} = \frac{\mathbf{v}}{\|\mathbf{v}\|},$$

$$\frac{\partial \mathbf{n}}{\partial \mathbf{x}} = \frac{1}{\|\mathbf{v}\|} \left(\mathbf{I} - \mathbf{n}\, \mathbf{n}^T \right) \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{1}{\|\mathbf{v}\|}\, \mathbf{P}\, \frac{\partial \mathbf{v}}{\partial \mathbf{x}}, \quad \text{with } \mathbf{P} = \mathbf{I} - \mathbf{n}\, \mathbf{n}^T. \tag{57}$$

Edge vector:

$$\mathbf{e} = \frac{\mathbf{x}_1 - \mathbf{x}_0}{\|\mathbf{x}_1 - \mathbf{x}_0\|}. \tag{58}$$

$$\frac{\partial \mathbf{e}}{\partial \mathbf{x}} = \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_0\|} \left(\mathbf{I} - \mathbf{e}\,\mathbf{e}^T\right) \frac{\partial\left(\mathbf{x}_1 - \mathbf{x}_0\right)}{\partial \mathbf{x}},$$

$$\frac{\partial \mathbf{e}}{\partial \mathbf{x}_1} = \frac{1}{\|\mathbf{x}_1 - \mathbf{x}_0\|} \left(\mathbf{I} - \mathbf{e}\,\mathbf{e}^T\right), \tag{59}$$

$$\frac{\partial \mathbf{e}}{\partial \mathbf{x}_0} = -\frac{1}{\|\mathbf{x}_1 - \mathbf{x}_0\|} \left(\mathbf{I} - \mathbf{e}\,\mathbf{e}^T\right). \tag{60}$$

To derive forces and their Jacobians we will use two tricks. First, these expressions require the projection of one triangle normal onto the tangent plane of the other normal. To write this concisely, we express one normal as a rotation of the other normal around the edge vector using the Rodrigues rotation formula. Bear in mind that the edge vector and the normals are orthogonal.

$$\mathbf{n}_b = \mathbf{n}_a\,\cos\psi + (\mathbf{e}\times\mathbf{n}_a)\,\sin\psi + \mathbf{e}^T\mathbf{n}_a\,(1-\cos\psi)\,\mathbf{e},$$

$$\mathbf{n}_b = \mathbf{n}_a\,\cos\psi + (\mathbf{e}\times\mathbf{n}_a)\,\sin\psi. \tag{61}$$

$$\mathbf{P}_a\,\mathbf{n}_b = (\mathbf{e}\times\mathbf{n}_a)\,\sin\psi. \tag{62}$$

$$\mathbf{n}_a = \mathbf{n}_b\,\cos\psi - (\mathbf{e}\times\mathbf{n}_b)\,\sin\psi + \mathbf{e}^T\mathbf{n}_b\,(1-\cos\psi)\,\mathbf{e},$$

$$\mathbf{n}_a = \mathbf{n}_b\,\cos\psi - (\mathbf{e}\times\mathbf{n}_b)\,\sin\psi. \tag{63}$$

$$\mathbf{P}_b\,\mathbf{n}_a = -(\mathbf{e}\times\mathbf{n}_b)\,\sin\psi. \tag{64}$$

Second, the expressions require the computation of cross products with two vectors on the tangent plane of each normal. This expression can also be formulated more concisely. Next, $\mathbf{u}$ and $\mathbf{e}$ form a plane normal to $\mathbf{n}$:

$$\mathbf{u}\times(\mathbf{e}\times\mathbf{n}) = \|\mathbf{u}\|\,\mathbf{n}\,\sin(\mathbf{u},\mathbf{e}\times\mathbf{n}) = \|\mathbf{u}\|\,\mathbf{n}\,(\sin((\mathbf{u},\mathbf{e})-\pi/2)) = -\|\mathbf{u}\|\,\mathbf{n}\,\cos(\mathbf{u},\mathbf{e}) = -\mathbf{u}^T\mathbf{e}\,\mathbf{n}. \tag{65}$$

Twist angle ψ between the two adjacent triangles:

$$\psi = \arccos(\mathbf{n}_a^T\mathbf{n}_b). \tag{66}$$

$$\cos\psi = \mathbf{n}_a^T\mathbf{n}_b,$$

$$-\sin\psi\,\frac{\partial\psi}{\partial\mathbf{x}} = \mathbf{n}_a^T\,\frac{\partial\mathbf{n}_b}{\partial\mathbf{x}} + \mathbf{n}_b^T\,\frac{\partial\mathbf{n}_a}{\partial\mathbf{x}},$$

$$\frac{\partial\psi}{\partial\mathbf{x}} = -\frac{1}{\sin\psi}\left(\mathbf{n}_a^T\,\frac{\partial\mathbf{n}_b}{\partial\mathbf{x}} + \mathbf{n}_b^T\,\frac{\partial\mathbf{n}_a}{\partial\mathbf{x}}\right),$$

$$\frac{\partial\psi}{\partial\mathbf{x}} = -\frac{1}{\sin\psi}\left(\frac{1}{\|\mathbf{v}_b\|}\,\mathbf{n}_a^T\,\mathbf{P}_b\,\frac{\partial\mathbf{v}_b}{\partial\mathbf{x}} + \frac{1}{\|\mathbf{v}_a\|}\,\mathbf{n}_b^T\,\mathbf{P}_a\,\frac{\partial\mathbf{v}_a}{\partial\mathbf{x}}\right),$$

$$\frac{\partial\psi}{\partial\mathbf{x}_i} = -\frac{1}{\sin\psi}\left(-\frac{1}{\|\mathbf{v}_b\|}\,(\mathbf{e}\times\mathbf{n}_b)^T\,\sin\psi\,\mathbf{x}_{bi}^* + \frac{1}{\|\mathbf{v}_a\|}\,(\mathbf{e}\times\mathbf{n}_a)^T\,\sin\psi\,\mathbf{x}_{ai}^*\right),$$

$$\frac{\partial\psi}{\partial\mathbf{x}_i} = \frac{1}{\|\mathbf{v}_b\|}\,\mathbf{n}_b^T\,\mathbf{e}^T\,\mathbf{x}_{bi} - \frac{1}{\|\mathbf{v}_a\|}\,\mathbf{n}_a^T\,\mathbf{e}^T\,\mathbf{x}_{ai}. \tag{67}$$

Roll energy for stitched yarns:

$$V = \frac{1}{2}\,k_t\,L\,\psi^2. \tag{68}$$

$$\frac{\partial V}{\partial\mathbf{x}} = k_t\,L\,\psi\,\frac{\partial\psi}{\partial\mathbf{x}}. \tag{69}$$

Forces on crossing points:

$$\mathbf{F}_{\mathbf{x}} = -\frac{\partial V}{\partial\mathbf{x}}^T = -k_t\,L\,\psi\,\frac{\partial\psi}{\partial\mathbf{x}}^T,$$

$$\mathbf{F}_{\mathbf{x}_i} = -k_t\,L\,\psi\,\left(\frac{1}{\|\mathbf{v}_b\|}\,\mathbf{x}_{bi}^T\,\mathbf{e}\,\mathbf{n}_b - \frac{1}{\|\mathbf{v}_a\|}\,\mathbf{x}_{ai}^T\,\mathbf{e}\,\mathbf{n}_a\right). \tag{70}$$

Force Jacobians:

$$
\begin{aligned}
\mathbf{F}_{\mathbf{x}_i} &= -k_t\, L\, \psi\, \left(\frac{1}{\|\mathbf{v}_b\|}\, \mathbf{x}_{bi}^T\, \mathbf{e}\, \mathbf{n}_b - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{x}_{ai}^T\, \mathbf{e}\, \mathbf{n}_a \right), \\
\frac{\partial \mathbf{F}_{\mathbf{x}_i}}{\partial \mathbf{y}} &= -k_t\, L\, \left(\frac{1}{\|\mathbf{v}_b\|}\, \mathbf{x}_{bi}^T\, \mathbf{e}\, \mathbf{n}_b - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{x}_{ai}^T\, \mathbf{e}\, \mathbf{n}_a \right) \frac{\partial \psi}{\partial \mathbf{y}} \\
&- k_t\, L\, \psi\, \left(-\frac{1}{\|\mathbf{v}_b\|^2}\, \mathbf{x}_{bi}^T\, \mathbf{e}\, \mathbf{n}_b\, \mathbf{n}_b^T\, \frac{\partial \mathbf{v}_b}{\partial \mathbf{y}} + \frac{1}{\|\mathbf{v}_b\|}\, \mathbf{x}_{bi}^T\, \mathbf{e}\, \frac{\partial \mathbf{n}_b}{\partial \mathbf{y}} + \frac{1}{\|\mathbf{v}_b\|}\, \mathbf{n}_b\, \mathbf{x}_{bi}^T\, \frac{\partial \mathbf{e}}{\partial \mathbf{y}} + \frac{1}{\|\mathbf{v}_b\|}\, \mathbf{n}_b\, \mathbf{e}^T\, \frac{\partial \mathbf{x}_{bi}}{\partial \mathbf{y}} \right) \\
&- k_t\, L\, \psi\, \left(\frac{1}{\|\mathbf{v}_a\|^2}\, \mathbf{x}_{ai}^T\, \mathbf{e}\, \mathbf{n}_a\, \mathbf{n}_a^T\, \frac{\partial \mathbf{v}_a}{\partial \mathbf{y}} - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{x}_{ai}^T\, \mathbf{e}\, \frac{\partial \mathbf{n}_a}{\partial \mathbf{y}} - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{n}_a\, \mathbf{x}_{ai}^T\, \frac{\partial \mathbf{e}}{\partial \mathbf{y}} - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{n}_a\, \mathbf{e}^T\, \frac{\partial \mathbf{x}_{ai}}{\partial \mathbf{y}} \right), \\
\frac{\partial \mathbf{F}_{\mathbf{x}_i}}{\partial \mathbf{x}_j} &= -k_t\, L\, \left(\frac{1}{\|\mathbf{v}_b\|}\, \mathbf{x}_{bi}^T\, \mathbf{e}\, \mathbf{n}_b - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{x}_{ai}^T\, \mathbf{e}\, \mathbf{n}_a \right) \left(\frac{1}{\|\mathbf{v}_b\|}\, \mathbf{x}_{bj}^T\, \mathbf{e}\, \mathbf{n}_b - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{x}_{aj}^T\, \mathbf{e}\, \mathbf{n}_a \right)^T \\
&- k_t\, L\, \psi\, \left(\frac{1}{\|\mathbf{v}_b\|^2}\, \mathbf{x}_{bi}^T\, \mathbf{e}\, \left(\mathbf{I} - 2\, \mathbf{n}_b\, \mathbf{n}_b^T \right) \mathbf{x}_{bj}^* + \frac{1}{\|\mathbf{v}_b\|}\, \mathbf{n}_b\, \mathbf{x}_{bi}^T\, \frac{\partial \mathbf{e}}{\partial \mathbf{x}_j} + \frac{1}{\|\mathbf{v}_b\|}\, \mathbf{n}_b\, \mathbf{e}^T\, \frac{\partial \mathbf{x}_{bi}}{\partial \mathbf{x}_j} \right) \\
&- k_t\, L\, \psi\, \left(-\frac{1}{\|\mathbf{v}_a\|^2}\, \mathbf{x}_{ai}^T\, \mathbf{e}\, \left(\mathbf{I} - 2\, \mathbf{n}_a\, \mathbf{n}_a^T \right) \mathbf{x}_{aj}^* - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{n}_a\, \mathbf{x}_{ai}^T\, \frac{\partial \mathbf{e}}{\partial \mathbf{x}_j} - \frac{1}{\|\mathbf{v}_a\|}\, \mathbf{n}_a\, \mathbf{e}^T\, \frac{\partial \mathbf{x}_{ai}}{\partial \mathbf{x}_j} \right).
\end{aligned} \tag{71}
$$

6 Parallel Contact

Energy of the warp segment $[\mathbf{q}_0, \mathbf{q}_1]$, assuming contact exists, i.e., $\Delta u = u_1 - u_0 < d$:

$$
V_{0,1} = \frac{1}{2}\, k_c\, L\, (\Delta_u - d)^2. \tag{72}
$$

Forces on warp coordinates:

$$
F_{u_0} = -F_{u_1} = k_c\, L\, (\Delta_u - d). \tag{73}
$$

Non-zero Jacobians:

$$
\frac{\partial F_{u_0}}{\partial u_0} = \frac{\partial F_{u_1}}{\partial u_1} = -\frac{\partial F_{u_0}}{\partial u_1} = -\frac{\partial F_{u_1}}{\partial u_0} = -k_c\, L. \tag{74}
$$